

# Linear entropy as entanglement fermionic measure for an electronic scattering in 2D semiconductor system

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**Bertinoro, 5 December, QMFPA**

# Entanglement in systems of identical particles

## ✿ *"Accessible Entanglement"*

Wiseman and Vaccaro PRL 91, 097902 (2003).

## ✿ *Use of density matrix mode-occupation representation*

Zanardi PRA 65,042101(2002).

## ✿ *Slater rank of the state*

Schliemann et al. PRA, 64, 022303 (2001).

## Linear entropy

✧ *Slater rank*

✧ *Linear entropy of reduced statistical operators*

*The aim of this talk is to show that the linear entropy is an efficient entanglement measure for a two-fermion system and its evaluation in binary collisions is easier than the ones performed with other approaches*

Any "pure" two-particle fermionic state can be written as

$$|\Psi_F\rangle = \sum_{i,j}^{2N} w_{ij} a_i^\dagger a_j^\dagger |0\rangle \quad , \quad \{a_i, a_j^\dagger\} = \delta_{ij}$$

where  $W$  is a complex and antisymmetric matrix ( $2N \times 2N$ ) while  $2N$  is the number of possible modes for each particle.

From the state  $|\Psi_F\rangle$  we can move to one-particle density matrix in this way

$$|\Psi_F\rangle \longrightarrow \rho_F = |\Psi_F\rangle\langle\Psi_F| \longrightarrow \rho_{\nu,\mu} = \frac{\text{Tr}(\rho_F a_\mu^\dagger a_\nu^\dagger)}{\text{Tr}(\rho_F \sum_\mu a_\mu^\dagger a_\mu^\dagger)} = (W^\dagger W)_{\mu,\nu}$$

The eigenvalues of the matrix  $\rho$  are  $|z_i|^2$

## Linear entropy

$$\varepsilon_L = 1 - \text{Tr}\rho^2$$

We note that the linear entropy can be expressed in terms of matrix  $W$ :

$$\varepsilon_L = 1 - \sum_{i,j}^{2N} \left| \sum_k^{2N} w_{ik} w_{kj}^* \right|^2$$

In order to calculate  $\varepsilon_L$  the definition and the allocation of the matrix  $\rho$  is not required. In this sense the linear entropy is easier to calculate than von Neumann entropy since no diagonalization of the matrix  $\rho$  is needed

Anyway by using the trace operation we can put  $\varepsilon_L$  in terms of eigenvalues of one particle density matrix



$$\varepsilon_L = 1 - \sum_k^N 2|z_k|^4$$

while the expression of von Neumann entropy is given by



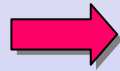
$$\varepsilon_{vN} = \ln 2 - \sum_k^N 2|z_k|^2 \ln 2|z_k|^2$$

## Again about linear entropy

Nonentangled states:  $\varepsilon_L = 1/2$  Entangled states:  $1/2 < \varepsilon_L \leq 1 - 1/(2N)$

The minimum value  $1/2$  indicates the unavoidable quantum correlations related to exchange symmetry as for the case of von Neumann entropy

$$\varepsilon_L > 1/2$$



Additional ignorance naturally connected with genuine entanglement

# Evaluation of the linear entropy and von Neumann entropy in a simple system

Let us consider the state of a simple two-fermion system with  $2N$  degrees of freedom

$$|\chi\rangle = \sqrt{\frac{1 + (N-1)(1-\alpha)^2}{2N}} (|12\rangle - |21\rangle) + \dots + \sqrt{\frac{\alpha(2-\alpha)}{2N}} (|2k-1\ 2k\rangle - |2k\ 2k-1\rangle) \dots$$

where  $2 \leq k \leq N$  and  $\alpha$  is a real parameter ranging between 0 and 1. The matrix  $W$  is a block diagonal matrix with  $2 \times 2$  blocks

$$W_1 = \begin{pmatrix} 0 & \sqrt{\frac{1+(N-1)(1-\alpha)^2}{2N}} \\ -\sqrt{\frac{1+(N-1)(1-\alpha)^2}{2N}} & 0 \end{pmatrix}$$

$$W_k = \begin{pmatrix} 0 & \sqrt{\frac{\alpha(2-\alpha)}{2N}} \\ -\sqrt{\frac{\alpha(2-\alpha)}{2N}} & 0 \end{pmatrix}$$

