Local operations, energy, and entanglement in quantum critical systems

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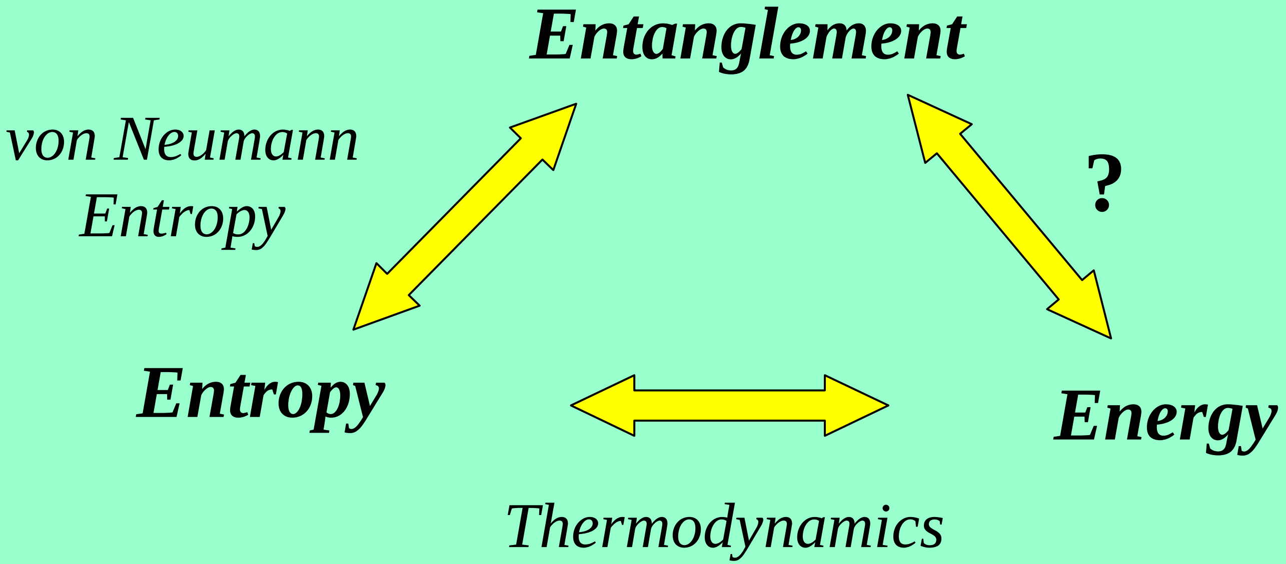
Entanglement and quantum phase transitions

Entanglement can be used to characterize quantum phase transitions from new and fruitful perspectives.

The study of quantum phase transitions can reveal structural aspects of nonlocal correlations in many-body systems.

Entangled representations can provide powerful schemes for the efficient numerical simulation of ground states of strongly correlated systems.

Entanglement and Energy



Can the study of quantum critical systems unveil possible relations between energy and entanglement?
And, if so, by what means?

Phase Transitions and Single-qubit Operations

Unitary single-qubit operation: perturbation that acts on the degrees of freedom of just one spin (site) in the lattice.

Why and what for?

- *Single-qubit operations are fundamental in quantum computation and information theory.*

- *Perturbing only one spin in the system does not change the entanglement properties, and at the same time it provides global informations on the state of the system.*

- *Operational approach: looking at the response to a perturbation is a basic tool to analyze physical properties.*

Working Scheme

Given a local unitary operation P_k *and the ground state of the system:*

$$|G\rangle \longrightarrow |\psi_k\rangle = P_k|G\rangle \longrightarrow Q = \langle\psi_k|\hat{O}|\psi_k\rangle - \langle G|\hat{O}|G\rangle$$

Fundamental elementary operations P_k *:*

$$\sigma_k^x, \quad \sigma_k^y, \quad \sigma_k^z$$

Choice of the observable $\hat{O}$*, a relevant example:*

Excitation Energy $\longrightarrow$ $\Delta E_{P_k} = \langle\psi_k|H|\psi_k\rangle - \langle G|H|G\rangle$

A Case Study: the antiferromagnetic XYZ Model

$$H = J\sum_i S_i^x S_{i+1}^x + \Delta_y S_i^y S_{i+1}^y + \Delta_z S_i^z S_{i+1}^z + h\sum_i S_i^z$$

$$J > 0; \quad 0 \le \Delta_y \le 1; \quad 0 \le \Delta_z \le 1;$$

Phase transition

$$h_c = h_c\left(J, \Delta_y, \Delta_z\right)$$

Δ_y=1 Kosterlitz-Thouless

Δ_y<1 Order parameter $M_x = \langle S_x \rangle$

Factorization

$$h_f = J\sqrt{\left(1+\Delta_y\right)\left(\Delta_y + \Delta_z\right)}$$

$$\left\langle S_i^\alpha S_j^\beta \right\rangle = \left\langle S_i^\alpha \right\rangle \left\langle S_j^\beta \right\rangle$$

J. Kurmann et al., Physica(Amsterdam) (1982).

$$h_c \ge h_f$$

Elementary Operations

Excitation energies associated to the single-qubit base operations

$$\sigma_k^x \quad \Delta E_x / J = 4\Delta_y G_{yy} + 4\Delta_z G_{zz} - 2M_z h / J$$

$$\sigma_k^y \quad \Rightarrow \quad \Delta E_y / J = 4G_{xx} + 4\Delta_z G_{zz} - 2M_z h / J$$

$$\sigma_k^z \quad \Delta E_z / J = 4G_{xx} + 4\Delta_y G_{yy}$$

Mean values and correlations

$$G_{\alpha\alpha} = \left\langle G \middle| \sigma_k^\alpha \sigma_{k+1}^\alpha \middle| G \right\rangle$$

