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General relativity

(notes for "Relativity" a.a. 2024/25)

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1 Foreword

General Relativity is a vast subject with many textbooks available. In this class, I will use two main textbooks to guide you through the basics. The first one is *S. Weinberg "Gravitation and Cosmology", John Wiley & Sons 1972*, which covers tensor analysis and Einstein's equation (chapters 3-7). The second one is *H. Ohanian and R. Ruffini "Gravitation and Spacetime", CUP 2013*, which offers more insights on classical tests, Schwarzschild black hole solution, and gravitational waves, and, of course, other topics.

These notes are supplementary and incomplete. They only cover some selected topics. Therefore, you should rely on the textbooks as your primary source of study.

2 The principle of equivalence of gravitation and inertia

You can find more details about this topic in Chapter 3 of [1], which is highly recommended.

Newton's law of universal gravitation tells us how massive bodies attract each other with gravitational forces. Suppose we have N particles with inertial masses $m_k^{(I)}$, gravitational masses $m_k^{(G)}$, and positions $\vec{x}_k$, with $k = 1, .., N$. Then, the gravitational force on the k-th particle is given by

$$m_k^{(I)} \frac{d^2\vec{x}_k}{dt^2} = G \sum_{l \neq k} m_k^{(G)} m_l^{(G)} \frac{\vec{x}_l - \vec{x}_k}{|\vec{x}_l - \vec{x}_k|^3} \ . \tag{1}$$

In addition, it is found that the inertial mass and the gravitational mass of a particle are equal, as confirmed by experiments. This means that we can use the same mass for both the acceleration and the attraction of a particle, i.e. $m_k^{(I)} = m_k^{(G)}$ for any k, which then simplifies from eq. (1).

The *principle of equivalence of gravitation and inertia* is based on this equality of masses and states that: *In any gravitational field, one can always find a local inertial frame (a free-falling frame) at any point in spacetime, such that near that point the laws of nature look like the ones in special relativity, where no gravitational field is present.*

This principle helps us to describe how gravity works and find the equations of motion that govern it (Einstein's equations). To apply this principle, we need to use tensor calculus, which gives us the tools to study arbitrary change of coordinates in spacetime. This is a branch of mathematics called differential geometry. Einstein illustrated the equivalence principle with the example of an elevator that is falling freely under gravity.

The force of gravity on a point particle

Let us use the above principle to find out how one can describe the force of gravity that acts on a point particle of mass m. In the reference frame with coordinates x^μ, one observes a particle that feels a gravitational force. We have to discover how to describe mathematically such a force. Thus, we use the *principle of equivalence*, which assures us that there must exist an inertial frame with coordinates ξ^α (a frame in free fall), such that locally (i.e. in a small neighborhood of the point where the particle is located and for a small amount of time around the time of observation) the particle satisfies the equations of motion of a free particle as known from the theory of special relativity

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{2}$$

where the proper time τ is computed using the Minkowski metric $\eta_{\alpha\beta}$

$$d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \ . \tag{3}$$

In this particular frame, no force of gravity is experienced locally, i.e. near the position of the pointlike particle. Then, we can go back to the original frame with coordinates x^μ and recognize how the gravitational force is described there. We use the relations between the coordinate systems, i.e. $\xi^\alpha = \xi^\alpha(x)$ and the inverse $x^\mu = x^\mu(\xi)$, to compute by the chain rule

$$0 = \frac{d^2\xi^\alpha}{d\tau^2} = \frac{d}{d\tau}\Big(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\Big) = \frac{\partial^2\xi^\alpha}{\partial x^\nu \partial x^\mu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} \ . \tag{4}$$

This equation is written more simply by multiplying with $\frac{\partial x^\lambda}{\partial \xi^\alpha}$ with a contraction on the index α, and using

$$\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial \xi^\alpha}{\partial x^\mu} = \frac{\partial x^\lambda}{\partial x^\mu} = \delta^\lambda_\mu \ . \tag{5}$$

One finds

$$\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0} \tag{6}$$

where we have defined the *affine connection*

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{7}$$

and renamed indices. The affine connection is symmetric under the exchange of the lower indices

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \tag{8}$$

because derivatives commute.