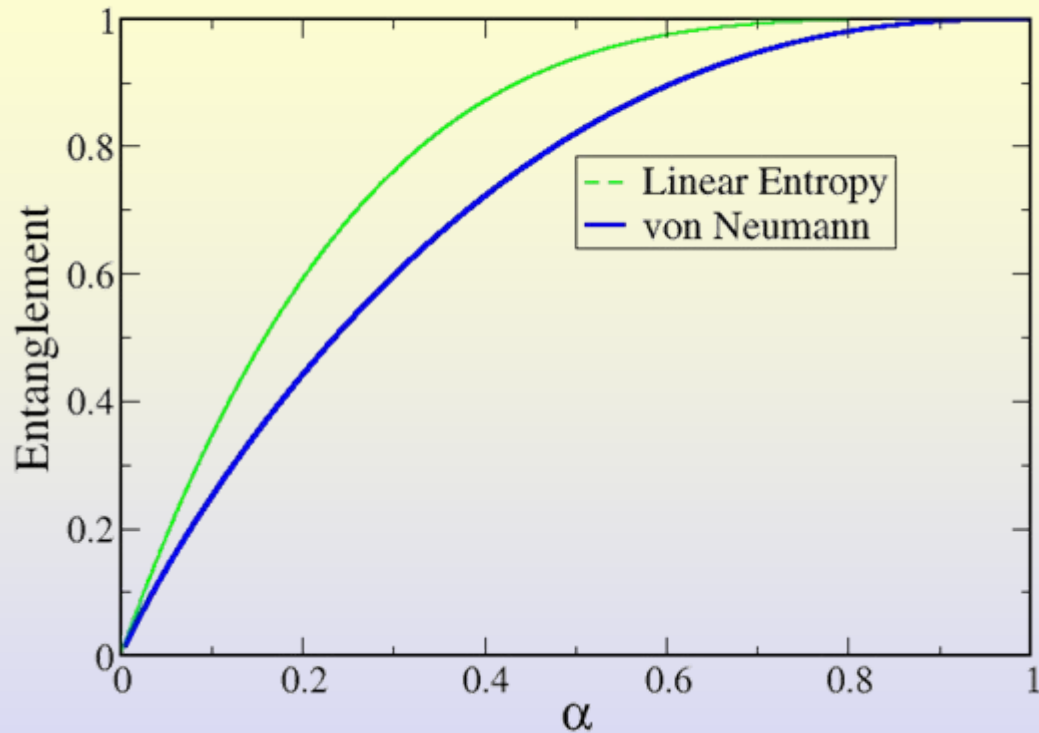
The linear entropy as a function of the parameter  $\alpha$  takes this form

$$\tilde{\varepsilon}_L^{\alpha} = 1 - (1 - \alpha)^4$$

where we normalized  $\varepsilon_L$  so that its values lie in the interval between 0 and 1.

