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Group theory

(Lecture notes - a.a. 2024/25)

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1 Introduction

Group theory is a useful mathematical language that helps describe the invariance properties of physical systems. These notes provide a quick introduction to group theory, primarily aimed at developing the tensorial language used by physicists. The main concepts of Lie groups are also introduced.

2 Preliminaries: Review of Linear Algebra

Let us consider a vector space V and a basis of vectors $|e_i\rangle$, so that an arbitrary vector $|v\rangle \in V$ can be expressed as a linear combination of them:

$$|v\rangle = v^i |e_i\rangle \tag{1}$$

where repeated indices are automatically summed over (Einstein summation convention). We assume V to be of finite dimension. Also, we use Dirac's bra-ket notation that should be familiar from quantum mechanics.

Linear operators

A linear operator A gives a new vector when acting on any given vector

$$\begin{aligned} A:\ V &\longrightarrow V \\ |v\rangle &\longrightarrow |v'\rangle = A|v\rangle\ . \end{aligned} \tag{2}$$

Using linearity, we can express the new vector as

$$\begin{aligned} |v'\rangle &= A|v\rangle = A(v^i|e_i\rangle) = v^i A|e_i\rangle = v^i|e'_i\rangle = v^i A^j{}_i|e_j\rangle \\ &= A^i{}_j v^j|e_i\rangle = v'^i|e_i\rangle \end{aligned} \tag{3}$$

where we have set $|e'_i\rangle = A|e_i\rangle = A^j{}_i|e_j\rangle$, since the transformed vectors of the basis can be expressed as linear combinations of the original basis. In the second line, we have renamed indices to extract the components of the transformed vector. $|v'\rangle = v'^i|e_i\rangle$. This allows us to recognize the matrix elements $A^i{}_j$ of the operator A and how it operates on the vector components v^i, namely $v'^i = A^i{}_j v^j$.

Physicist's notation

Physicists often use only the components v^i to indicate the vector $|v\rangle$, assuming that a basis (or reference frame) has been chosen. Thus, the linear transformation above is written as

$$\boxed{v'^i = A^i{}_j v^j} \tag{4}$$

where $A^i{}_j$ are the entries of the matrix that performs the linear transformation on the column vector v^j. Note that the second index of the matrix is summed over with the index of the vector components.

This linear transformation can be expressed in matrix language using column vectors and matrices. Considering the example of a two-dimensional vector space, where indices can take only two values, we write

$$v^i \longrightarrow v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \tag{5}$$

and similarly,

$$A^i{}_j \longrightarrow A = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix} \tag{6}$$

where the first index (conventionally written as an upper index) is the row index, while the second index (conventionally written as a lower index) is the column index. Then, eq. (4) is cast as

$$\boxed{v' = Av} \, . \tag{7}$$

Matrix multiplication

Since we will use extensively square matrices, mostly interpreted as linear operators acting on a vector space, let us review some of their properties. For square matrices, one can define a product and several other operations. We review these operations using $2 \times \textit{two}$ matrices, as the extension to higher dimensions is straightforward. The product of two such matrices

$$C = AB \tag{8}$$

is defined as usual by the row-by-column multiplication rule:

$$C = \begin{pmatrix} C^1{}_1 & C^1{}_2 \\ C^2{}_1 & C^2{}_2 \end{pmatrix} = \begin{pmatrix} A^1{}_1 & A^1{}_2 \\ A^2{}_1 & A^2{}_2 \end{pmatrix} \begin{pmatrix} B^1{}_1 & B^1{}_2 \\ B^2{}_1 & B^2{}_2 \end{pmatrix} = \begin{pmatrix} A^1{}_1 B^1{}_1 + A^1{}_2 B^2{}_1 & A^1{}_1 B^1{}_2 + A^1{}_2 B^2{}_2 \\ A^2{}_1 B^1{}_1 + A^2{}_2 B^2{}_1 & A^2{}_1 B^1{}_2 + A^2{}_2 B^2{}_2 \end{pmatrix} .$$

This is written more compactly as

$$C^i{}_j = \sum_{k=1}^{2} A^i{}_k B^k{}_j$$

or using Einstein's convention as

$$C^i{}_j = A^i{}_k B^k{}_j \, . \tag{9}$$

Thus, we see how the product of matrices is written using components. Note that the product of matrices is non-commutative, so $A^i{}_k B^k{}_j \neq B^i{}_k A^k{}_j$, that is $AB \neq BA$. However, $A^i{}_k B^k{}_j = B^k{}_j A^i{}_k$ as numbers commute.

Dual space

The dual space $\tilde{V}$ of an original vector space V is defined as the space of linear maps that produce a number out of any vector $|v\rangle \in V$. An element of the dual space $\langle w| \in \tilde{V}$ is defined by its action on the vectors $|v\rangle \in V$ which we write as follows

$$\begin{aligned} \langle w| : \; V &\longrightarrow \mathbb{R} \\ |v\rangle &\longrightarrow \langle w|v\rangle \end{aligned} \tag{10}$$

where we have used the Dirac's bra-ket notation. In this definition, we have used the field of real numbers $\mathbb{R}$, but any other field could have been chosen for similar definitions, as the field of complex numbers $\mathbb{C}$, which is used in quantum mechanics. The set of all such elements defines the dual space $\tilde{V}$, which is itself a vector space. Setting

$$\langle w| = w_i \langle \tilde{e}^i| \tag{11}$$

for a dual basis $\langle \tilde{e}^i|$ chosen such that[1]

$$\langle \tilde{e}^i | e_j \rangle = \delta^i_j \tag{12}$$

one may write

$$\langle w|v\rangle = \big(w_i \langle \tilde{e}^i|\big)\big(v^j |e_j\rangle\big) = w_i v^j \langle \tilde{e}^i|e_j\rangle = w_i v^j \delta^i_j = w_i v^i$$

that is

$$\langle w|v\rangle = w_i v^i \ . \tag{13}$$

Note that, in using components, the index position tells if we are considering a vector (as in the case of v^i) or a dual vector (as in the case of w_i).

The "scalar" $w_i v^i$ obtained by contracting a dual vector w_i with a vector v^i is invariant under a change of basis of the vector space, i.e. if one considers a new basis $|e'_i\rangle$. This must be so as the vectors are independent of how one expresses them through a basis. To understand algebraically this statement, let us consider that, because of the completeness of the basis vectors, the two bases must be related through an invertible matrix B with components $B_i{}^j$

$$|e'_i\rangle = B_i{}^j |e_j\rangle \tag{14}$$

so that an arbitrary vector $|v\rangle$ can be expressed equivalently as

$$|v\rangle = v^i|e_i\rangle = v'^i|e'_i\rangle \tag{15}$$

with the new components v'^i fixed by

$$v'^i = A^i{}_j v^j \tag{16}$$

where

$$A^i{}_j B_i{}^k = \delta^k_i \tag{17}$$

i.e. $A^T B = \mathbb{1}$ and thus $A = B^{-1\,T}$.

Similarly, the dual basis vectors must transform as

$$\langle \tilde{e}'^i| = A^i{}_j \langle \tilde{e}^j| \tag{18}$$

so that $\langle \tilde{e}'^i|e'_j\rangle = \delta^i_j$, while the components of the dual vectors transform as

$$w'_i = B_i{}^j w_j \ . \tag{19}$$

[1] δ^i_j is the Kronecker delta defined by

$$\delta^i_j = \begin{cases} 1 & \text{if } i = i \\ 0 & \text{otherwise} \ . \end{cases}$$

Thus, one can verify that

$$\langle w|v\rangle = w_i v^i = w'_i v'^i \tag{20}$$

checking that this number can be computed independently by using the primed or unprimed components. This is the invariance we were talking about.

The vector spaces V and $\tilde{V}$ are isomorphic, being of the same dimensions, but this isomorphism is not unique. A canonical isomorphism, relating a vector of V to a vector of the dual space $\tilde{V}$ in a unique way, can be established if there is a metric on the original vector space V.

Metric

We define a metric g, not necessarily positive definite, as a bilinear function that produces a real number out of two vectors of V, that we take to be real for the moment. That is,

$$\begin{aligned} g:\ V\times V &\longrightarrow \mathbb{R} \\ |w\rangle, |v\rangle &\longrightarrow g(|w\rangle, |v\rangle) \end{aligned} \tag{21}$$

so that $g(|w\rangle, |v\rangle) \in \mathbb{R}$. This metric function is defined to be linear in both entries so that we can write

$$g(|w\rangle, |v\rangle) = g(w^i|e_i\rangle, v^j|e_j\rangle) = w^i v^j g(|e_i\rangle, |e_j\rangle) \equiv w^i v^j g_{ij} \tag{22}$$

where we have set $g_{ij} \equiv g(|e_i\rangle, |e_j\rangle)$. These are the components of the metric tensor, which we assume to be invertible, i.e. g_{ij} must be invertible as a matrix.

Having a metric, one can define a canonical isomorphism between V and $\tilde{V}$, relating in a unique way a vector of V to a vector of $\tilde{V}$. We identify $\langle w|$, the dual of the vector $|w\rangle$, by writing

$$\langle w| = g(|w\rangle,\ \cdot\ ) \tag{23}$$

which operates as

$$\langle w|v\rangle \equiv g(|w\rangle, |v\rangle) \in \mathbb{R}\ . \tag{24}$$

Using linearity, one can expand it as follows

$$\langle w|v\rangle = g(|w\rangle, |v\rangle) = g(w^i|e_i\rangle, v^j|e_j\rangle) = w^i v^j g(|e_i\rangle, |e_j\rangle) = w^i v^j g_{ij} = w_i v^i \tag{25}$$

where in the last step we have recognized the components $w_i = g_{ij} w^j$ of the dual vector $\langle w|$ with respect to the dual basis $\langle \tilde{e}^i|$ of eqs. (11)–(12). When the metric is positive definite, one often considers an orthonormal basis in which

$$g_{ij} = \langle e_i|e_j\rangle = \delta_{ij}\ . \tag{26}$$

Note that g_{ij} can be written as a matrix, but its index structure shows that it cannot be interpreted as a linear operator acting on V. Rather, it acts on the tensor product $V \times V$ as seen in (21). It can be interpreted as a tensor, as we shall see.

Thus, in physicist's notation, the vector w^i is related to the dual vector w_i by lowering its index with the metric, i.e.,

$$w_i = g_{ij} w^j\ . \tag{27}$$

The inverse relation makes use of the inverse metric $g^{ij} \equiv (g^{-1})^{ij}$, which satisfies

$$g^{ij} g_{jk} = \delta^i{}_k \tag{28}$$

so that $w^i = g^{ij} w_j$. The canonical isomorphism between a vector space and its dual space depends on the introduction of a metric. This fact is heavily utilized in the description of general relativity with tensors.

Let us consider some examples. The euclidean space E_N of N dimensions with coordinates $x \in \mathbb{R}^N$ can be considered as a vector space. It can be endowed with a scalar product that defines a metric. It can be presented in equivalent ways as

$$s^2 = x^T x = x^T \mathbb{1}\, x = \delta_{ij} x^i x^j \tag{29}$$

to expose the euclidean metric tensor δ_{ij}. This metric relates the vector x^i to its dual vector x_i by $x_i = \delta_{ij} x^j$. For this particular case, vector and dual vectors coincide, as $x^i = x_i$ numerically for any value of the index i.