$$M_\alpha = \left\langle G \middle| \sigma_k^\alpha \middle| G \right\rangle$$

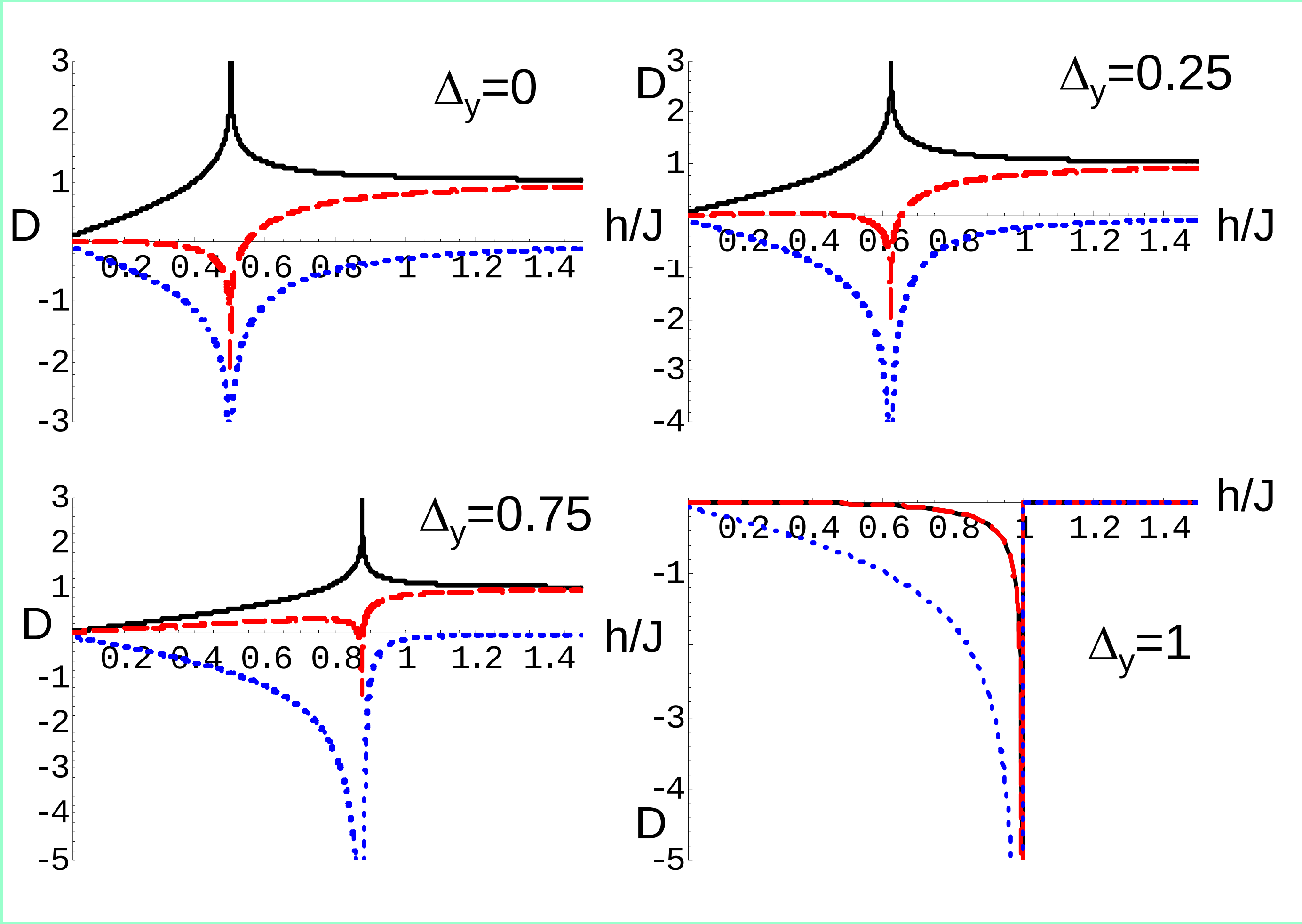
A simple example

Case of $\Delta z=0$ (XY model). Behaviour of the derivatives of the excitation energies as functions of h/J: Singularity at the critical points

ΔE_x=Black line

ΔE_y=Red line

ΔE_z=Blue line



A useful theorem

If the ground state is factorized, there exists a single-qubit operation that leaves it unchanged (invariant operation U):

$$\Delta E\left(U\right) = 0$$

Viceversa, if the ground state is entangled, and the system is traslationally invariant, there exists no local operation that leaves it unchanged. In fact, if such operation existed, one would have

$$\otimes_k P_k \left|G\right\rangle = \left|G\right\rangle$$

and hence one reaches the contradictory conclusion that the ground state is factorized. Therefore, for an entangled ground state, it is always:

$$\Delta E\left(P_k\right) > 0$$

The vanishing of the excitation energy is a necessary and sufficient condition for the factorizability of the ground state.

Finding the invariant operation

XYZ Hamiltonian: competition between the magnetization along the x axis and the external field on the z axis. For this reason, when the ground state is factorized we can consider:

$$U = \sin(\theta)\sigma_k^x + \cos(\theta)\sigma_k^z$$

and the associated excitation energy:

$$\Delta E(U)/J = \sin(\theta)^2(\Delta E_x/J) + \cos(\theta)^2(\Delta E_z/J) + \sin(\theta)\cos(\theta)\left[4(1-\Delta_z)G_{zx} - 2M_x h/J\right]$$

Imposing the vanishing of the excitation energy ΔE(U) one determines the direction:

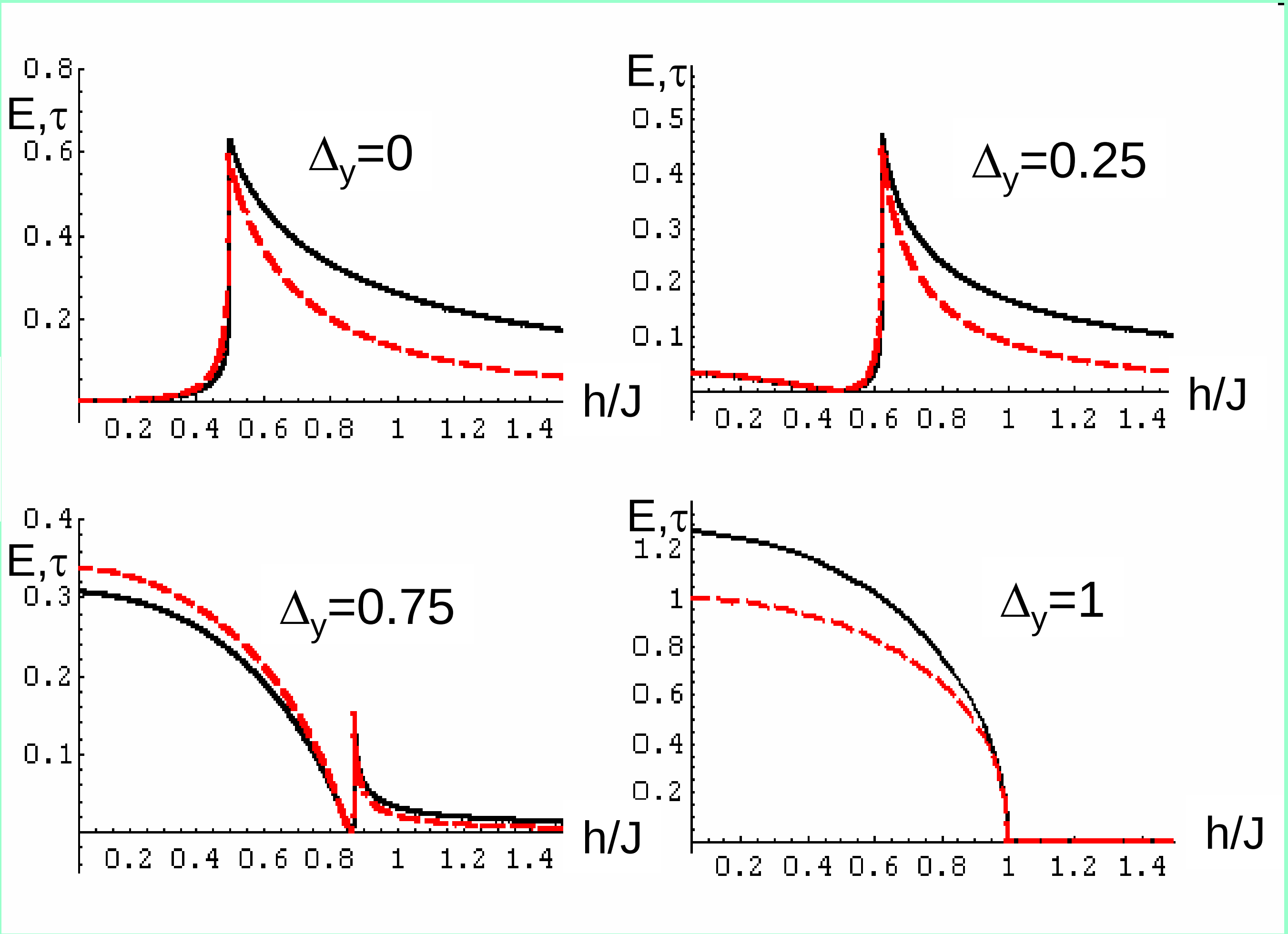
$$\tan(\theta) = \frac{2JM_x}{h - 2J\Delta_z M_z}$$