The expression for the normalized von Neumann entropy is given by

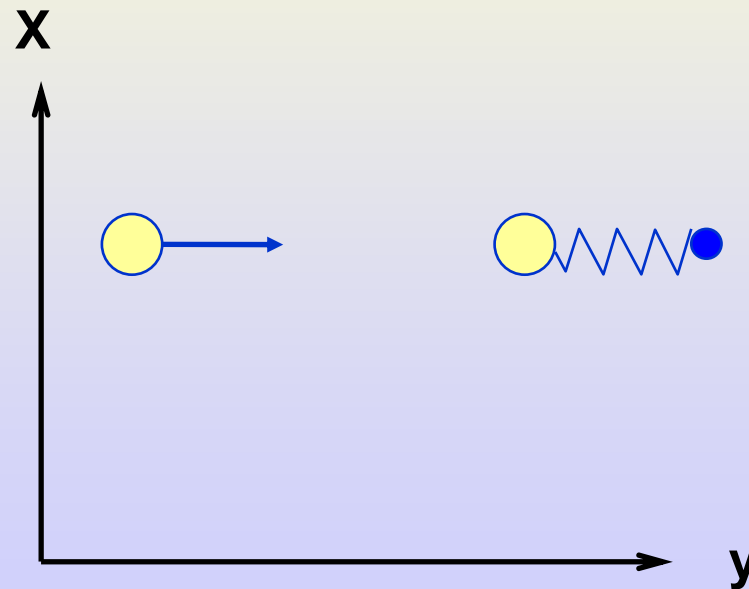
$$\tilde{\varepsilon}_{vN}^{\alpha} = \frac{-1}{N \ln N} \left( (N - 1)\alpha(2 - \alpha) \ln \frac{\alpha(2 - \alpha)}{N} \right. \\ \left. + (1 + (1 - \alpha)^2(N - 1)) \ln \frac{1 + (1 - \alpha)^2(N - 1)}{N} \right)$$



The comparison between linear entropy and von Neumann entropy for the state  $|\chi\rangle$ . We see that LE entropy is always greater than vNE apart from the initial and the final values when they coincide

# Physical model in 2D

An electron propagating in a 2D system interacting through a Coulomb potential with an electron bound to a specific site by a harmonic potential



# The Hamiltonian

$$\mathcal{H} = -\frac{\hbar^2}{2m} \nabla_{\mathbf{r}_A}^2 - \frac{\hbar^2}{2m} \nabla_{\mathbf{r}_B}^2 + \frac{e^2}{\xi |\mathbf{r}_A - \mathbf{r}_B|} + \frac{1}{2} m \omega^2 (\mathbf{r}_A - \mathbf{r}_0)^2 + \frac{1}{2} m \omega^2 (\mathbf{r}_B - \mathbf{r}_0)^2$$

It does not contain spin terms

Initial condition

 Incoming electron

$$\psi(\mathbf{r}, t_0) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(\mathbf{r} - \mathbf{r}_0)^2}{4\sigma^2} + i\mathbf{k} \cdot \mathbf{r}\right)$$

Minimum uncertainty wave packet

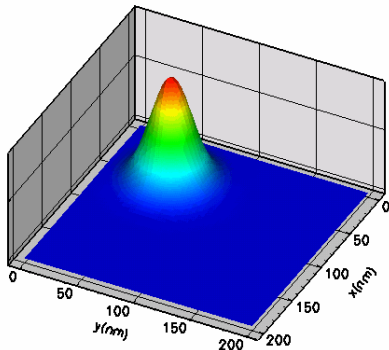
 Bound electron

$$\phi(\mathbf{r}, t_0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{m\omega(\mathbf{r} - \mathbf{r}_1)^2}{2\hbar}\right)$$

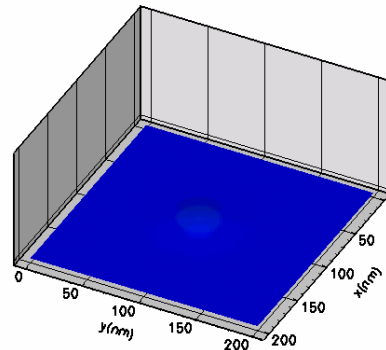
Ground state of the harmonic potential

As a consequence of the scattering the state of the bound particle changes. Here we plot the time evolution the square modulus of projection the antisymmetrized two particle wavefunction  $\Psi(\mathbf{r}, \mathbf{r}') = \psi(\mathbf{r}) \phi(\mathbf{r}') - \phi(\mathbf{r}) \psi(\mathbf{r}')$  on the eigenstates of the harmonic oscillator

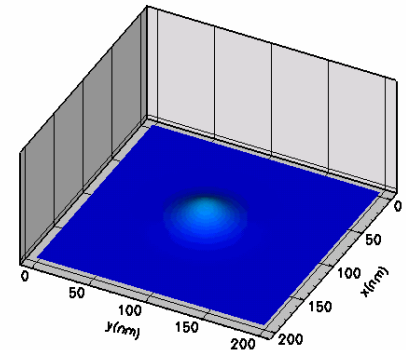
Projection on the ground state of the HO