The Minkowski space M_4 of 4 dimensions with coordinates $x \in \mathbb{R}^4$ is endowed with a scalar product that is written in various ways as follows

$$s^2 = x^T \eta\, x = \eta_{\mu\nu} x^\mu x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 \tag{30}$$

where the metric $\eta_{\mu\nu}$ has non-vanishing components $\eta_{00} = -1$, and $\eta_{11} = \eta_{22} = \eta_{33} = 1$, that is

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{31}$$

We have chosen the greek indices to take the values $0, 1, 2, 3$, with 0 indicating the time-like direction. Note that one should be careful to distinguish upper indices from powers, as the notation may be potentially ambiguous. In practice, any confusion is always resolved by looking at the context. The Minkowski metric is not positive definite. Nevertheless, it allows to map vectors x^μ to dual vectors $x_\mu = \eta_{\mu\nu} x^\nu$, so that the scalar product above may be written also as $s^2 = x^\mu x_\mu$. Notice now that $x^\mu \neq x_\mu$. This construction is at the basis of Special Relativity when treated with tensors.

Similar definitions of dual spaces and metrics can be extended to complex vector spaces, such as the Hilbert space of quantum mechanics, with the field $\mathbb{R}$ replaced by the field of the complex numbers $\mathbb{C}$.

Let us describe for example $\mathbb{C}^N$, considered as a complex vector space of N complex dimensions. Its dual space, which we name $\tilde{\mathbb{C}}^N$, is defined as the space of linear maps $\langle w|$ from $\mathbb{C}^N$ to $\mathbb{C}$

$$\begin{aligned} \langle w| : \ \mathbb{C}^N &\longrightarrow \mathbb{C} \\ |z\rangle &\longrightarrow \langle w|z\rangle = w_i z^i \ . \end{aligned} \tag{32}$$

where the components z^i and w_i are all complex numbers. A canonical map that relates the two spaces is obtained by introducing a (complex) metric defined by the scalar product

$$s^2 = z^\dagger z = z_i^* z^i \tag{33}$$

where $z \in \mathbb{C}^N$ have components z^i for $i = 1, ..., N$, with z_i^* denoting their complex conjugates.

Transposition of matrices

To introduce this concept, let us first consider a matrix with lower indices only. Setting

$$A = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \tag{34}$$

one can define the transposed matrix by exchanging rows and columns. Thus, the transpose of the matrix A is given by

$$A^T = \begin{pmatrix} A^T_{11} & A^T_{12} \\ A^T_{21} & A^T_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} \tag{35}$$

i.e.

$$A^T_{ij} = A_{ji} \tag{36}$$

which means precisely exchanging rows with columns. For products of matrices, one finds that

$$\begin{aligned} (AB)^T &= B^T A^T \\ (AB)^{-1} &= B^{-1} A^{-1} \\ \det AB &= \det A \det B \end{aligned} \tag{37}$$

where we recall that the inverse of a matrix exists only if its determinant is nonvanishing. To familiarize ourselves with these notations, let us observe the following product written using indices:

$$A = B^T C \qquad \longleftrightarrow \qquad A_{ij} = B^T_{ik} C_{kj} = B_{ki} C_{kj} \ .$$

where in the last expression, one uses only the entries of B rather than those of B^T. This example shows that one must be careful in reconstructing a product of matrices in an expression written using indices.

Similarly, one defines the transpose of an operator A with an index structure $A^i{}_j$ as follows:

$$(A^T)_i{}^j = A^j{}_i \tag{38}$$

showing that A^T cannot be interpreted as a linear operator on the original vector space V, but rather as an operator acting on the dual space $\tilde{V}$. The index structure of A^T shows precisely that.

3 Definition of a group

Let us define a group $G = \{g\}$ as a set of elements g that satisfy the following properties:

1. There exists a composition law: given $g_1, g_2 \in G$, then $g_1 \cdot g_2 = g_3 \in G$.
2. There exists an identity element: $\exists e \in G$ such that $g \cdot e = e \cdot g = g$.
3. There exists an inverse element: if $g \in G$, then $\exists g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.
4. Associativity: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.

Discrete groups are those that contain a finite number of elements. For example, the group $Z_2 \equiv \{1, -1\}$ with the usual multiplication law defines a group with two elements. *Lie groups* are groups with an infinite number of elements, where the elements depend continuously on certain parameters. For example, rotations around the z-axis of our three-dimensional space form a Lie group whose elements are parameterized by an angle $\theta \in [0, 2\pi]$. For Lie groups, one can consider infinitesimal transformations, i.e. transformations that are, in a sense, very close to the identity and lead to the concept of Lie algebras. *Abelian groups* are those groups whose elements commute under the composition law: $g_1 \cdot g_2 = g_2 \cdot g_1$ for every elements g_1 and g_2 belonging to the group. If this does not happen, the group is said to be *non-abelian*.

3.1 Examples

Some examples of discrete groups are:

- The *cyclic group* Z_n, the finite group generated by the powers of an element a of the group, $Z_n = \{e, a, a^2, \ldots, a^{n-1}\}$, with the condition that $a^n = a^0 = e$. It is a group isomorphic to the n-th roots of unity $e^{\frac{2\pi i}{n}k}$, with $k = 0, 1, \ldots, n-1$. It is an abelian group for any n.
- The group of permutations of n objects, which contains $n!$ elements. It is denoted by S_n and is called the *symmetric group*. One can check that $S_2 = Z_2$, while S_3 contains six elements and is the simplest example of a non-abelian group.

Some examples of Lie groups are:

- $GL(N, \mathbb{R})$, the group of real $N \times N$ matrices with determinant $\neq 0$.
- $SL(N, \mathbb{R})$, the group of real $N \times N$ matrices with determinant $= 1$.
- $O(N)$, the group of real orthogonal $N \times N$ matrices. It describes the invariances of the scalar product $x^T x$ with $x \in \mathbb{R}^N$.
- $SO(N)$, the group of real orthogonal $N \times N$ matrices with determinant $= 1$.
- $GL(N, \mathbb{C})$, the group of complex $N \times N$ matrices with determinant $\neq 0$.
- $SL(N, \mathbb{C})$, the group of complex $N \times N$ matrices with determinant $= 1$.
- $U(1) = \{z \in \mathbb{C} \mid |z| = 1\} = \{e^{i\theta} \mid \theta \in [0, 2\pi]\}$, the group of phases. It describes the invariances of the product $z^* z$ with $z \in \mathbb{C}$.
- $U(N)$, the group of unitary $N \times N$ matrices. It describes the invariances of the scalar product $z^\dagger z$ with $z \in \mathbb{C}^N$.
- $SU(N)$, the group of unitary $N \times N$ matrices with determinant $= 1$.

There are relationships between these groups, for example: $U(1) = SO(2)$; $O(N) = Z_2 \otimes SO(N)$; $U(N) = U(1) \otimes SU(N)$.

4 Representations

We now introduce the concept of group representation. A *representation* of an abstract group G is a "realization" of the multiplicative relations of the group G in a corresponding group of square matrices, where the product is given by the usual matrix multiplication. These matrices should be thought of as *linear operators* that act on a *vector space* V, whose dimension is called the *dimension* of the representation. Explicitly, a representation is given by a mapping

$$\begin{aligned} R: \; G &\longmapsto \text{Square Matrices} \\ g &\longmapsto R(g) \end{aligned} \tag{39}$$

such that

1. $R(g_1)R(g_2) = R(g_1 \cdot g_2)$
2. $R(e) = \mathbb{1}$ with $\mathbb{1}$ as the identity matrix.

From this, it also follows that $R(g^{-1})R(g) = R(e) = \mathbb{1}$, hence $R(g^{-1}) = R^{-1}(g)$. Associativity is automatic because matrix multiplication is associative. Thus, all the properties of the group are explicitly realized by the matrices of a representation.

By thinking of the matrices of a representation as operators that act on a vector space V of dimension N, the matrices are $N \times N$ matrices, and the representation is said to be of dimension N.

As a very simple example of a representation, consider the cyclic group $Z_2 = \{e, a\}$ with the relation that $a^2 = e$. Then, a simple two-dimensional representation is given by the following 2×2 matrices

$$R(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; , \quad R(a) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \; . \tag{40}$$

It is easily checked that the matrices of the representation satisfy all the properties of the abstrarct group $Z_2 = \{e, a\}$. As we shall understand soon, this representation is reducible, as it contains two copies of the more simple representation defined in terms of 1×1 matrices, i.e. numbers,

$$R(e) = 1 \; , \quad R(g) = -1 \; . \tag{41}$$

In the list of Lie groups introduced in the previous examples, we have used matrices to define the groups. Thus, these matrices give rise immediately to a particular representation: the *defining representation* (also called the *fundamental representation*). The elements of the group in the defining representation naturally perform transformations on vectors belonging to a vector space V, the vector space on which the matrices act as linear operators. Let v^a denote the components of a vector. The matrix $R(g)$, which represents the element g of the abstract group G, transforms this vector as follows

$$v^a \quad \stackrel{g \in G}{\longrightarrow} \quad v'^a = [R(g)]^a{}_b \, v^b \tag{42}$$

where, as usual, $[R(g)]^a{}_b$ describes, as the indices a and b vary, the elements of the matrix $R(g)$. The row index a is the first index and is conventionally placed in the upper position, and the column index b is the second index and is conventionally placed in the lower position. In this way, the vectors in the vector space V are transformed by operations associated with the group

G. Repeated indices are summed over all their possible values, and the convention is used that in the sum, one index is in the upper position and the other one in the lower position.

At this point, the problem arises of studying how many and what kinds of representations of a given group exist. In particular, it is useful to know which are their dimensions. This problem is of great importance for physical applications because the "vectors" of a representation (generically called "tensors") can be used to conveniently describe physical quantities associated with models where G acts as a symmetry group.

In general, *equivalent* representations are defined as those that are related by similarity transformations: $R(g)$ and $\tilde{R}(g)$ are equivalent representations if

$$\tilde{R}(g) = A\, R(g)\, A^{-1} \qquad \forall g \in G \tag{43}$$

where A is a matrix independent of g. This equivalence relation allows us to consider equivalent representations as essentially the same representation. Indeed, the similarity transformation simply represents a change of basis in the vector space V: the matrices of the different equivalent representations identify the same linear operator expressed in different bases.

A *reducible* representation is a representation equivalent to a representation whose matrices are block diagonal, for example, $R(g)$ is reducible if

$$\tilde{R}(g) = A\, R(g)\, A^{-1} = \left(\begin{array}{c|c|c} R_1(g) & 0 & 0 \\ \hline 0 & R_2(g) & 0 \\ \hline 0 & 0 & R_3(g) \end{array}\right) \qquad \forall g \in G \tag{44}$$

for an appropriate matrix A, and it is said that $R(g)$ is reducible to the three representations $R_1(g)$, $R_2(g)$, $R_3(g)$. In this example, the vector space V on which the reducible representation $R(g)$ acts is naturally decomposed as a direct sum of the three vector spaces on which the representations $R_1(g)$, $R_2(g)$, $R_3(g)$ act, i.e., $V = V_1 \oplus V_2 \oplus V_3$. This reducibility is thus written as $R(g) = R_1(g) \oplus R_2(g) \oplus R_3(g)$.

An *irreducible* representation is a representation that cannot be decomposed as above[2].

In the classification of the possible representations of a group G, it is useful to consider only inequivalent irreducible representations, as all other representations follow from them. Given a fixed integer N, it is not guaranteed that an irreducible representation of dimension N exists. In general, only for certain values of N will there be representations of a fixed group G (sometimes even more than one with the same dimension).

A *unitary* representation is a representation in terms of unitary matrices (operators). Unitary representations are very useful in applications of quantum mechanics, where the symmetries of a quantum system are described by unitary operators acting in Hilbert space (an infinite-dimensional vector space endowed with a positive-definite norm).