The excitation energy at Δ_z=0 (XY model)

Comparing the behaviours of $\Delta E(U)$ and the one-tangle τ_1 as functions of h/J

$\Delta E(U)$=Black line

τ_1=Red line

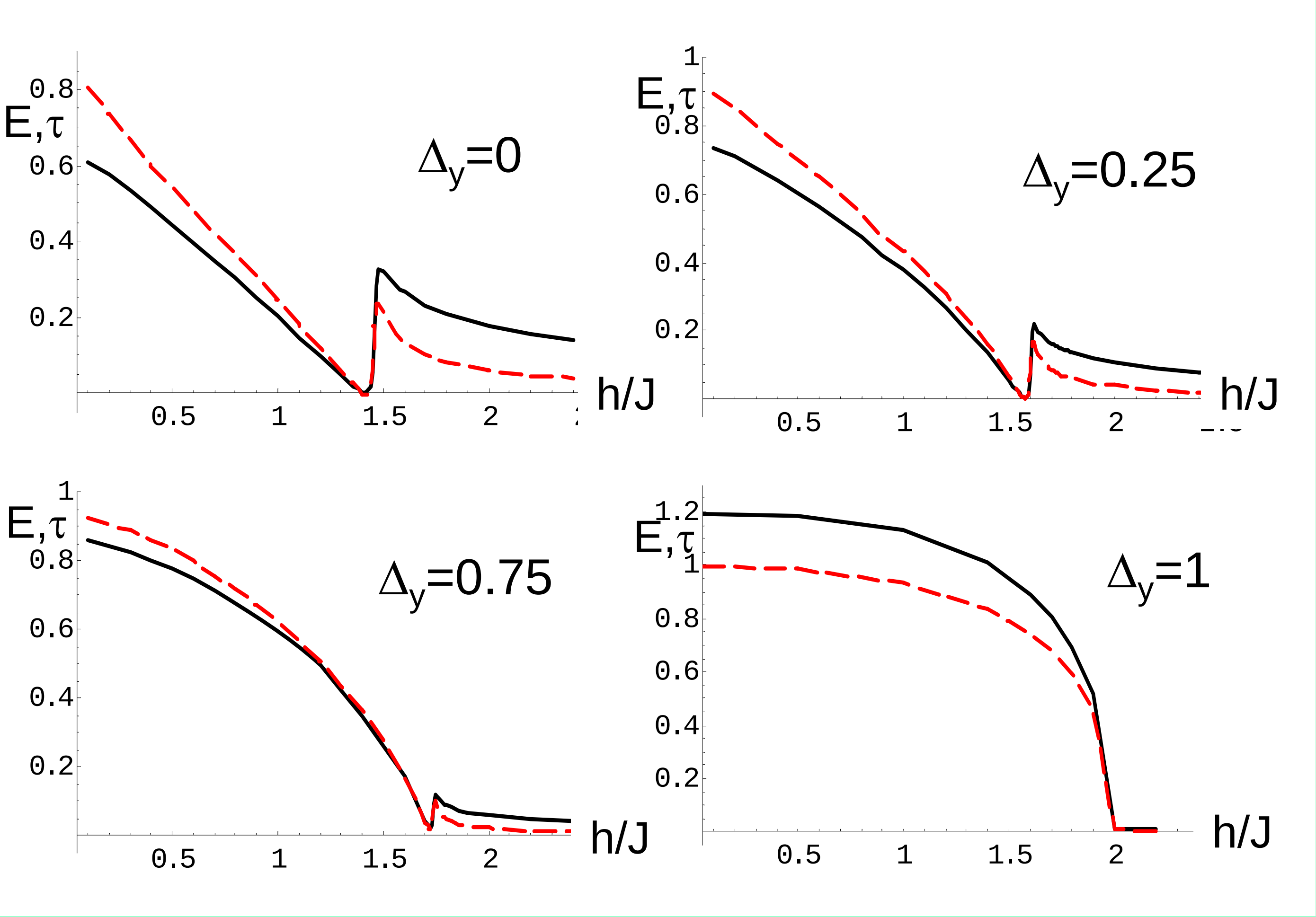


The excitation energy at Δ_z=1 (XYX model)

Comparing the behaviours of $\Delta E(U)$ and the one-tangle τ_1 as functions of h/J

$\Delta E(U)$=Black line

τ_1=Red line



Summary on the properties of the excitation energy $\Delta E(U)$

For an ample class of spin 1/2 models:

1) Necessary and sufficient separability criterion:
The ground state is factorized if and only if $\Delta E(U)=0$.

2) Local singular maximum of $\Delta E(U)$ at the critical point for the transitions with order parameter

3) $\Delta E(U)$ monotonic in the global measures of the entanglement between a single spin and the rest of the system (one-tangle and von Neumann block entropy)

↓

Excitation Energy $\Delta E(U)$ associated to the invariant operation establishes a first direct connection between energy and entanglement

Quantum Phase Transitions and Local Operations in Fermionic Models – 1

Extended Hubbard model in 1-D

$$H = -\sum_i \left(C_i^+ C_{i+1} + h.c \right) + U \sum_i n_{i\uparrow} n_{i\downarrow} + V \sum_i n_i n_{i+1}$$

Four degrees of freedom at each lattice site:

$$|0\rangle; \quad |\uparrow\rangle; \quad |\downarrow\rangle; \quad |\uparrow\downarrow\rangle.$$

Qudit analogue of single-qubit operation: Local unitaries that act on the internal degrees of freedom associated to a single site.