Projection on the first excited state of the HO



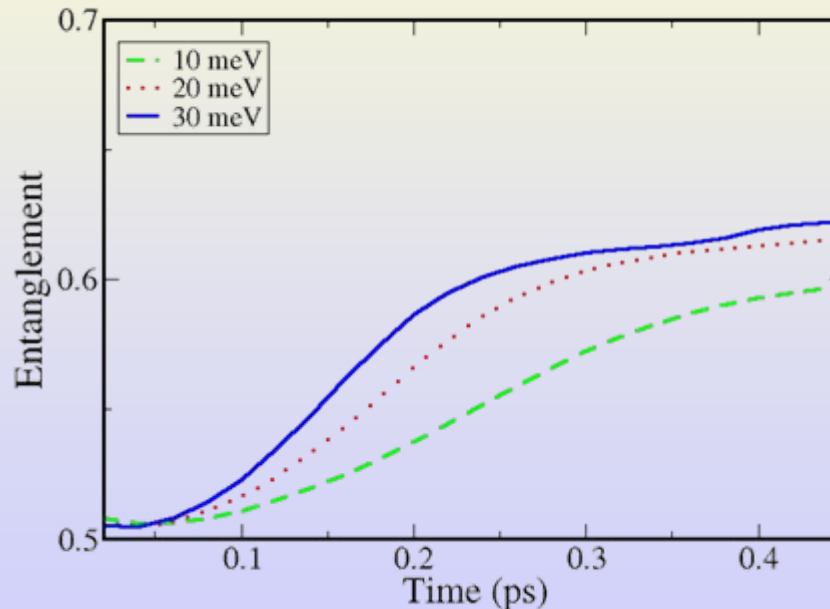
Projection on the second excited state of the HO



Two fermions, same spin.

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |\psi\phi\rangle - |\phi\psi\rangle \right) |\uparrow\uparrow\rangle$$

$\hbar\omega = 2 \text{ meV}$

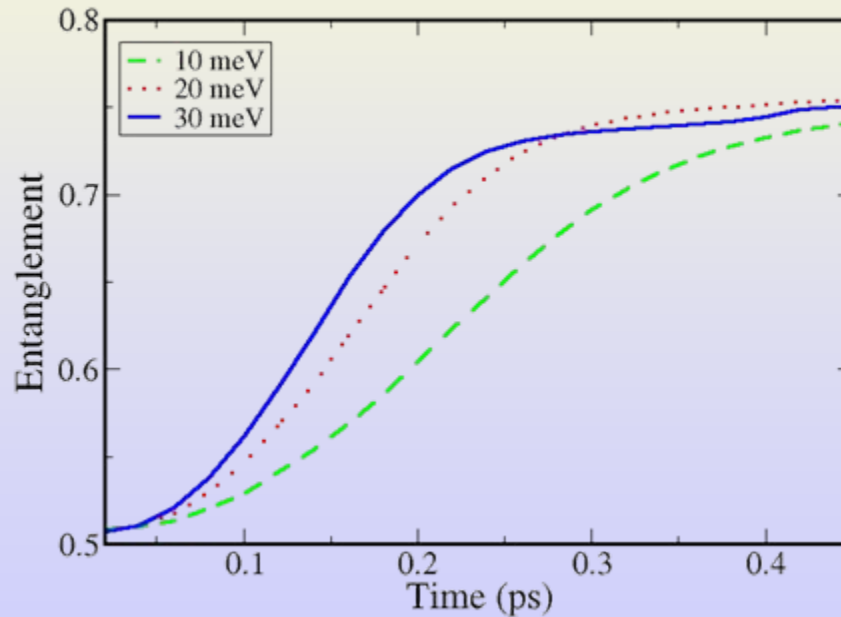


- ❖ Time evolution of the entanglement for different values of the incoming electron initial energy  $E_k$ .
- ❖ The stationary value of the entanglement is higher for higher energies

Two fermions, different spin, non factorized state.