4.1 Upper and Lower Indices, Dotted Indices

In the previous examples, we defined Lie groups using matrices that directly identify a representation, the so-called defining (or fundamental) representation. We denote the dimension of the defining representation by N. As mentioned earlier, we can think of the $N \times N$ matrices of this representation as operators acting on a vector space V of dimension N. We denote the

[2]For some groups, there can exist reducible representations of a particular type, formed by upper triangular matrices, but we overlook this subtlety in a first exposition to group theory, as the simplest groups we are interested in do not show such a phenomenon.

vectors in V by their components v^a, where the index $a = 1, 2, \ldots, N$. The vectors $v^a \in V$ are transformed by the matrices $[R(g)]^a{}_b$ of the representation. By definition, a generic vector v^a transforms under the action of the group G as follows:

$$v^a \quad \overset{g \in G}{\longrightarrow} \quad v^{a\prime} = [R(g)]^a{}_b \, v^b \tag{45}$$

(note that the convention is used where repeated indices are automatically summed over all their possible values). Vectors that transform in the manner described above are defined to have upper indices. Vectors whose components have upper indices belong to vector spaces equivalent to V and transform the same way under the action of G, as described by (45).

Given the defining representation $R(g)$ that acts on the vector space V, which corresponds to transforming vectors with upper indices, we can immediately construct three other representations:

$R(g)^*$, the complex conjugate representation acting on V^*

$R(g)^{-1\,T}$, the inverse transposed representation acting on the dual space $\tilde{V}$

$R(g)^{-1\,\dagger}$, the inverse Hermitian conjugate[3] representation acting on $\tilde{V}^*$.

The vectors they act on have the following index structures by convention, respectively:
vectors with "dotted upper indices" $v^{\dot{a}}$ (vectors in the complex conjugate space V^*),
vectors with "lower indices" v_a (vectors in the dual space $\tilde{V}$),
vectors with "dotted lower indices" $v_{\dot{a}}$ (vectors in the complex conjugate dual space $\tilde{V}^*$).
In formulae:

$$\begin{aligned} v^{\dot{a}} \quad &\overset{g \in G}{\longrightarrow} \quad v'^{\dot{a}} = [R(g)^*]^{\dot{a}}{}_{\dot{b}} \, v^{\dot{b}} \\ v_a \quad &\overset{g \in G}{\longrightarrow} \quad v'_a = [R(g)^{-1\,T}]_a{}^b \, v_b \\ v_{\dot{a}} \quad &\overset{g \in G}{\longrightarrow} \quad v'_{\dot{a}} = [R(g)^{-1\,\dagger}]_{\dot{a}}{}^{\dot{b}} \, v_{\dot{b}} \; . \end{aligned} \tag{46}$$

It is immediate to verify that these are representations of the group G if $R(g)$ is one. The different index structure associated with these matrices reflects the fact that they are operators acting on different vector spaces.

Invariant quantities under the action of the group G can be obtained by taking the scalar product between vectors with upper indices (sometimes called contravariant) and those with lower indices (sometimes called covariant), whether dotted or undotted. One can verify the following identities

$$\begin{aligned} v_a w^a \quad &\overset{g \in G}{\longrightarrow} \quad v'_a w'^a = v'^T w' = (R(g)^{-1\,T}\, v)^T R(g) w = v^T R(g)^{-1} \, R(g) w = v^T w = v_a w^a \\ x_{\dot{a}} y^{\dot{a}} \quad &\overset{g \in G}{\longrightarrow} \quad x'_{\dot{a}} y'^{\dot{a}} = x'^T y' = (R(g)^{-1\,\dagger} x)^T R(g)^* y = x^T R(g)^{-1\,*} \, R(g)^* y = x^T y = x_{\dot{a}} y^{\dot{a}} \end{aligned} \tag{47}$$

Exercize: rederive these equations using only the index notation.

In general, it makes no group-theoretic sense to contract indices of the vectors described above in any other way ("contracting" refers to the operation of equating two indices and summing over all possible values that these indices can assume).

However, some of these different representations may be equivalent to each other, i.e., related by a similarity transformation. Indeed, for <u>real</u> representations, $R(g)^* = R(g)$, so

[3]Given a matrix R, its Hermitian conjugate (or adjoint) $R^\dagger$ is defined as the complex conjugate of the transpose, $R^\dagger = R^{\mathrm{T}*}$.

$v^{\dot{a}} \sim v^a$ and $v_{\dot{a}} \sim v_a$, where the symbol $\sim$ means "transforms like". Thus, in this case, there is no need to introduce dotted indices. For <u>*unitary*</u> representations, $R(g)^{-1} = R(g)^{\dagger}$, and therefore $R(g)^{-1\dagger} = R(g)$, so $v_{\dot{a}} \sim v^a$ and $v^{\dot{a}} \sim v_a$. Again, there is no need to use dotted indices. Finally, for unitary and real (i.e., <u>*real orthogonal*</u>) representations, all four of the above-described representations are equivalent: there is no need to use dotted indices or lower indices.[4]

4.2 Other Representations: Tensor Representations and Tensors

Other representations can be obtained from the tensor product of the previously described representations. By definition, these representations act on "tensors," which are elements of vector spaces obtained from the tensor product of copies of V, V^*, $\tilde{V}$, and $\tilde{V}^*$. Therefore, tensors, by definition, have a certain number of upper and lower indices, with transformation properties defined by the nature associated with those indices.

For example, a tensor $F^{ab}{}_{c}{}^{\dot{d}}{}_{\dot{e}}$ is, by definition, an object with N^5 components that transform exactly like the product of the components of the previously defined vectors (tensor product):

$$F^{ab}{}_{c}{}^{\dot{d}}{}_{\dot{e}} \sim v^a u^b w_c x^{\dot{d}} y_{\dot{e}} \; . \tag{48}$$

Thus, the tensor $F^{ab}{}_{c}{}^{\dot{d}}{}_{\dot{e}}$ represents (the components of) an element of a vector space of dimension N^5 (because each index can take N values; it corresponds to an element of the vector space $V \otimes V \otimes \tilde{V} \otimes V^* \otimes \tilde{V}^*$ and we can write $F^{ab}{}_{c}{}^{\dot{d}}{}_{\dot{e}} \in V \otimes V \otimes \tilde{V} \otimes V^* \otimes \tilde{V}^*$). Under the action of the group G, it transforms as follows:

$$F^{ab}{}_{c}{}^{\dot{d}}{}_{\dot{e}} \quad \xrightarrow{g \in G} \quad F'^{\,ab}{}_{c}{}^{\dot{d}}{}_{\dot{e}} = [R(g)]^a{}_f \, [R(g)]^b{}_g \, [R(g)^{-1\,T}]_c{}^h \, [R(g)^*]^{\dot{d}}{}_{\dot{m}} \, [R(g)^{-1\,\dagger}]_{\dot{e}}{}^{\dot{n}} \, F^{fg}{}_{h}{}^{\dot{m}}{}_{\dot{n}} \; . \tag{49}$$

This linear transformation law identifies a representation of dimension N^5 (the N^5 components are mixed among themselves by an $N^5 \times N^5$ matrix, implicitly defined by the above formula, thus providing a representation of the group).

Typically, tensors correspond to reducible representations, i.e. are transformed by reducible representations. The problem of decomposing representations into irreducible ones now arises. One way to decompose a representation is to study the tensors on which they act. A first decomposition operation is to separate the tensors by considering their symmetry properties under permutations of indices of the same nature (it is therefore useful to know the properties of the permutation group of n objects, i.e. the symmetric group S_n).

For example, the tensor T^{ab} can be separated into its symmetric part ($S^{ab} = S^{ba}$) and its antisymmetric part ($A^{ab} = -A^{ba}$) as follows:

$$T^{ab} = \underbrace{\frac{1}{2}(T^{ab} + T^{ba})}_{S^{ab}} + \underbrace{\frac{1}{2}(T^{ab} - T^{ba})}_{A^{ab}} \; . \tag{50}$$

[4]The finite-dimensional spinorial representations of the Lorentz group $SO(3,1)$ are two-dimensional and are neither unitary nor real (spinorial representations are double-valued representations and correspond to true representations of the universal covering of the Lorentz group $SL(2,\mathbb{C})$). In this case, all four different types of indices are useful (although only two of these four representations are inequivalent). The use of dotted and undotted indices for these two-component spinors (Weyl spinors) in four spacetime dimensions was introduced by Van der Waerden. This is why the use of dotted indices is sometimes referred to as the "Van der Waerden notation".

It is easy to convince oneself that these parts with distinct symmetries do not mix under group transformations. Indeed, one can calculate the transformed symmetric part under an arbitrary group transformation and verify that it remains symmetric:

$$\begin{aligned} S^{ab} \quad \stackrel{g\in G}{\longrightarrow} \quad S'^{ab} &= [R(g)]^a{}_c [R(g)]^b{}_d\, S^{cd} \\ &= [R(g)]^a{}_c [R(g)]^b{}_d\, S^{dc} \\ &= [R(g)]^b{}_d [R(g)]^a{}_c S^{dc} = S'^{ba} \ . \end{aligned} \tag{51}$$

Similarly, one can verify that

$$\begin{aligned} A^{ab} \quad \stackrel{g\in G}{\longrightarrow} \quad A'^{ab} &= [R(g)]^a{}_c [R(g)]^b{}_d\, A^{cd} \\ &= [R(g)]^a{}_c [R(g)]^b{}_d\, (-A^{dc}) \\ &= -[R(g)]^b{}_d [R(g)]^a{}_c A^{dc} = -A'^{ba} \ , \end{aligned} \tag{52}$$

so that the transformed antisymmetric part remains antisymmetric. Symmetric parts and antisymmetric parts are never mixed by group transformations, so the tensor representation identified by the tensor T^{ab} is reducible. In a compact notation, we can denote the representation that transforms the tensor $T^{ab} \sim T$ as $R_T(g)$ so that

$$T' = R_T(g)\, T \ . \tag{53}$$

This representation is reducible:

$$\begin{pmatrix} S' \\ A' \end{pmatrix} = \underbrace{\left(\begin{array}{c|c} R_S(g) & 0 \\ \hline 0 & R_A(g) \end{array}\right)}_{R_T(g)} \begin{pmatrix} S \\ A \end{pmatrix} \tag{54}$$

where $T \sim \binom{S}{A}$ indicates the decomposition into symmetric and antisymmetric parts.

These parts may be further reduced if there are other invariant operations (such as the possibility of taking scalar products as in (47)). For the simpler representations, it is easy to study any further reducibility on a case-by-case basis.

Note that the Kronecker delta tensors $\delta^a{}_b$ and $\delta^{\dot a}{}_{\dot b}$, which are the matrix elements of the identity operators, remain invariant under group transformations if their indices are transformed according to their nature. For example,

$$\begin{aligned} \delta^a{}_b \quad \stackrel{g\in G}{\longrightarrow} \quad (\delta')^a{}_b &= [R(g)]^a{}_c [R(g)^{-1\,T}]_b{}^d \delta^c{}_d = [R(g)]^a{}_c [R(g)^{-1\,T}]_b{}^c = [R(g)]^a{}_c [R(g)^{-1}]^c{}_b = \\ &= [R(g) R(g)^{-1}]^a{}_b = \delta^a{}_b \ . \end{aligned} \tag{55}$$

These are called *invariant tensors*. In contrast, δ_{ab} does not identify any invariant tensor (unless there are special relations between the various types of indices): if we define a tensor that coincides with δ_{ab} in a “reference frame,” under a group transformation (a “change of reference frame”) the components of the tensor change value.

The existence and number of invariant tensors depend on the group G under consideration. For example, the group $SO(N)$ admits an invariant tensor defined by the completely antisymmetric symbol $\epsilon^{a_1 \dots a_n}$, where the indices are those of the fundamental representation. This follows from the fact that the matrices of the group $SO(N)$ have determinant 1. Similarly, the group $SU(N)$ admit the invariant tensors given by the completely antisymmetric symbols $\epsilon^{a_1 \dots a_n}$ and $\epsilon_{a_1 \dots a_n}$.