Quantum Phase Transitions and Local Operations in Fermionic Models – 2

Working Scheme

$$\left|G\right\rangle \Longrightarrow \left|\psi\right\rangle = P_k \left|G\right\rangle \Longrightarrow Q = \left\langle \psi \middle| \hat{O} \middle| \psi \right\rangle - \left\langle G \middle| \hat{O} \middle| G \right\rangle$$

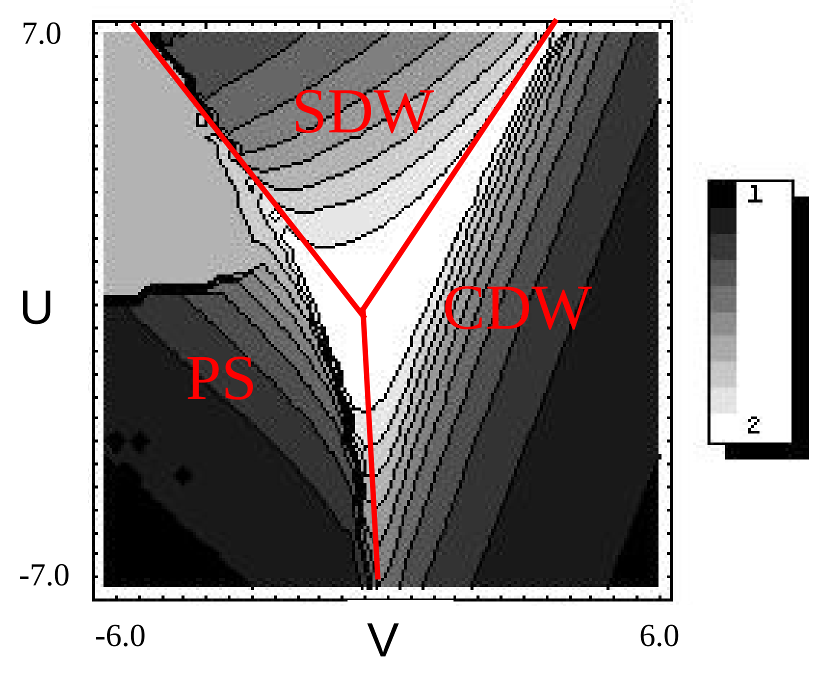
Local operation analogous to σ_k^z $\Longrightarrow$ $P_k = \left(1 - 2n_{k\uparrow}\right)\left(1 - 2n_{k\downarrow}\right)$

$$\Delta E_{P_k} = \left\langle \psi_k \middle| H \middle| \psi_k \right\rangle - \left\langle G \middle| H \middle| G \right\rangle = 16 \left\langle C^{+}_{k,\uparrow} C_{k+1,\uparrow} \right\rangle$$

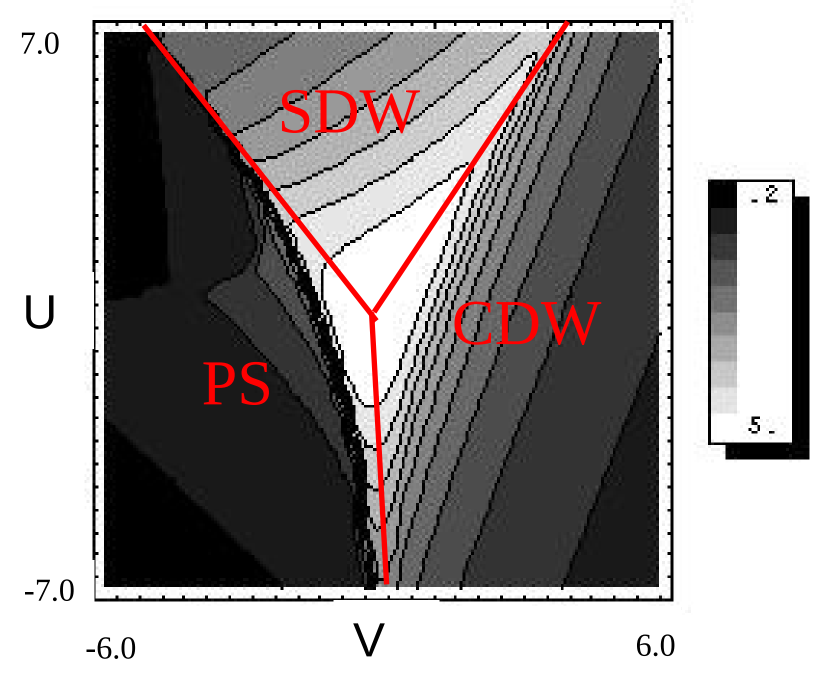
Quantum Phase Transitions and Local Operations in Fermionic Models – 3