$$|\Upsilon\rangle = \frac{1}{\sqrt{2}} \left( |\psi\phi\rangle|\uparrow\downarrow\rangle - |\phi\psi\rangle|\downarrow\uparrow\rangle \right)$$

$\hbar\omega = 2 \text{ meV}$

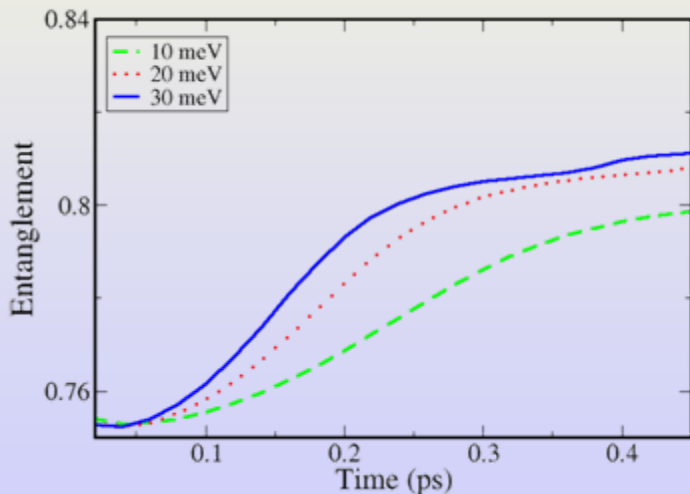


The final value of the entanglement reaches its maximum value for  $E_K = 20 \text{ meV}$

## Two fermions, different spin, factorized state.

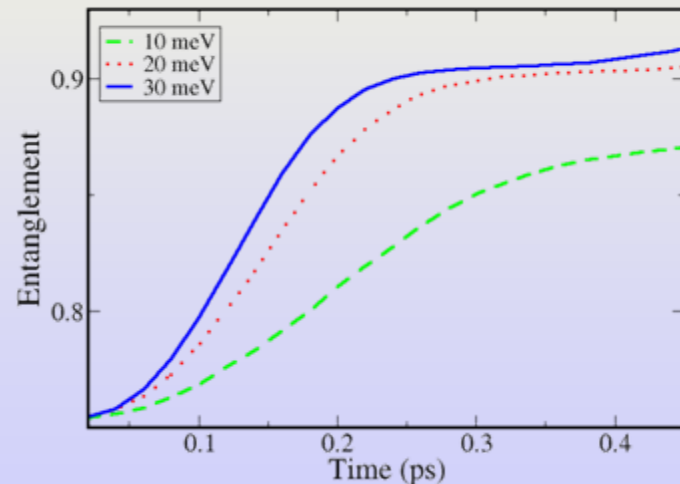
$$|\Xi\rangle = \frac{1}{2} \left( |\psi\phi\rangle - |\phi\psi\rangle \right) \left( |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right)$$

Triplet spin state



$$|\Phi\rangle = \frac{1}{2} \left( |\psi\phi\rangle + |\phi\psi\rangle \right) \left( |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right)$$