4.3 Representations of $SO(N)$

We describe here the simplest representations of $SO(N)$, the special orthogonal group of real $N \times N$ matrices

$$SO(N) = \{\text{real } N \times N \text{ matrices } R \mid R^T R = \mathbb{1}, \det R = 1\} \; . \tag{56}$$

This is the group that leaves invariant the scalar product of vectors $\vec{v}, \vec{w} \in \mathbb{R}^N$, defined by $\vec{v} \cdot \vec{w} = \delta_{ab} v^a w^b$, where the metric δ_{ab} is recognized to be an invariant tensor (indices up and down are equivalent for $SO(N)$, so that $\delta_{ab} = \delta^a{}_b$, and we already know that $\delta^a{}_b$ is an invariant tensor). More directly, using matrix notation, we compute

$$\begin{aligned} & v' = Rv \; , \quad w' = Rw \\ & \vec{v} \cdot \vec{w} = v^T w \quad \to \quad v'^T w' = (Rv)^T R w = v^T R^T R w = v^T w \; . \end{aligned} \tag{57}$$

Equivalently, using components

$$\begin{aligned} & v'_a = R_{ab} v_b \; , \quad w'_a = R_{ab} w_b \\ & \vec{v} \cdot \vec{w} = v_a w_a \quad \to \quad v'_a w'_a = R_{ab} v_b \, R_{ac} w_c = v_b \, R_{ab} R_{ac} \, w_c = v_b \underbrace{R^T_{ba} R_{ac}}_{\delta_{bc}} w_c = v_b w_b = v_a w_a \end{aligned} \tag{58}$$

where we used again the fact that upper and lower indices are equivalent.

Thus, the defining representation (also called vector representation) acts on the vectors v^a. As already described, the four basic representations are all equivalent as $v^a \sim v_a \sim v^{\dot{a}} \sim v_{\dot{a}}$. We denote this representation by N, i.e., by its dimension. The tensor product $N \otimes N$ identifies the tensor representation that acts on tensors with two indices T^{ab} and thus corresponds to a representation of dimension N^2. It is a reducible representation. To extract the irreducible representations that it contains, we proceed as follows. We have already seen that the tensor T^{ab} can be separated into the symmetric part S^{ab} (of dimension $\frac{N(N+1)}{2}$) and the antisymmetric part A^{ab} (of dimension $\frac{N(N-1)}{2}$). The symmetric part is still reducible because one can construct a scalar (an invariant under the group transformations) by taking its trace:

$$S \equiv \delta_{ab} S^{ab} = S^a{}_a \; . \tag{59}$$

It is easily seen that this is a scalar, as we already know that the contraction of an upper index with a lower index produces a scalar:

$$S \quad \stackrel{g \in SO(N)}{\longrightarrow} \quad S' = S \tag{60}$$

It identifies a trivial one-dimensional representation: $R_{scal}(g) = 1$. We can separate the trace from the symmetric tensor S^{ab} in the following way:

$$S^{ab} = \underbrace{S^{ab} - \frac{1}{N} \delta^{ab} S}_{\hat{S}^{ab}} + \frac{1}{N} \delta^{ab} S \tag{61}$$

where we have defined the traceless symmetric tensor $\hat{S}^{ab}$ (which satisfies $\hat{S}^a{}_a = 0$). Thus, collecting all pieces, we have succeeded in separating the tensor T^{ab} into its irreducible parts:

$$T^{ab} = \frac{1}{N} \delta^{ab} S + A^{ab} + \hat{S}^{ab} \tag{62}$$

They transform independently without ever mixing. Indicating the irreducible representations with their respective dimensions, the above translates into the following expression:

$$N \otimes N = 1 \oplus \frac{N(N-1)}{2} \oplus \left(\frac{N(N+1)}{2} - 1 \right) . \tag{63}$$

It can be shown that there are no further reductions. The representation acting on antisymmetric tensors with two indices A^{ab}, the $\frac{N(N-1)}{2}$, is also called the adjoint representation: its dimension corresponds to the number of independent parameters of the group, given by the angles describing the rotations in the a–b planes (with $a \neq b$).

In summary, for $SO(N)$, we understand that there exist the following irreducible representations, indicated by their dimension:

$$1, N, \frac{N(N-1)}{2}, \left(\frac{N(N+1)}{2} - 1 \right), \dots \tag{64}$$

where 1 is the trivial representation (the scalar), N is the vector representation (also called defining or fundamental), the $\frac{N(N-1)}{2}$ is the adjoint representation, the $\frac{N(N+1)}{2} - 1$ is the traceless symmetric representation, etc.

In the specific case of $SO(3)$, the formula in (63) reduces to:

$$3 \otimes 3 = 1 \oplus 3 \oplus 5 . \tag{65}$$

We see that in this special case, the adjoint representation coincides with the vector representation: indeed the dimensions are the same, and a full proof is simple to produce. Translated into the language of quantum mechanics, this formula tells us that combining spin 1 (the vector representation "3") with another spin 1 yields spin 0 (the "1" representation, the scalar), spin 1 (again the "3" representation), and spin 2 (the "5" representation). Equivalently, defining the quantum numbers l by setting $n = 2l + 1$ for $n = 1, 3, 5$, this relation can be written as:

$$(l = 1) \otimes (l = 1) \ = \ (l = 0) \oplus (l = 1) \oplus (l = 2)$$

which is the formula for adding quantum angular momentums. In quantum mechanics, orbital angular momentum is quantized and is fixed by an integer quantum number $l = 0, 1, 2, 3, \dots$, indicating that the projection of the angular momentum along a fixed axis can only take $2l+1$ values. The $(2l+1)$-representation is the one acting on the traceless, symmetric tensor with l indices, $\hat{S}^{a_1, a_2, \dots, a_l}$. The electron orbiting the nucleus can have angular momentum with $l = 0$ (S orbital), angular momentum with $l = 1$ (P orbital), angular momentum with $l = 2$ (D orbital), etc. Continuing with the study of angular momentum in quantum mechanics, one discovers that intrinsic angular momenta (spins) are characterized by integer and half-integer values of the quantum number, i.e. $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. The rules for composing angular momentum in quantum mechanics correspond precisely to the decomposition of a tensor product into irreducible representations mentioned above. Strictly speaking, the representations with half-integer spin (spinors) are not truly representations of the $SO(N)$ group, as they are double valued (a rotation of 2π is not the identity but minus the identity). They are representations of the covering group as well as of the $SO(N)$ Lie algebra, a concept that we shall introduce shortly.

In the case of $SO(4)$, or the Lorentz group $SO(3, 1)$, the formula (63) reduces to:

$$4 \otimes 4 = 1 \oplus 6 \oplus 9 . \tag{66}$$

The 6-dimensional representation is the adjoint representation. It is the one that acts on the electromagnetic field, which indeed has six independent components that are mixed under Lorentz transformations. The electromagnetic field is described by an antisymmetric tensor with two indices $F^{\mu\nu}$. In the case of the Lorentz group, upper and lower indices are equivalent, and the Minkowski metric is used to pass from one to the other (the metric describes the similarity transformation that connects the two representations).

4.4 Representations of $SU(N)$

Consider now $SU(N)$, the special unitary group of $N \times N$ matrices

$$SU(N) = \{\text{complex } N \times N \text{ matrices } U \mid U^\dagger U = \mathbb{1},\ \det U = 1\}\ . \tag{67}$$

This is the group that preserves the inner product of vectors $\vec{v}, \vec{w} \in \mathbb{C}^N$ defined by $\vec{v}^* \cdot \vec{w} = v_a^* w^a = \delta^a{}_b v_a^* w^b$, where * denotes the complex conjugate. The metric $\delta^a{}_b$ is an invariant tensor (see eq. (55)).

Starting from the fundamental representation, N (corresponding to the vectors v^a), we immediately obtain another representation, the complex conjugate representation (transforming the vectors $v^{\dot{a}} \sim v_a$). It is denoted by $\overline{N}$.

Now, let's find other irreducible representations by considering the tensor product

$$N \otimes N = \frac{N(N+1)}{2} \oplus \frac{N(N-1)}{2} \tag{68}$$

which corresponds to the decomposition of the tensor T^{ab} into its symmetric and antisymmetric parts, $T^{ab} = S^{ab} + A^{ab}$. This decomposition is exhaustive (note that it is not possible to take traces to form scalars on these tensors because δ_{ab} is not an invariant tensor for $SU(N)$: to see this, simply transform the tensor δ_{ab} as dictated by the structure of its indices and see that it is not invariant). Hence, we have discovered the existence of two new representations and know their dimensions.

Consider now

$$N \otimes \overline{N} = 1 \oplus (N^2 - 1) \tag{69}$$

which corresponds to the decomposition of the tensor $T^a{}_b$ into its trace part (the scalar) and its traceless part. This is possible because we know that contracting a raised index with a lowered index produces a scalar. In formulas, this separation is written as

$$T^a{}_b = \frac{1}{N}\delta^a{}_b T + \hat{T}^a{}_b \tag{70}$$

where $T \equiv T^a{}_a$ and $\hat{T}^a{}_b \equiv T^a{}_b - \frac{1}{N}\delta^a{}_b T$. Note that the tensor $\delta^a{}_b$ is an invariant tensor (it corresponds to the metric of the complex vector space $\mathbb{C}^N$). Thus, we have discovered the existence of the representation of dimension $N^2 - 1$, the so-called *adjoint representation*.

Other invariant tensors of $SU(N)$ are the completely antisymmetric tensors with N indices, $\epsilon_{a_1 a_2 \dots a_N}$ and $\epsilon^{a_1 a_2 \dots a_N}$ (this can be demonstrated using the fact that the group matrices have determinants equal to one). They can be used to study the reduction (or equivalence) of other tensorial representations.

In summary, for $SU(N)$, we have seen that there exist the following irreducible representations:

$$1, N, \bar{N}, N^2 - 1, \frac{N(N-1)}{2}, \frac{N(N+1)}{2}, \overline{\frac{N(N-1)}{2}}, \overline{\frac{N(N+1)}{2}}, \tag{71}$$

where 1 is the trivial representation (the scalar), N is the fundamental representation (or defining), $\bar{N}$ is the antifundamental (complex conjugate of the fundamental), and N^2-1 is the adjoint representation, which is real, etc.

Let's make this explicit for the case of $SU(2)$. We have

$$2 \otimes 2 = 1 \oplus 3 \ , \qquad 2 \otimes \bar{2} = 1 \oplus 3 \tag{72}$$

These formulas suggest that perhaps 2 and $\bar{2}$ are equivalent representations, i.e. $\bar{2} \sim 2$. This is indeed the case: using the invariant tensor ϵ_{ab} we may relate the two representations by setting $w_a = \epsilon_{ab} v^b$, then under a group transformation we see that

$$w'_a = \epsilon'_{ab} v'^b = \epsilon_{ab} v'^b \tag{73}$$

which indicates that, up to a change of basis given by the ϵ_{ab} tensor, the vectors v^a and w_a transform in the same way. Here, we used the fact that ϵ_{ab} is an invariant tensor. The explicit proof is as follows: if $R \in SU(2)$ then

$$\epsilon'^{ab} = R^a{}_c R^b{}_d \epsilon^{cd} = k \epsilon^{ab} \tag{74}$$

for some coefficient k. This follows from the fact that an antisymmetric 2×2 matrix has only one independent component. To determine k, we calculate

$$\epsilon'^{12} = R^1{}_c R^2{}_d \epsilon^{cd} = R^1{}_1 R^2{}_2 - R^1{}_2 R^2{}_1 = \det R = 1 \ . \tag{75}$$

So $k = 1$ and $\epsilon'^{ab} = \epsilon^{ab}$.