Comparing the behaviours of $\Delta E(P_K)$ and the von Neumann entropy

von Neumann entropy



Excitation energy



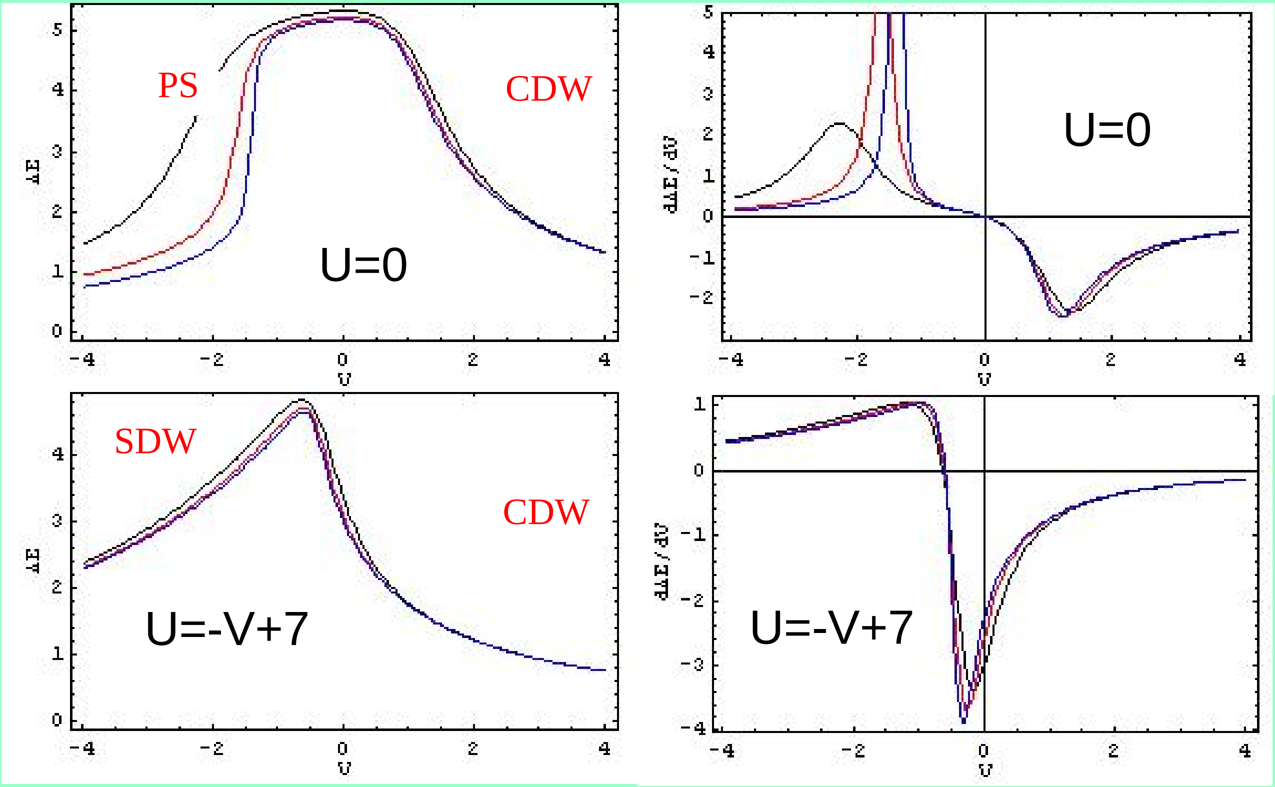
Exact Diagonalization : Sites N=10

Quantum Phase Transitions and Local Operations in Fermionic Models – 4

Scaling Analysis

N=6 -> Black line
N=8 -> Red line
N=10 -> Blue line

Excitation energy ***Derivative***



Summary

1) *We have introduced an energy measure inspired by quantum information concepts that is able to detect quantum phase transitions in a wide class of spin 1/2 models.*

2) *Among all the possible excitation energies it is possible to single out a particular one, whose vanishing yields a necessary and sufficient condition for the separability of the ground state.*

3) *Preliminary studies indicate that this framework can be extended to models of interacting particles.*

Outlook and Perspectives

1) *Deeper understanding of the very close connection between single-qubit excitation energies and global measures of entanglement: possible operational characterization of the one-tangle.*

2) *Extension to higher spins and models with higher spatial dimension, e. g. planar and ladder systems.*

3) *Generalization of the concept of invariant operation to models of interacting particles.*

References

Local operations and phase transitions

S.M. Giampaolo, F. Illuminati and S. De Siena,
quant-ph/0604047

Excitation energies and ground state entanglement

S.M. Giampaolo, F. Illuminati, P. Verrucchi and S. De Siena,
quant-ph/0611035

Transizioni di fase quantistiche

Transizione di fase classica veicolata da parametri macroscopici termodinamici (temperatura, densità, pressione, ecc.)

Transizione di fase quantistica veicolata dai parametri interni dell'Hamiltoniana del sistema:

$$H = H_1 + gH_2$$

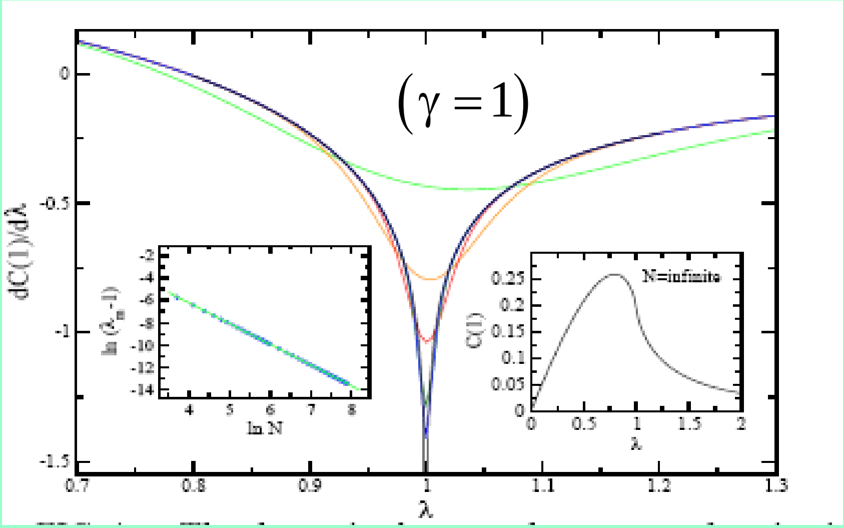
Transizione di fase quantistica del secondo ordine

Lunghezza di correlazione divergente $\xi^{-1} \sim \Lambda \left| g - g_c \right|^{\nu}$

Transizioni di fase quantistiche ed Entanglement - I

Osterloh, Amico, Falci, and Fazio, Nature 2002.
Osborne and Nielsen, Phys. Rev. A 2002.

$$H = -\frac{J}{2}\sum_i \left(1+\gamma\right)\sigma_i^x\sigma_{i+1}^x + \left(1-\gamma\right)\sigma_i^y\sigma_{i+1}^y - h\sum_i \sigma_i^z$$



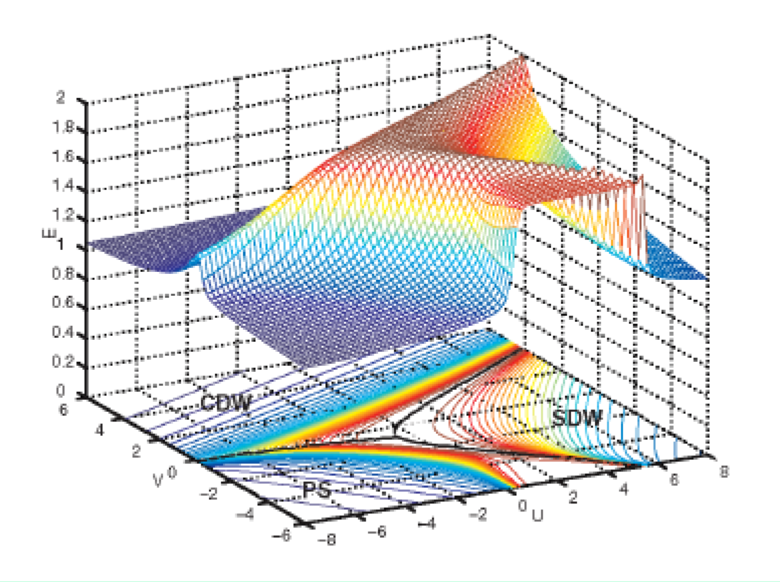
L'analisi dello scaling con le dimensioni del sistema e dell'andamento della concurrence in funzione del rapporto J/h è in grado di caratterizzare completamente la transizione di fase.