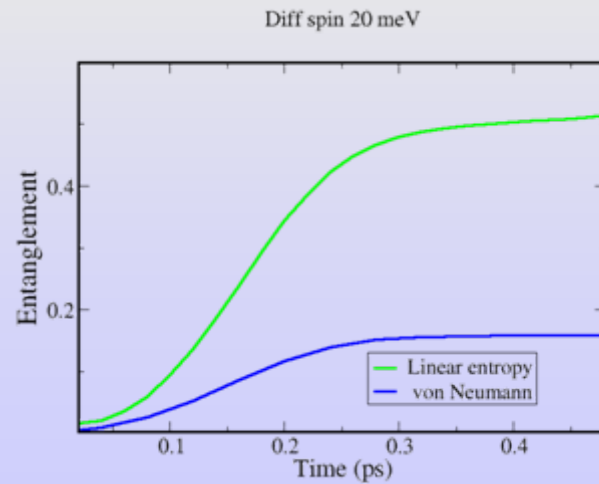
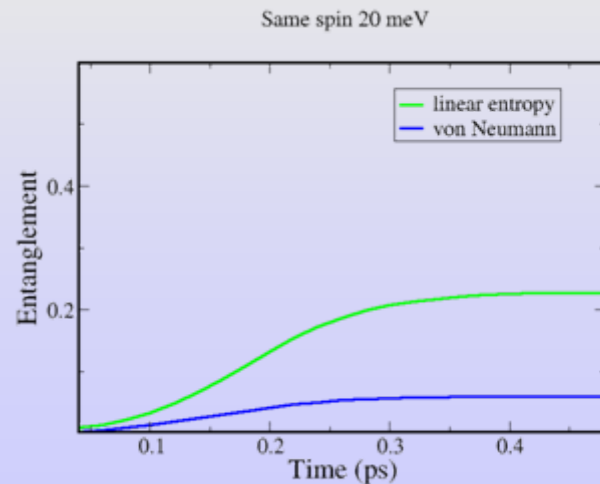
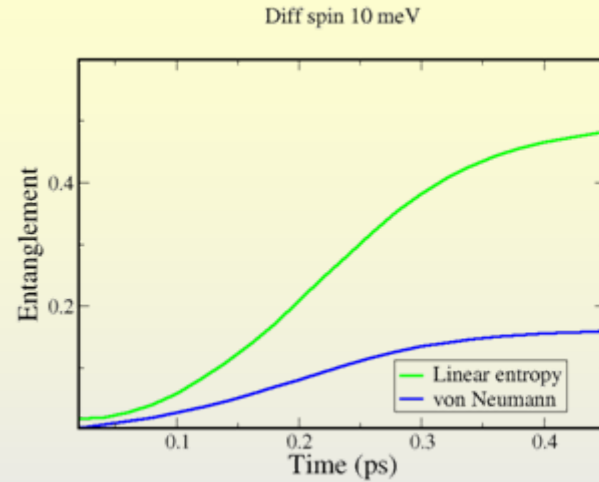
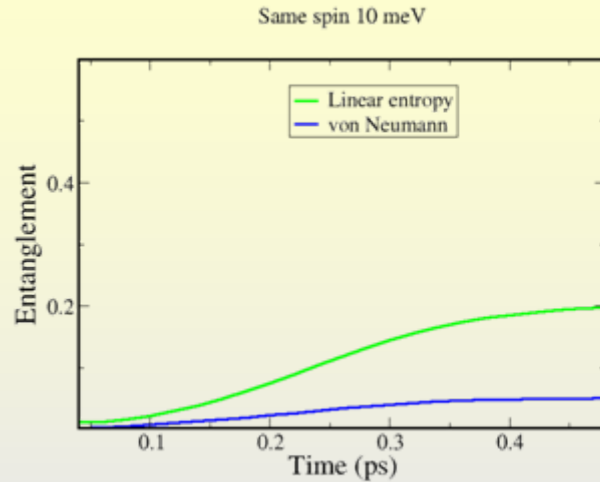
Singlet spin state



The linear entropy of triplet spin state can be obtained from the one of the same spin as  $\varepsilon^{\Xi} = 0.5 \cdot (1 + \varepsilon^{\Psi})$

The initial value of the entanglement is  $3/4$ . The triplet and singlet spin state are initially entangled

# Comparison between normalized linear and von Neumann entropies



We observe that both measures increase with time but the linear entropy is greater than the von Neumann one. Time of entanglement formation is the same for the two notions.

# Conclusions

- ❖ *We have analyzed one possible entanglement measure for two-fermion system based on the linear entropy of the one particle-density matrix. It agrees with Slater rank criterion*
- ❖ *We evaluate the entanglement as LE in an electron-electron scattering in a 2D semiconductor system. Our results confirm a dependence of the entanglement on spin components of the state. The times of entanglement formation correspond to the ones obtained in the previous work by using the vNE.*
- ❖ *From a computational point of view the evaluation of the entanglement in terms of LE is easier and much faster than one performed by means of vNE*

F. Buscemi, P. Bordone and A. Bertoni, PRA 73 052312 (2006)

F. Buscemi, P. Bordone and A. Bertoni, quant-ph/0611223