Translating (72) into the language of quantum mechanics, it means that combining spin $\frac{1}{2}$ (the representation "2") with itself yields spin 0 (the representation "1", the scalar) and spin 1 (the representation "3"). Indeed, defining $j = 2s+1$ for $s = 0, \frac{1}{2}, 1$, this relation can be equivalently written as

$$(j = \tfrac{1}{2}) \otimes (j = \tfrac{1}{2}) \ = \ (j = 0) \oplus (j = 1) \ .$$

The group $SU(2)$ describes space rotations, including the possibility of having half-integer spins associated with fermionic particles. In mathematical terms, one says that the group $SU(2)$ is the universal cover of the group $SO(3)$.

Now, let us make explicit also the case of $SU(3)$. It has physical applications both as the flavor symmetry group $SU(3)_{\text{flavor}}$ which mixes the three "flavors" of quarks (up, down, strange), and as color symmetry group $SU(3)_{\text{color}}$ which mixes the three colors of each quark (conventionally red, green, blue). We have

$$3 \otimes \bar{3} = 1 \oplus 8 \tag{76}$$

In $SU(3)_{\text{flavor}}$, 3 and $\bar{3}$ correspond to the up, down, and strange quarks and their antiquarks

$$q^a = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \sim 3 \ , \qquad \bar{q}_a = \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \sim \bar{3} \ . \tag{77}$$

Flavor symmetry means that we can redefine flavors through $SU(3)$ group transformations without changing anything in the description of physical phenomena. In the static quark model of mesons, which are hadrons composed of bound states of quark-antiquark ($q\bar{q}$), the symmetry implies that only singlets or octets of flavor can emerge. The mesonic octet containing the pions is the main example: there are eight mesons with identical properties, and one could not distinguish them from each other if the symmetry were exact (same masses, same spin, etc.). In reality, the symmetry is only approximate, so there are some small differences (e.g., they have slightly different mass, also they have different charges and electromagnetism violates this symmetry).

Another application concerns the color of quarks and is associated with another $SU(3)$ group, called $SU(3)_{\text{color}}$. Each quark flavor has three colors (red, green, blue); for example, for the up quark we can group them in a vector

$$u^a = \begin{pmatrix} u^{\text{red}} \\ u^{\text{green}} \\ u^{\text{blue}} \end{pmatrix} \sim 3 \,. \tag{78}$$

Color symmetry means that we can redefine colors through $SU(3)$ group transformations without changing anything (color symmetry is exact). The information contained in the relation (76) is that it is possible to combine the colors of a quark with the colors of an antiquark (the anticolors) to form a colorless state (the scalar) or states with eight possible different color combinations: indeed, quark/antiquark of the same flavor can combine into a photon (the scalar, or singlet, of color) or into a gluon (there are eight different possibilities, so that one says that the gluons form an octet of color).

Moreover,

$$3 \otimes 3 = 6 \oplus \bar{3} \,. \tag{79}$$

The possible ambiguity in understanding whether the tensor A^{ab}, which has three components, corresponds to 3 or $\bar{3}$ is resolved in favor of the latter option considering that $A^{ab} \sim A^{ab}\epsilon_{abc} \sim V_c$ (since ϵ_{abc} is an invariant tensor for $SU(3)$). This relation in $SU(3)_{\text{color}}$ tells us that combining the colors of two quarks is not possible to obtain a colorless state (the scalar).

With a bit more effort, one can also deduce (considering the symmetries of the tensor T^{abc}) that

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10 \tag{80}$$

where the 1 corresponds to the completely antisymmetric part of T^{abc}, the 10 to the completely symmetric part of T^{abc}, and the two 8s to parts of the tensor with mixed symmetry. In applications in the static quark model of baryons, hadrons composed of bound states of three quarks (qqq), the symmetry $SU(3)_{\text{flavor}}$ predicts that families of similar particles can only exist with 1, 8, or 10 components (not all need to exist: some combinations might not appear for other reasons). There are several octets (the 8), like the eight baryons which have similar properties concerning the strong interactions (a particular octet contains the proton and the neutron). Their antiparticles also form octets. There is also a famous decuplet of baryons, whose wave functions are symmetric in the flavors of the three constituent quarks. These wave functions transform into the 10 of SU(3) under flavor symmetry transformations. The corresponding anti-baryons group into $\overline{10}$. Applying the relation (80) to color, the fact that the 1 appears on the right side is interpreted as the possibility of combining the colors of three quarks to form a colorless state (e.g. the proton is made of three quarks; in general mesons and baryons must be color scalars due to a dynamical process called *color confinement*).

4.5 Representations of $U(1)$

Let us also consider the case of representations of the group $U(1)$, which also plays a significant role in physics. The group $U(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi]\}$ is the group of phases. It can be shown that all its irreducible unitary representations are one-dimensional complex representations which are characterized by an integer number, positive or negative, called the "charge". The defining representation represents an element of the group $U(1)$ with the phase $e^{i\theta}$ which "rotates" naturally a complex one-dimensional vector v ($v \in \mathbb{C}$, where $\mathbb{C}$ denotes the field of complex numbers, which we interpret here as a one-dimensional complex vector space)

$$v \quad \stackrel{g\in U(1)}{\longrightarrow} \quad v' = e^{i\theta} v \; , \qquad v \in \mathbb{C} \; . \tag{81}$$

Thus, the vector space of the defining representation is one-dimensional and complex, and the matrices of the representation are complex 1×1 matrices (i.e., complex numbers).

Objects that transform as tensor products of the defining representation

$$v_{(q)} \sim \underbrace{vv \cdots v}_{q \text{ times}} = v^q \tag{82}$$

with q an integer give rise to the *representation of charge* q

$$v_{(q)} \quad \stackrel{g\in U(1)}{\longrightarrow} \quad v'_{(q)} = e^{iq\theta} v_{(q)} \; . \tag{83}$$

The number q can also be negative, as seen by tensoring the antifundamental representation acting on $\bar{v}$.

The tensor product of a representation with charge q_1 with a representation with charge q_2 yields the representation with charge $q_1 + q_2$. The symmetry group $U(1)$ is used in physics when there are *quantized additive quantum numbers*. Since all its representations are one-dimensional, to distinguish the various inequivalent representations, one indicates the charge q of the representation rather than its dimension.

What has been analyzed so far also allows us to interpret the possible charges (generalized charges, such as electric charge, color charge, etc.) of particles and associate them with a representation of the corresponding symmetry group. For example, the Standard Model of elementary particles contains the symmetry group $SU(3) \times SU(2) \times U(1)$ (called the gauge symmetry group). The fermions of the Standard Model have generalized charges under these groups. Let us describe them. We can indicate these charges using a notation of the form $(SU(3), SU(2))_{U(1)}$, where for the non-abelian groups we specify the representation by its corresponding dimension, while for the abelian part we use the $U(1)$ charge, called hypercharge. Anticipating that fermions can be decomposed into right-handed (R) and left-handed (L) fermions, with possibly different charges, one has so far discovered in Nature elementary fermions with the following charges

$\begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix}$	ν_{eR}	e_R	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	u_R	d_R
$\begin{pmatrix} \nu_{\mu L} \\ \mu_L \end{pmatrix}$	$\nu_{\mu R}$	μ_R	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	c_R	s_R
$\begin{pmatrix} \nu_{\tau L} \\ \tau_L \end{pmatrix}$	$\nu_{\tau R}$	τ_R	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$	t_R	b_R
$(1,2)_{-\frac{1}{2}}$	$(1,1)_0$	$(1,1)_{-1}$	$(3,2)_{\frac{1}{6}}$	$(3,1)_{\frac{2}{3}}$	$(3,1)_{-\frac{1}{3}}$

(84)

The group $SU(3)$ is called the color group, and the quarks transform in the fundamental representation, the 3, and thus have three "colors", while the corresponding antiparticles, the antiquarks, transform in the complex conjugate representation, the $\bar{3}$, and thus have three "anticolors". Leptons do not feel the strong force and, therefore, are singlets under the color group. The group $SU(2)$ is called the weak isospin group, and the $SU(2)$ doublets have been written above in the form of column vectors: they transform in the two-dimensional representation, the 2, and thus have weak isospin $I = \frac{1}{2}$, with the third component $I_3 = \frac{1}{2}$ for the upper element of the vector and $I_3 = -\frac{1}{2}$ for the lower one. Note that the 2 is equivalent to the $\bar{2}$, both identifying the same representation with weak isospin equal to $\frac{1}{2}$. $U(1)$ is the hypercharge group. If we denote by Y the hypercharge of a particle, the corresponding electric charge Q is given by $Q = I_3 + Y$, where I_3 denotes the third component of the weak isospin. From the above table, one may extract which are the electric charges of these elementary particles.

5 Lie Groups and Lie Algebras

A Lie group is, by definition, a group whose elements depend continuously on some parameters. By studying the infinitesimal group transformations, i.e. those transformations that differ slightly from the identity, one obtains the so-called *Lie algebra* of the group, an algebra that summarizes essential information about the group. In particular, the Lie algebra captures the non-abelian structure of the group. To introduce these topics, we first study some of the simplest yet most commonly used groups in physics and then list general properties and definitions.

5.1 SO(2)

Consider the familiar group of rotations in two-dimensional Euclidean space, the group $SO(2)$ of real orthogonal 2×2 matrices with determinant equal to 1. These matrices generate the transformations of a vector

$$\vec{x} \quad \longrightarrow \quad \vec{x}\,' = R\,\vec{x} \tag{85}$$

or in tensor notation $x^i \to x'^i = R^i{}_j x^j$ with $i, j = 1, 2$. This is the defining (or vector) representation. The rotations that mix the two components of the vector $\vec{x} = (x, y) = (x^1, x^2)$ depend on an angle θ and can be written as

$$R(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \quad \stackrel{\theta \ll 1}{\longrightarrow} \quad \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} \; = \; \mathbb{1} + \theta \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{iT} + \cdots \tag{86}$$

where the matrix T is the operator that "generates" the infinitesimal part of the transformation

$$T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} . \tag{87}$$

The imaginary unit i in (86) is conventional but allows us to present the generator T as a Hermitian matrix (whose eigenvalues are real).

The group is abelian; its elements commute, $R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1)$, and obviously

$$[T, T] = 0 \tag{88}$$

where $[\cdot,\cdot]$ denotes the commutator ($[A,B]=AB-BA$). This is called the Lie algebra of $SO(2)$. In general, the Lie algebra of a group is generated by the commutators of its infinitesimal generators.

Finite transformations can be obtained by iterating infinitesimal transformations. If the parameter θ is not infinitesimal, consider $\frac{\theta}{n}$ with n large enough to make it infinitesimal. Then, one can write

$$\Big[R(\theta)\Big] \approx \Big[R\Big(\tfrac{\theta}{n}\Big)\Big]^n \approx \Big(\mathbb{1}+i\tfrac{\theta}{n}T\Big)^n \quad \stackrel{n\to\infty}{\longrightarrow} \quad e^{i\theta T} = \mathbb{1}\cos(\theta)+iT\sin(\theta) \tag{89}$$

which reproduces the finite transformation in (86). The notation $e^{i\theta T}$, which contains the infinitesimal generator T and the continuous Lie parameter θ of the group, is the *exponential representation* of the elements of the group $SO(2)$. It generalizes to arbitrary Lie groups.

Here, we have obtained the Lie algebra of the group $SO(2)$ by considering the defining representation of $SO(2)$, which is enough to recognize the abstract $SO(2)$ Lie algebra. Then, one can study the various representations of the $SO(2)$ Lie algebra in terms of other matrices and classify inequivalent representations.

Note that by defining the complex number $z=x+iy$, the $SO(2)$ transformation of (x,y) takes the form of $U(1)$ a phase transformation

$$\begin{aligned} z' = x'+iy' &= (x\cos(\theta)+y\sin(\theta)) + i(-x\sin(\theta)+y\cos(\theta)) \\ &= (\cos(\theta)-i\sin(\theta))(x+iy) = e^{-i\theta}z\ . \end{aligned} \tag{90}$$

The groups $SO(2)$ and $U(1)$ are equivalent, $SO(2)\sim U(1)$.