Transizioni di fase quantistiche ed Entanglement - II

Gu, Deng, Li and Lin, Phys. Rev. Lett. 2004

$$H = -\sum_{i,\sigma}\left(C^{+}_{i,\sigma}C_{i+1,\sigma} + h.c.\right) + U\sum_{i} n_{i,\uparrow}n_{i,\downarrow} + V\sum_{i} n_{i}\,n_{i+1}$$

L'entropia di von Neumann può essere utilizzata per determinare le transizioni di fase quantisitche in modelli di fermioni interagenti

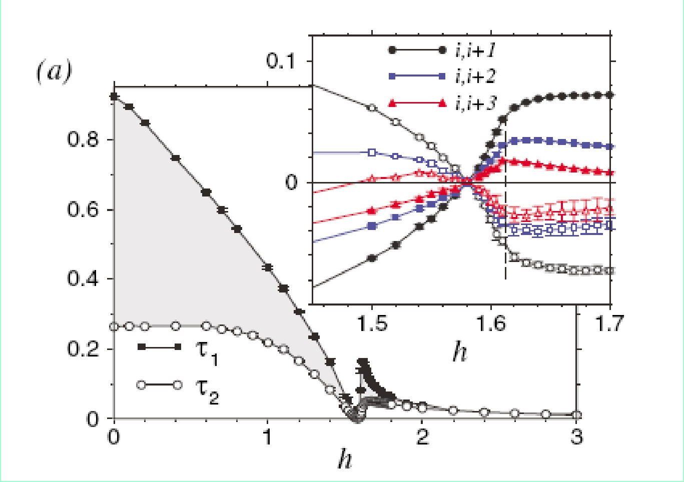


Transizioni di fase quantistiche ed Entanglement - III

Roscilde, Verrucchi, Fubini, Haas, and Tognetti, Phys. Rev. Lett 2004

$$H = J\sum_i \sigma_i^x\sigma_{i+1}^x + \Delta\sigma_i^y\sigma_{i+1}^y + \sigma_i^z\sigma_{i+1}^z - h\sum_i \sigma_i^z$$

L'analisi dell'andamento della concurrence in funzione del rapporto J/h è in grado di rilevare i punti di fattorizzazione dello stato fondamentale. Misure locali non sono in generale in grado né di individuare né di caratterizzare transizioni di fase.



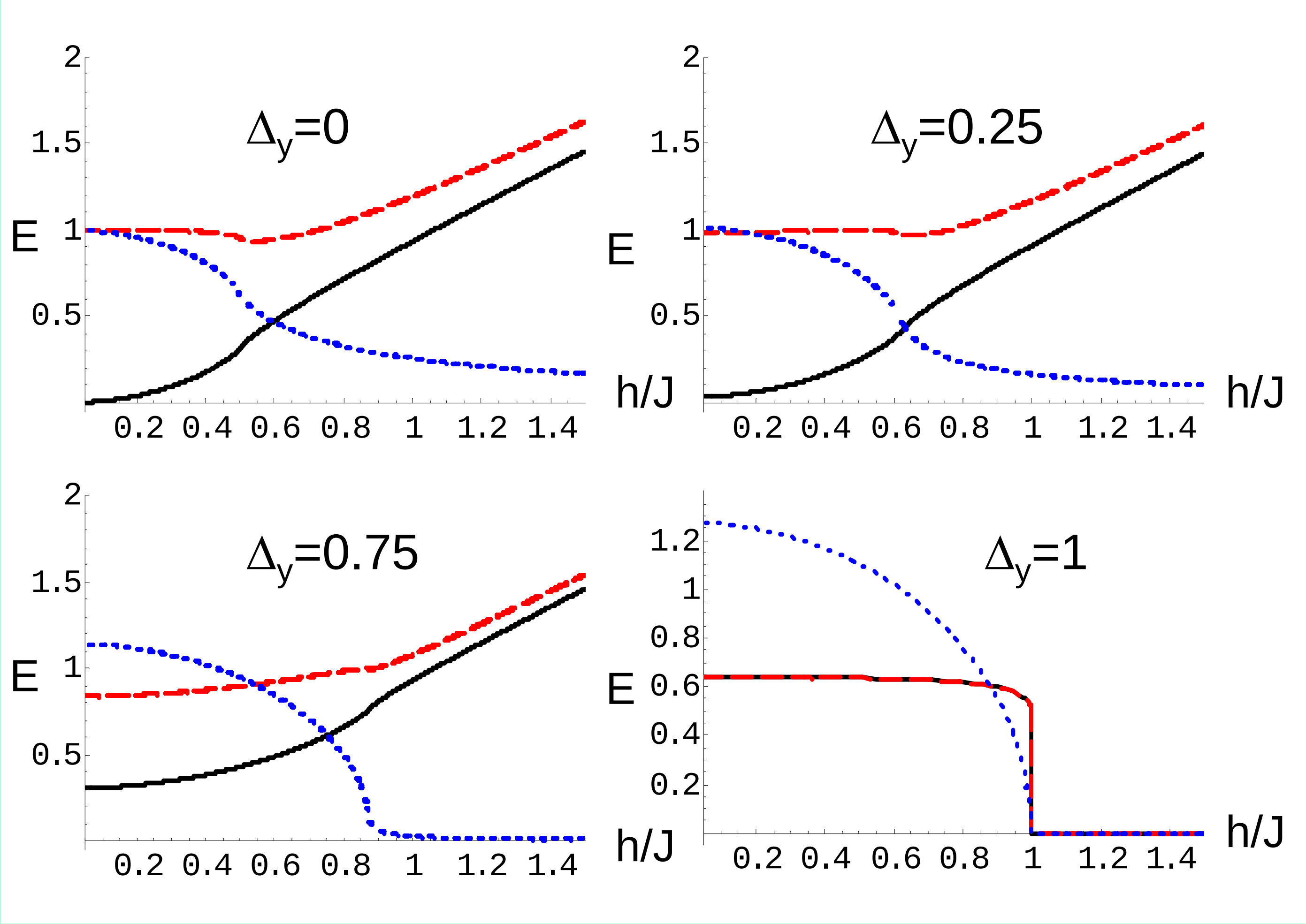
Gu, Tian and Lin quant-ph/0511243

Model (QPT point)	GS LC	ES LC	concurrence
XXZ chain($\Delta = -1$)	Yes		singular
$J_1 - J_2$ model($J_2 = 0.5$)	Yes		singular
XXZ chain($\Delta = 1$)	No	Yes	maximum, not singular
spin ladder($J = 0$)	No	Yes	maximum, not singular
XXZ 2&3D($\Delta = 1$)	No	Yes	maximum, singular
$J_1 - J_2$ model($J_2 \simeq 0.241$)	No	Yes	not maximum
Ising model($\lambda = 1$)	No	No	singular, not maximum

Energie di eccitazione nel caso Δ_z=0 (XY)