5.2 SO(3)

Consider now the group of rotations in three-dimensional space, the group $SO(3)$ of real orthogonal 3×3 matrices with determinant equal to 1. These matrices generate transformations of a three-dimensional vector $\vec{x}\to\vec{x}\,'=R\,\vec{x}$.

Consider the rotations around the three Cartesian axes with coordinates $(x,y,z)=(x^1,x^2,x^3)$

$$R_x(\theta_x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & \sin(\theta_x) \\ 0 & -\sin(\theta_x) & \cos(\theta_x) \end{pmatrix} \quad \stackrel{\theta_x\ll 1}{\longrightarrow} \quad \mathbb{1}+\theta_x \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{iT^1} + \cdots \tag{91}$$

$$R_y(\theta_y) = \begin{pmatrix} \cos(\theta_y) & 0 & -\sin(\theta_y) \\ 0 & 1 & 0 \\ \sin(\theta_y) & 0 & \cos(\theta_y) \end{pmatrix} \quad \stackrel{\theta_y\ll 1}{\longrightarrow} \quad \mathbb{1}+\theta_y \underbrace{\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}}_{iT^2} + \cdots \tag{92}$$

$$R_z(\theta_z) = \begin{pmatrix} \cos(\theta_z) & \sin(\theta_z) & 0 \\ -\sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \stackrel{\theta_z\ll 1}{\longrightarrow} \quad \mathbb{1}+\theta_z \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{iT^3} + \cdots \tag{93}$$

so that the generators T^i of the infinitesimal transformations are given by

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ . \tag{94}$$

The corresponding Lie algebra is easily computed by calculating the commutators of the matrices just identified

$$[T^i, T^j] = i\epsilon^{ijk}T^k \ . \tag{95}$$

The right-hand side is not zero, indicating that the group is non-abelian (the group elements do not commute). The constants ϵ^{ijk} are called the structure constants of the $SO(3)$ group because they encode the non-abelian structure of the group. A finite element of the group can be parameterized in exponential form as

$$R(\vec{\theta}) = e^{i\vec{\theta}\cdot\vec{T}} = e^{i\theta_i T^i} \tag{96}$$

where θ_i are the independent parameters of the group (a rotation of angle $\theta = \sqrt{\vec{\theta}\cdot\vec{\theta}}$ around the axis of the unit vector $\hat{\theta} = \frac{\vec{\theta}}{\theta}$).

To understand the role of the Lie algebra, let us study the product

$$R(\alpha)R(\beta)R^{-1}(\alpha)R(\beta) \tag{97}$$

that would be the identity for an abelian group. For infinitesimal parameters and working at the linear order in both α and β, one finds

$$R(\alpha)R(\beta)R^{-1}(\alpha)R(\beta) = \mathbb{1} + \alpha_i\beta_j[T^i, T^j] + \cdots \tag{98}$$

which is nonvanishing for the non-abelian group $SO(3)$: the Lie algebra captures the non-commutative structure of the Lie group. In addition, one understands that the result must correspond to an infinitesimal group transformation, just like the left-hand side, so that the commutator $[T^i, T^j]$ must be proportional to a generator, as indeed verified in (95).

We have obtained the Lie algebra using the defining representation, and now we can consider it as the abstract Lie algebra of the group $SO(3)$ and study its different irreducible representations, as done for the representations of the group. From the representations of the group studied previously, one obtains the corresponding representations of the associated Lie algebra. Conversely, exponentiating the matrices of a representation of the Lie algebra yields finite transformations that provide a representation of the group[5].

Let us comment on the $SO(3)$ Lie algebra and relate it to known topics studied in quantum mechanics. In equation (95), we recognize the algebra of the quantum angular momentum operator. Renaming $T^i \to L^i$, we recognize the familiar algebra of the angular momentum (in units of $\hbar = 1$)

$$[L^i, L^j] = i\epsilon^{ijk}L^k \ . \tag{99}$$

The study of its irreducible unitary representations is solved explicitly using the methods of quantum mechanics: the known result is that these irreducible representations are those given by the spherical harmonics $|l, m\rangle \sim Y_{lm}$, which for fixed l form a basis of the spin l representation. It is $(2l+1)$-dimensional, as for fixed l the possible values of m are $2l+1$:

$$Y'_{lm} = [R_{(l)}(\theta)]_{lm}{}^{ln}\, Y_{ln} \ , \qquad l \text{ fixed}, \quad m, n \in [-l, -l+1, ..., 0, ..., l-1, l] \ . \tag{100}$$

In the case of spinorial representations (i.e., with half-integer spin, i.e. with $l \to j$ and j half-integer), a rotation by 2π (which for $SO(3)$ coincides with the identity) is represented by the

[5] Except for possible topological obstructions that might prevent the representation from being truly single-valued. This situation is exemplified by the spinor representations of $SO(3)$, which, as we will see later, are true (i.e., single-valued) representations of the $SU(2)$ group only.

matrix $-\mathbb{1}$, and thus we speak of a 2-valued representation (one needs to rotate by another 2π to get back to the identity). As we will see, these spinorial representations are true representations of the $SU(2)$ group, which has the same Lie algebra as $SO(3)$ and therefore has the same local structure but different global properties.

To appreciate future developments (such as the Lie algebras of $SO(N)$ and $SO(N,M)$), let's rewrite the matrices identifying the generators in the vector representation (94) and the corresponding Lie algebra in (95) in an alternative way. We can rename the generator T^1 as T^{23}, as it generates a rotation in the 2-3 plane, and so on: $T^2 \equiv T^{31}$, $T^3 \equiv T^{12}$. The matrix elements in (94) can be written as

$$(T^1)^i{}_j \equiv (T^{23})^i{}_j = -i(\delta^{2i}\delta^3{}_j - \delta^{3i}\delta^2{}_j) \tag{101}$$

and similarly for T^{31} and T^{12}. Thus, the general expression obtained is

$$(T^{kl})^i{}_j = -i(\delta^{ki}\delta^l{}_j - \delta^{li}\delta^k{}_j) \tag{102}$$

which can be used to recalculate the Lie algebra of $SO(3)$. Rewritten on this basis, the Lie algebra (95) becomes

$$[T^{kl}, T^{mn}] = -i\delta^{lm}T^{kn} + i\delta^{km}T^{ln} + i\delta^{ln}T^{km} - i\delta^{kn}T^{lm} \ . \tag{103}$$

Note the presence of the Euclidean (inverse) metric δ^{ij} in this relation. Written in this form, the Lie algebra is valid for the generic group $SO(N)$, provided that the indices range from 1 to N. Moreover, by replacing the metric δ_{ij} with a Minkowski metric η_{ij}, appropriate for a spacetime with N spatial and M temporal dimensions, one obtains the Lie algebra of $SO(N,M)$.

5.3 U(1)

Consider the group $U(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi]\}$, the group of phases defined via its defining representation. For infinitesimal transformations

$$e^{i\theta} = 1 + i\theta + \cdots \tag{104}$$

the infinitesimal generator is given by $T = 1$ (which we can think of as a 1×1 matrix), which produces the Abelian Lie algebra of the $U(1)$ group given by the commutator

$$[T, T] = 0 \ . \tag{105}$$

In the charge q representation, where the element $e^{i\theta}$ is represented by $e^{iq\theta}$, the infinitesimal generator is represented by $T = q$ and satisfies the same Lie algebra (105). Therefore, we can think of the Lie algebra $[T, T] = 0$ as the abstract Lie algebra corresponding to the $U(1)$ group, which is represented by different matrices in different representations. Since the irreducible representations of the $U(1)$ group are all one-dimensional, all these matrices are 1×1 matrices and thus are simply numbers. In the charge q representation, the generator of $U(1)$ is represented by $T = q$. It is also common to use the notation Q (which often denotes a charge) instead of T for the generator of the $U(1)$ group. The groups $U(1)$ and $SO(2)$ identify the same Abelian Lie group, as already described.

5.4 SU(2)

Let's now analyze the group $SU(2)$, the group of 2×2 unitary matrices with unit determinant:

$$SU(2) = \{U \text{ complex matrices } 2\times 2 \mid U^\dagger = U^{-1},\ \det U = 1\}\ . \tag{106}$$

We can write the matrices that differ infinitesimally from the identity matrix as

$$U = \mathbb{1} + iT \qquad\qquad T^i{}_j \ll 1\ . \tag{107}$$

Now, the requirement that $U^\dagger = \mathbb{1} - iT^\dagger$ coincides with $U^{-1} = \mathbb{1} - iT$ implies that the matrices T must be Hermitian:

$$T^\dagger = T \tag{108}$$

while the requirement for unit determinant, $\det U = 1 + i\,\mathrm{tr}\,T = 1$, implies that these matrices must be traceless:

$$\mathrm{tr}\,T = 0\ . \tag{109}$$

A basis of Hermitian traceless 2×2 matrices is given by the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{110}$$

so we can express an arbitrary matrix T as a linear combination of the σ^a:

$$T = \theta_a \frac{\sigma^a}{2} \equiv \theta_a T^a \qquad\qquad a = 1,2,3\ . \tag{111}$$

The normalization has been chosen to satisfy

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}\ . \tag{112}$$

With this normalization, the infinitesimal generators $T^a = \frac{\sigma^a}{2}$ give rise to the following $SU(2)$ Lie algebra

$$[T^a, T^b] = i\epsilon^{abc}T^c \tag{113}$$

which is recognized to coincide with the Lie algebra of $SO(3)$. This shows that locally they are similar (they have the same structure constants), although globally there are differences: using the language of differential geometry, we can say that the group $SU(2)$ is a double cover of the group $SO(3)$. This difference is seen explicitly in the defining representation of $SU(2)$ (the spin-$\frac{1}{2}$ or 2 representation). A finite rotation is obtained by exponentiating infinitesimal transformations to make them finite:

$$U(\theta_a) = \exp(i\theta_a \frac{\sigma^a}{2})\ . \tag{114}$$

In particular, a finite rotation around the z-axis is obtained by choosing $\theta_3 = \theta$ and $\theta_1 = \theta_2 = 0$, to find a matrix $U_3(\theta)$ given by

$$\begin{aligned} U_3(\theta) = e^{i\theta\frac{\sigma^3}{2}} &= \mathbb{1} + i\theta\frac{\sigma^3}{2} + \frac{1}{2!}\Big(i\theta\frac{\sigma^3}{2}\Big)^2 + \frac{1}{3!}\Big(i\theta\frac{\sigma^3}{2}\Big)^3 + \frac{1}{4!}\Big(i\theta\frac{\sigma^3}{2}\Big)^4 + \cdots \\ &= \mathbb{1} + i\Big(\frac{\theta}{2}\Big)\sigma^3 - \frac{1}{2!}\Big(\frac{\theta}{2}\Big)^2\mathbb{1} - i\frac{1}{3!}\Big(\frac{\theta}{2}\Big)^3\sigma^3 + \frac{1}{4!}\Big(\frac{\theta}{2}\Big)^4\mathbb{1} \\ &= \mathbb{1}\Big(1 - \frac{1}{2!}\Big(\frac{\theta}{2}\Big)^2 + \frac{1}{4!}\Big(\frac{\theta}{2}\Big)^4 + \cdots\Big) + i\sigma^3\Big(\frac{\theta}{2} - \frac{1}{3!}\Big(\frac{\theta}{2}\Big)^3 + \cdots\Big) \\ &= \mathbb{1}\cos\Big(\frac{\theta}{2}\Big) + i\sigma^3\sin\Big(\frac{\theta}{2}\Big) = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}\ . \end{aligned} \tag{115}$$

Setting $\theta = 2\pi$ gives the transformation

$$U_3(\theta = 2\pi) = -\mathbb{1} \tag{116}$$

which does not coincide with the identity in $SU(2)$. The identity transformation is obtained only for $\theta = 4\pi$. As known from quantum mechanics, all irreducible unitary representations of $SU(2)$ are characterized by a quantum number j that can be either an integer or a half-integer. They are of dimension $2j+1$.

Historical Note: Pauli introduced the matrices in (110) to describe the electron's spin, defining the spin operator $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, which acts on a two-component wave function (spinor).

5.5 SU(3)

The same analysis performed to extract the infinitesimal generators of $SU(2)$ applies also to the general $SU(N)$ group, whose generators are then seen to be traceless, hermitian, $N\times N$ matrices. There are N^2-1 of such matrices, so that there are N^2-1 independent Lie parameters for the group $SU(N)$. In particular, the eight infinitesimal generators of $SU(3)$ in the fundamental representation are given by the Gell-Mann matrices λ^a, which form a basis of hermitian 3×3 traceless matrices (generalizing the Pauli matrices σ^i for $SU(2)$):

$$T^a = \frac{\lambda^a}{2} \qquad a = 1, \ldots, 8 \tag{117}$$

where

$$\begin{aligned}
&\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
&\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\
&\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{118}$$

These matrices are normalized so that

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{119}$$

just as was done for $SU(2)$, see eq. (112). An arbitrary element of the $SU(3)$ group in the fundamental representation is thus described by 3×3 matrices of the form $U(\theta) = \exp(i\theta_a \frac{\lambda^a}{2})$, where θ_a with $a = 1, 2, \ldots, 8$ are the eight parameters of the group. By calculating the Lie algebra, one finds the structure constants f^{abc} that correspond the $SU(3)$ group

$$[T^a, T^b] = i f^{abc} T^c \tag{120}$$

They are totally antisymmetric and given by:

$$\begin{aligned}
&f^{123} = 1 \\
&f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2} \\
&f^{458} = f^{678} = \frac{\sqrt{3}}{2}
\end{aligned} \tag{121}$$

while all other f^{abc} not related to these by permuting indices are zero. As an exercise, try to verify some of these numbers. This group has important applications in the description of *color* associated with strong interactions and in the quark model that classifies the hadrons composed of the three lightest *flavors* of quarks (up, down, strange).

5.6 General Case

We summarize for arbitrary Lie groups what was illustrated above through examples. A Lie group is, by definition, a group of transformations that depend continuously on some parameters. By studying the infinitesimal transformations of the group, i.e., transformations that differ only slightly from the identity, we recognize the generators, operators that "generate" the infinitesimal transformations. They identify the so-called *Lie algebra* of the group, which summarizes information about the group.

In general, an element $g(\theta)$ of a Lie group G (or, more precisely, of the component connected to the identity) can be parametrized in the following way:

$$g(\theta) = e^{i\theta_a T^a} \in G \qquad a = 1, \dots, \dim G \tag{122}$$

where the parameters θ_a are real numbers that parametrize the various elements of the group. They are chosen so that for $\theta_a = 0$ one gets the identity $g = \mathbb{1}$. The operators T^a are the generators of the group. Considering the group as a group of $N \times N$ matrices for some N (for example, the defining representation), the generators are also $N \times N$ matrices. They generate infinitesimal transformations when $\theta_a \ll 1$. Simply expand the exponential function in a Taylor series and keep the lowest order terms:

$$g(\theta) = \mathbb{1} + i\theta_a T^a + \cdots . \tag{123}$$

By studying the relations that capture the composition properties of the group using infinitesimal transformations (which are generically non-commutative), one obtains the Lie algebra of the group G:

$$[T^a, T^b] = i f^{ab}{}_c T^c . \tag{124}$$

The constants $f^{ab}{}_c$ are called *structure constants* of the group and characterize it. Groups with the same Lie algebra may only differ in their topology but are locally similar. It is useful to mention the Jacobi identities:

$$f^{ab}{}_d f^{dc}{}_e + f^{bc}{}_d f^{da}{}_e + f^{ca}{}_d f^{db}{}_e = 0 \tag{125}$$

which are quadratic relations satisfied by the structure constants and emerge as a consequence of the operator identities:

$$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0 . \tag{126}$$

The structure constants can be used to define the adjoint representation $T^a_{(A)}$ of the Lie algebra, given by the formula:

$$(T^a_{(A)})^b{}_c = -i f^{ab}{}_c . \tag{127}$$

It is verified to be a representation of the Lie algebra thanks to the Jacobi identities. It is a real representation because the structure constants are real numbers, and it a representation of dimensions equal to the dimensions of the group, since the indices $a, b, c = 1, 2, \dots, \dim G$.

Finally, it is useful to mention the Baker-Campbell-Hausdorff formula for the product of exponentials of two linear operators A and B:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]+\cdots} \tag{128}$$

where the dots indicate higher-order terms, always expressible in terms of commutators. This formula shows that the knowledge of the Lie algebra is sufficient to reconstruct the (generally non-commutative) product of the elements of the corresponding Lie group.

To summarize, let us list and review some of the main definitions and properties of Lie algebras:

(i) $g = \exp(i\theta_a T^a) \in G \qquad a = 1, \ldots, \dim G$

(ii) $[T^a, T^b] = i f^{ab}{}_c T^c$

(iii) $\mathrm{tr}(T^a T^b) = \frac{1}{2}\gamma^{ab}$ (generators in the fundamental representation)

(iv) $[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0 \quad \Rightarrow \quad f^{ab}{}_d f^{dc}{}_e + f^{bc}{}_d f^{da}{}_e + f^{ca}{}_d f^{db}{}_e = 0$

(v) $f^{abc} = f^{ab}{}_d \gamma^{dc}$ (completely antisymmetric tensor)

Point (i) describes the exponential parametrization of an arbitrary element of the group that is connected to the identity. The index a takes as many values as the dimensions of the group. An element of the group is parametrized by the parameters θ_a with $a = 1, \ldots, \dim G$.
Point (ii) corresponds to the Lie algebra satisfied by the infinitesimal generators T^a. The constants $f^{ab}{}_c$ are called structure constants and characterize the group G. They are antisymmetric on indices a and b.
Point (iii) identifies (the inverse of) a metric γ^{ab} called the "Killing metric". This metric is positive-definite only for compact and simple Lie groups, such as $SU(N)$ or $SO(N)$. Being positive, it is often normalized to the Kronecker delta: $\gamma^{ab} = \delta^{ab}$.
Point (iv) amounts to the so-called "Jacobi identities" satisfied by the structure constants. They can be used to construct the adjoint representation of the Lie algebra. Denoting by $(T^a_{(A)})^b{}_c$ the matrix elements of the generators of the adjoint representation $T^a_{(A)}$, we have $(T^a_{(A)})^b{}_c = -i f^{ab}{}_c$. The Jacobi identities imply that this is a representation. It is real and of dimensions equal to the dimensions of the group since the indices $a, b, c = 1, 2, \ldots, \dim G$. By exponentiation, it gives rise to a representation of the group.
In point (v), the Killing metric is used to raise an index of the structure constants. Then, f^{abc} are completely antisymmetric in all indices: antisymmetry in the indices a and b is obvious from (ii), while antisymmetry in the indices b and c is deduced by taking the trace of the Jacobi identities in (iv) and using (ii) and (iii).

Finally, we conclude with the statement of a theorem which we shall not prove: *The unitary irreducible representations of compact groups are finite-dimensional, while the unitary representations of non-compact groups must be infinite-dimensional.*

Thus, compact groups such as $SO(N)$ and $SU(N)$ have unitary finite-dimensional irreps. Non-compact groups, such as the Lorentz group $SO(3,1)$ and the Poincaré group $ISO(3,1)$, have unitary representations that must be infinite-dimensional. For applications in relativistic field theory, it is useful to have some knowledge of:

(i) the finite-dimensional representations of the Lorentz group. They are not unitary and are used to label the quantum fields that define a given relativistic QFT,

(ii) the unitary representations of the Poincaré group, which are infinite-dimensional and are realized in the Hilbert space of quantum field theories via unitary operators.

These points are very briefly commented upon at the very end of the next section.

6 Special relativity, the Lorentz group, and representations

Let us review the main points of special relativity, keeping in mind group theory applied to the Lorentz group $SO(3,1)$.

The standard Lorentz transformation that relates the spacetime coordinates of two inertial frames in relative motion with constant velocity v along the x axis are given by

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{129}$$

where $\beta \equiv \frac{v}{c}$ and $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. Taking the relative velocity v to be positive, we see that $0 \le \beta < 1$ and $1 \le \gamma < \infty$. Denoting by x the column 4-vector with components x^μ

$$x = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \to \quad x^\mu \ , \tag{130}$$

we write more compactly the Lorentz transformation in the equivalent forms as

$$x' = \Lambda x \ , \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu \ . \tag{131}$$

This transformation is seen to leave invariant the light cone at the origin. More generally, it leaves invariant the modulus square of the 4-vector x^μ, which is defined in the following way

$$\begin{aligned} s^2 = -c^2t^2 + x^2 + y^2 + z^2 &= \begin{pmatrix} ct, & x, & y, & z \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \\ &= x^T \eta\, x = \eta_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu \end{aligned} \tag{132}$$

where η is the *Minkowski metric.* It is also used to lower indices on vectors and tensors.

The general Lorentz group is defined as the group of linear transformations that leave invariant the scalar s^2

$$s^2 = s'^2 \qquad \Rightarrow \qquad x^T \eta\, x = x'^T \eta\, x' = x^T \Lambda^T \eta\, \Lambda\, x \qquad \Rightarrow \qquad \Lambda^T \eta\, \Lambda = \eta$$

or, equivalently, using components

$$\eta_{\mu\nu}x^\mu x^\nu = \eta_{\mu\nu}x'^\mu x'^\nu = \eta_{\mu\nu}(\Lambda^\mu{}_\alpha x^\alpha)(\Lambda^\nu{}_\beta x^\beta) = \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta x^\alpha x^\beta = \eta_{\alpha\beta}x^\alpha x^\beta \quad \Rightarrow \quad \eta_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta = \eta_{\alpha\beta} \ .$$

This invariance allows us to define the group of Lorentz transformations as

$$O(3,1) = \{\text{real } 4\times 4 \text{ matrices } \Lambda \mid \Lambda^T\eta\Lambda = \eta\} \ . \tag{133}$$

This group contains the space-inversion (the parity transformation P) as well as time-inversion (the time reversal T), which can be eliminated from the group by defining the proper orthochronous Lorentz group

$$SO^+(3,1) = \{\text{real } 4\times 4 \text{ matrices } \Lambda \mid \Lambda^T\eta\Lambda = \eta, \det\Lambda = 1, \Lambda^0{}_0 \geq 1\} \tag{134}$$

also called the restricted Lorentz group. By relativistic invariance, one generically refers to an invariance under the latter as parity and time reversal are usually treated separately.

Tensors are defined as usual for the Lorentz group. They are used to describe physical quantities and their transformation properties under changes of inertial frames. An example is the 4-momentum p^μ

$$p^\mu = (p^0, \vec{p}) = \left(\frac{E}{c}, \vec{p}\right) = \left(\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{135}$$

that transforms as a 4-vector and whose modulus square satisfies

$$p^\mu p_\mu = -m^2c^2 \ . \tag{136}$$

This last relation states that

$$E^2 = \vec{p}^{\,2}c^2 + m^2c^4 \ . \tag{137}$$

Similarly, the electric and magnetic fields $\vec{E}$ and $\vec{B}$ are recognized to be the components of an antisymmetric tensor field $F^{\mu\nu}$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \ . \tag{138}$$

here written using Gaussian or Heaviside-Lorentz units. Under a Lorentz transformation, the electromagnetic tensor transforms according to the tensor laws as

$$F'^{\mu\nu} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta F^{\alpha\beta} \ . \tag{139}$$

From it, one can construct the scalar

$$F^{\mu\nu}F_{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2) \tag{140}$$

which is proportional to the free lagrangian of the electromagnetic field.