Andamento delle energie di eccitazione in funzione di h/J

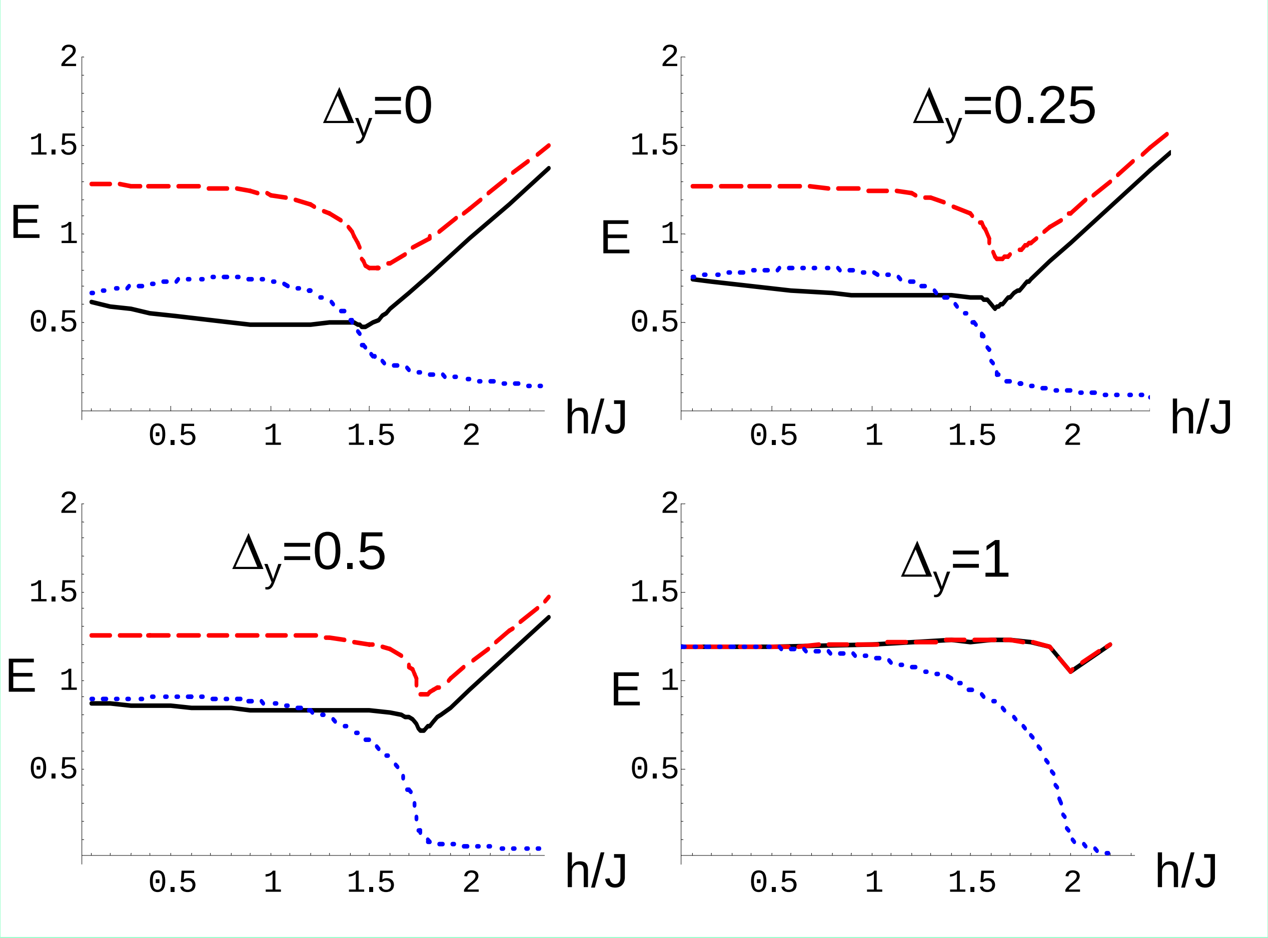
ΔE_x=*Black line*
ΔE_y=*Red line*
ΔE_z=*Blue line*



Energie di eccitazione nel caso Δ_z=1 (XYX)

Andamento delle energie di eccitazione in funzione di h/J

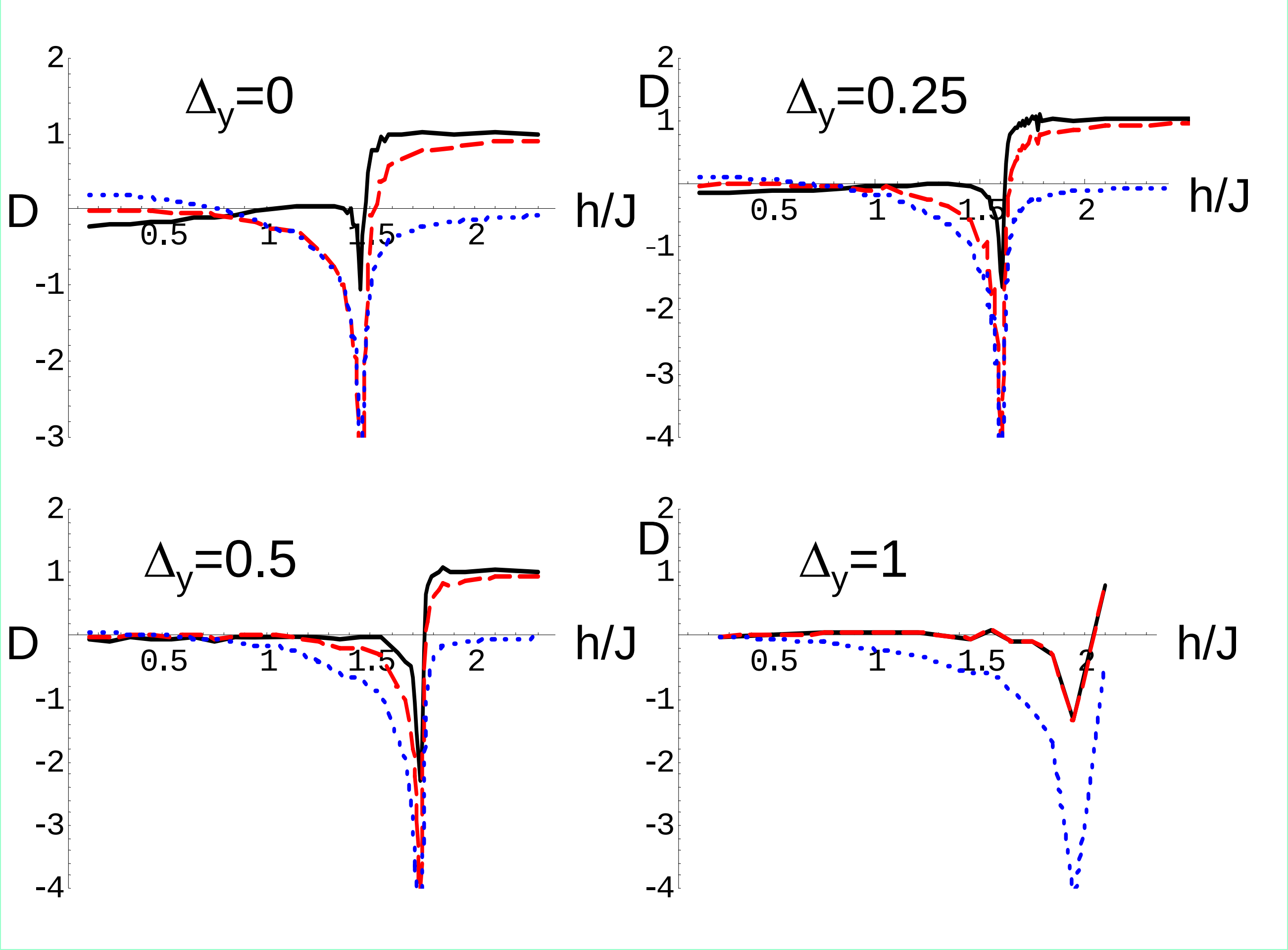
ΔE_x=Black line
ΔE_y=Red line
ΔE_z=Blue line