The space-time derivatives naturally form a vector with a lower index

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \tag{141}$$

so that $\partial_\mu x^\mu = 4$ is a scalar ($\partial'_\mu x'^\mu = \partial_\mu x^\mu = 4$).

Then, the inhomogeneous Maxwell's equations are written in a covariant form as

$$\partial_\mu F^{\mu\nu} = -\frac{1}{c} J^\nu \tag{142}$$

where $J^\mu = (J^0, \vec{J}) = (c\rho, \vec{J})$ is the 4-vector charge-current density. The homogenous Maxwell's equations take instead the following covariant form

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \; . \tag{143}$$

6.1 Finite Dimensional Representations of the Lorentz Group

First, it is useful to derive the Lie algebra of the Lorentz group. For infinitesimal transformations, we can write

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu \; , \qquad |\omega^\mu{}_\nu| \ll 1 \tag{144}$$

and imposing the condition that defines Lorentz transformations ($\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu$), we obtain that $\omega^\mu{}_\nu$ must satisfy the antisymmetric condition

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \tag{145}$$

(indices are lowered as usual, $\omega_{\mu\nu} = \eta_{\mu\lambda}\omega^\lambda{}_\nu$). Thus, they contain six independent parameters identified with the $\omega_{\mu\nu}$ with fixed indices $\mu < \nu$.

Then, in matrix notation, we can re-write an arbitrary infinitesimal Lorentz transformation by making explicit the infinitesimal parameters that multiply the corresponding generators

$$\Lambda = \mathbb{1} + \frac{i}{2}\omega_{\alpha\beta} M^{\alpha\beta} \; . \tag{146}$$

The six matrices $M^{\alpha\beta}$ with $\alpha < \beta$ are the independent generators of the Lorentz group. In the defining representation (the "four-vector" representation), they are given by

$$(M^{\alpha\beta})^\mu{}_\nu = -i(\eta^{\alpha\mu}\delta^\beta_\nu - \eta^{\beta\mu}\delta^\alpha_\nu) \tag{147}$$

so that eq. (146) reproduces eq. (144). For example, some of these generators can be written explicitly as

$$M^{12} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \; , \qquad M^{01} = \left(\begin{array}{c|ccc} 0 & i & 0 & 0 \\ \hline i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{148}$$

where M^{12} generates infinitesimal rotations about the z-axis, while M^{01} generates a boost along the x-axis.

Although it might seem tedious, it is straightforward to calculate the Lie algebra

$$[M^{\mu\nu}, M^{\alpha\beta}] = -i\eta^{\nu\alpha}M^{\mu\beta} + i\eta^{\mu\alpha}M^{\nu\beta} + i\eta^{\nu\beta}M^{\mu\alpha} - i\eta^{\mu\beta}M^{\nu\alpha} \; . \tag{149}$$

This is also valid for any generic group $SO(N,M)$ if one identifies $\eta_{\mu\nu}$ with the corresponding metric: in particular, to obtain $SO(3)$ one sets $\eta_{\mu\nu} \to \delta_{ij}$ and defining $J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}$ one recovers the form of the $SO(3)$ Lie algebra given in eq. (95) with $T^i \equiv J^i$.

Returning to the specific case of $SO(3,1)$, one can rewrite the algebra in a more useful form that allows us to deduce immediately its finite-dimensional representations. Separating the indices into time and space parts $\mu = (0, i)$, and defining the following basis for the generators of the Lorentz group

$$J^i = \frac{1}{2}\epsilon^{ijk}M^{jk} \ , \qquad K^i = M^{i0} \tag{150}$$

the Lie algebra (149) can be rewritten as

$$[J^i, J^j] = i\epsilon^{ijk}J^k \ , \qquad [J^i, K^j] = i\epsilon^{ijk}K^k \ , \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k \tag{151}$$

where the generators J^i generate the spatial rotation subgroup $SO(3)$. Finally, defining the complex linear combinations

$$N^i = \frac{1}{2}(J^i - iK^i) \ , \qquad \bar{N}^i = \frac{1}{2}(J^i + iK^i) \tag{152}$$

the algebra can be rewritten as

$$[N^i, N^j] = i\epsilon^{ijk}N^k \ , \qquad [\bar{N}^i, \bar{N}^j] = i\epsilon^{ijk}\bar{N}^k \ , \qquad [N^i, \bar{N}^j] = 0 \tag{153}$$

which shows that the algebra of $SO(3,1)$ is equivalent to that of $SU(2) \times SU(2)$, up to different hermiticity relations (araising because $SO(3,1)$ is not compact, while $SU(2)$ is). Since $SO(3,1)$ reduces to two independent copies of $SU(2)$, the well-known finite-dimensional representations of the latter can be used to find the finite-dimensional representations of $SO(3,1)$: they are classified by two integer or half-integer numbers (j_1, j_2) corresponding to the representations of the two subgroups $SU(2)$ generated by N^i and $\bar{N}^i$. Furthermore, recalling (152), the spin operator corresponds to $J^i = N^i + \bar{N}^i$, so that the highest spin content of the representation is given by $j = j_1 + j_2$. These representations are finite-dimensional but are not unitary due to the necessity of taking complex combinations of the generators in (152).

In quantum field theory, fields with these Lorentz representations are used to describe particles with fixed spin, for example

$$\begin{aligned}
&(0,0) \quad \longrightarrow \quad \text{scalar } \phi \\
&(\tfrac{1}{2},0) \quad \longrightarrow \quad \text{left-handed Weyl fermion } \psi_{L} \sim \xi_a \\
&(0,\tfrac{1}{2}) \quad \longrightarrow \quad \text{right-handed Weyl fermion } \psi_{R} \sim \eta_{\dot{a}} \\
&(\tfrac{1}{2},0) \oplus (0,\tfrac{1}{2}) \quad \longrightarrow \quad \text{Dirac fermion } \psi \sim \psi_\alpha \\
&(\tfrac{1}{2},\tfrac{1}{2}) \quad \longrightarrow \quad \text{spin-1 field } A_\mu \ .
\end{aligned} \tag{154}$$

Just as $SO(3) \to SU(2)$ allows to view the spinorial representations of $SO(3)$ as single-valued representations of $SU(2)$, a similar phenomenon happens for $SO(3,1) \to SL(2,C)$: the Lie algebras of $SO(3,1)$ and $SL(2,C)$ coincide and the latter group is the covering group of the former.

6.2 Unitary Representations of the Poincaré Group

The Poincaré group extends the Lorentz group with spacetime translations. It transforms the position four-vector as follows

$$x^\mu \ \to \ x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{155}$$

where $\Lambda^\mu{}_\nu$ describes a Lorentz transformation and a^μ a spacetime translation. This group is sometimes referred to as the $ISO(3,1)$ group, the inhomogeneous special orthogonal group, where the inhomogeneity refers to the translations.

The Lie algebra of the Poincaré group can be written as

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [M^{\mu\nu}, P^\lambda] &= -i\eta^{\nu\lambda}P^\mu + i\eta^{\mu\lambda}P^\nu \\ [M^{\mu\nu}, M^{\alpha\beta}] &= -i\eta^{\nu\alpha}M^{\mu\beta} + i\eta^{\mu\alpha}M^{\nu\beta} + i\eta^{\nu\beta}M^{\mu\alpha} - i\eta^{\mu\beta}M^{\nu\alpha} \end{aligned} \tag{156}$$

where P^μ are the generators of the translations and $M^{\mu\nu}$ are the generators of the Lorentz transformations[6]. Its unitary irreducible representations are infinite-dimensional and have been classified by Wigner in 1939. They are classified according to the values of the so-called Casimir operators $P^2 \equiv P_\mu P^\mu$ and $W^2 \equiv W_\mu W^\mu$, where $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}P^\nu M^{\alpha\beta}$ is the so-called Pauli-Lubanski vector. It is seen, using equations (156), that P^2 and W^2 commute with all elements of the Poincaré algebra: they are invariant under infinitesimal transformations of the Poincaré group. Thus, they take constant values inside an irreducible representation, just like $\vec{J}^2$ takes a constant value inside a fixed representation of the rotation group with generators J^i. The unitary representations of the Poincaré group are classified by the following values of the Casimir operators:

- $P^2 = -m^2 < 0$, $W^2 = m^2 s(s+1)$ with $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2...$: it corresponds to quantum particles of mass m and spin s. This unitary representation is associated with a Hilbert space that contains the allowed states of a relativistic particle with mass m and spin s.
- $P^2 = 0$, $W^2 = 0$ and with $W_\mu = \pm s P_\mu$ where $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2...$: massless particles with helicity s.
- $P^2 = 0$, $W^2 = k^2 > 0$: massless "particles" with infinitely many states of "polarization" that can vary continuously: they do not seem to have any immediate application to field theory (at least at the perturbative level).
- $P^2 = -m^2 > 0$: tachyonic representations, never used in physics (inconsistent with standard physical interpretations).
- $P_\mu = 0,\ W_\mu = 0$: trivial (scalar) representation $\to$ vacuum (no particles).

For example, the physical case of mass m and spin s (i.e. the case with $P^2 = -m^2 > 0$ and $W^2 = m^2 s(s+1)$) corresponds to an infinite-dimensional vector space that is constructed as the Hilbert space spanned by vectors of the form

$$|\vec{p}, s_3\rangle\ , \qquad \vec{p} \in \mathbb{R}^3, \quad s_3 = -s, ..., +s \tag{157}$$

which are the eigenstates of the linear momentum operator $\hat{\vec{p}}$ and of the component of the spin operator along the z-axis $\hat{S}_3$. Unitary operators representing the Poincaré group transformations act on this infinite-dimensional Hilbert space.

[6] The Lorentz part of this algebra was found previuosly using the defining representation of the Lorentz group. The Poincaré group, as given above, is not defined in terms of matrices only. A way of finding its Lie algebra is to consider its generators, that perform the infinitesimal transformation, in a quantum mechanical Hilbert space where $P^\mu = \hat{p}^\mu$ and $M^{\mu\nu} = \hat{x}^\mu\hat{p}^\nu - \hat{x}^\nu\hat{p}^\mu$, with the elementary commutators given by $[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu$. This allows to deduce the Lie algebra of the Poincaré group given above.

Exercises

Please test your understanding of the material presented in the lecture notes by solving the following exercises:

1. *Prove that the matrices of the groups* $GL(N,\mathbb{R})$, $SL(N,\mathbb{R})$, $O(N)$, *and* $SO(N)$ *satisfy the group axioms.*

2. *Calculate the electric charges of the elementary particles listed in table* (84) *using the quantum numbers provided in the last row of the same table.*

3. *Show that the generators* $(T^a)^\alpha{}_\beta$ *in the fundamental representation of* $SU(N)$ *define an invariant tensor when all indices are transformed accordingly. You only need to consider infinitesimal transformations.*

4. *Show that for a simple Lie group, the expression* $C_2 = T^a T^b \gamma_{ab}$ *is a Casimir operator, i.e. it commutes with all generators* T^a. *Here,* γ_{ab} *denotes the Killing metric whose inverse* γ^{ab} *is defined by* $\mathrm{tr}(T^a T^b) = \frac{1}{2}\gamma^{ab}$.

5. *Derive the transformation rules for the electric and magnetic fields,* $\vec{E}$ *and* $\vec{B}$, *under the Lorentz transformation in eq.* (129), *using the tensor law from eq.* (139).

6. *Verify that the Lorentz matrices in eq.* (146) *reproduce the result of eq.* (144) *after applying eq.* (147).

7. *Determine the Lie algebra of the Poincaré group by following the hints provided in footnote 6.*