Derivate delle energie di eccitazione nel caso Δ_z=1 (XYX)

Andamenti delle derivate rispetto ad h/J delle energie d'eccitazione in funzione di h/J: Singolarità al punto critico

ΔE_x=Black line
ΔE_y=Red line
ΔE_z=Blue line



Relazioni tra energia ed entanglement - 1

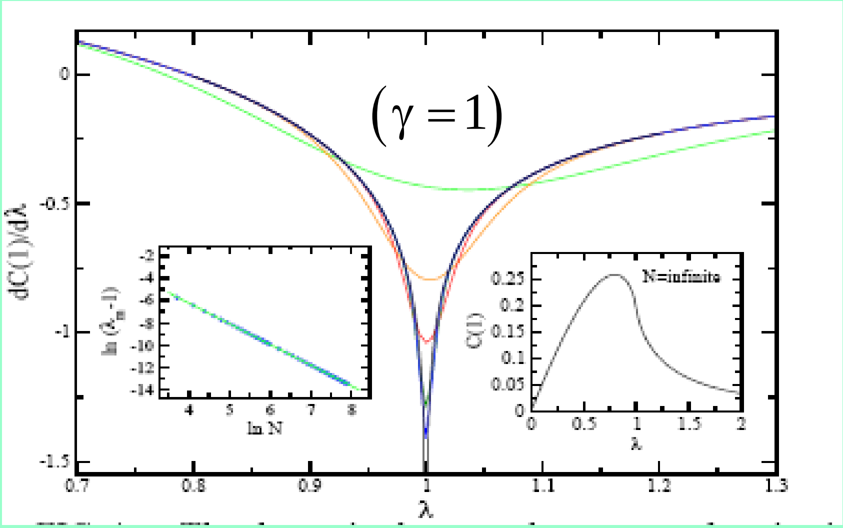
Osterloh, Amico, Falci, and Fazio, Nature 2002.

$$H = -\frac{J'}{2}\sum_i (1+\gamma)\sigma_i^x\sigma_{i+1}^x + (1-\gamma)\sigma_i^y\sigma_{i+1}^y - h\sum_i \sigma_i^z$$

$$\frac{dC(1)}{d\lambda} = \mathrm{cost} + c_c(\gamma)\ln|\lambda - \lambda_c|$$

$$\lambda = J'/2h$$

$$c_c(\gamma) = \frac{c_c(1)}{\gamma} \qquad c_c(1) = 0.27$$



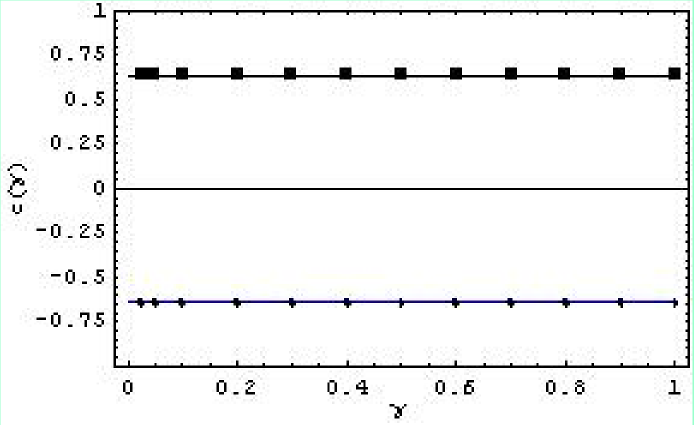
Relazioni tra energia ed entanglement - 2

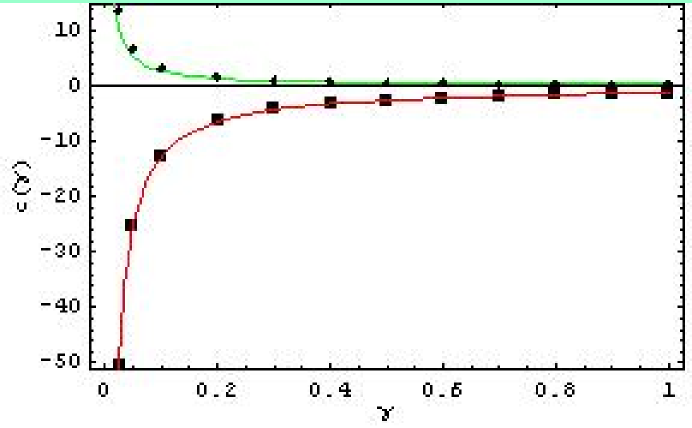
$$J = J'(1+\gamma); \quad \Delta_y = \frac{1-\gamma}{1+\gamma}; \quad \Delta_z = 0;$$

$$\frac{d\Delta E_\alpha}{d\lambda} = \cos t + c_\alpha(\gamma)\ln|\lambda - \lambda_c|$$

$$- c_x(\gamma) = c_y(\gamma) = 0.64$$

$-c_c(\gamma)$





$$c_z(\gamma) = \frac{c_z(1)}{\gamma} \qquad c_z(1) = -1.27$$

$$c_z(\gamma) = -c_c(\gamma) - \frac{1}{\gamma}$$

Relazioni tra energia ed entanglement - problemi

La relazione appena vista ha almeno due problemi.

Non è sufficientemente generale. Non vale nel caso $\Delta_z=1$.

Essendo sia la concurrence sia le energie di eccitazione funzioni delle correlazioni, la relazione, quando esiste, appare poco significativa.

Relazioni tra energia ed entanglement: Nuova strada

Sinora abbiamo studiato le energie di eccitazione associate alle operazioni fondamentali di singolo qubit. Ma sono effettivamente quelle più significative?

$$H = -J\sum_i S_i^x S_{i+1}^x + \Delta_y S_i^y S_{i+1}^y + \Delta_z S_i^z S_{i+1}^z + h\sum_i S_i^z$$

J. Kurmann et al., Physica(Amsterdam) (1982).

Fattorizzazione: $h/J = \infty; \quad h/J = \sqrt{\left(1+\Delta_y\right)\left(\Delta_y + \Delta_z\right)}$

Per questi due valori del campo esterno lo stato fondamentale dell'Hamiltoniana è fattorizzato

Variazione della direzione di perturbazione

Analisi dell'andamento dell'angolo θ al variare del parametro Hamiltoniano per le diverse classi di universalità:

XY Black line
XYX Red line
K.T. Blue line