Now, one must express the proper time (3) in terms of the new coordinates

$$d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = -\eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}dx^\mu \frac{\partial \xi^\beta}{\partial x^\nu}dx^\nu \tag{9}$$

which we rewrite in the form

$$\boxed{d\tau^2 = -g_{\mu\nu}(x) dx^\mu dx^\nu} \tag{10}$$

where the *metric tensor* $g_{\mu\nu}$ is defined by

$$g_{\mu\nu}(x) = \eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu} \ . \tag{11}$$

Eqs. (6) and (11) are the equations that govern the motion of the particle under the force of gravity.

There is a direct relation between the metric tensor (which we interpret as the potential of the gravitational force) and the affine connection (the coefficients that determine the gravitational force on the particle). The relation is found as follows: one differentiates eq. (11) with respects to x^λ

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \eta_{\alpha\beta}\frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu} \;, \tag{12}$$

that using (7) is rewritten as

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \Gamma^\rho_{\lambda\mu} g_{\rho\nu} + \Gamma^\rho_{\lambda\nu} g_{\mu\rho} \;. \tag{13}$$

Then, computing

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu} = 2 g_{\rho\nu}\Gamma^\rho_{\lambda\mu} \tag{14}$$

and using the inverse metric $g^{\sigma\nu}$, that satisfies $g^{\sigma\nu}g_{\nu\rho} = \delta^\sigma_\rho$, one finds

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\sigma\nu}\Big(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu}\Big) \;. \tag{15}$$

It is convenient to use the shorthand notation $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and rename indices to write this formula in the form

$$\boxed{\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})} \;. \tag{16}$$

The Newtonian limit

To relate the above equations to Newton's theory, let us look at the simple case of a slow-moving particle in a weak and stationary gravitational field. Since the particle moves slowly, we can ignore all but the zero-component of the 4-velocity in (6)

$$\frac{dx^\mu}{d\tau} = \Big(\frac{dx^0}{d\tau}, \frac{d\vec{x}}{d\tau}\Big) = (c\gamma, \vec{v}\gamma) = (\gamma, \vec{\beta}\gamma) \tag{17}$$

since $|\vec{\beta}| \ll 1$ for a slow motion (in our units $c = 1$), and find (recalling that we denote $x^0 \equiv t$ in this frame)

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{00}\frac{dt}{d\tau}\frac{dt}{d\tau} = 0 \;. \tag{18}$$

For a stationary field, the time derivatives of the metric vanish, and one is left with

$$\Gamma^\mu_{00} = -\frac{1}{2} g^{\mu\nu}\partial_\nu g_{00} \;. \tag{19}$$

For a weak field, we may use nearly Cartesian coordinates and write the metric $g_{\mu\nu}$ as a weak perturbation of the Minkowski metric $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \;. \tag{20}$$

At lowest order in $h_{\mu\nu}$ we find

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00} \qquad \to \qquad \begin{cases} \Gamma^0_{00} = 0 \\ \Gamma^i_{00} = -\frac{1}{2}\partial_i h_{00} \end{cases} \tag{21}$$

and the equations of motion (18) simplify further to

$$\begin{cases} \dfrac{d^2x^0}{d\tau^2} \equiv \dfrac{d^2t}{d\tau^2} = 0 & (22) \\ \dfrac{d^2\vec{x}}{d\tau^2} = \dfrac{1}{2}\vec{\nabla}h_{00}\left(\dfrac{dt}{d\tau}\right)^2 . & (23) \end{cases}$$

The first equation is solved by $t = \alpha\tau + \alpha_0$ for some constants α and α_0. It tells us that dt is proportional to $d\tau$, i.e. $dt = \alpha d\tau$. Using this proportionality, we find that the second equation may be rewritten as

$$\boxed{\frac{d^2\vec{x}}{dt^2} = \frac{1}{2}\vec{\nabla}h_{00}} \tag{24}$$

that has the Newtonian form

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi \tag{25}$$

with the Newtonian potential ϕ related to the metric deformation by

$$h_{00} = -2\phi \; . \tag{26}$$

For example, consider the case of the Newtonian potential ϕ created by a large mass M placed at the origin

$$\phi = -\frac{GM}{r} \; . \tag{27}$$

Then, the '00' component of the metric takes the form

$$\boxed{g_{00} = \eta_{00} + h_{00} = -1 - 2\phi = -1 + \frac{2GM}{r}} \; . \tag{28}$$

Note that ϕ is adimensional (we use $c = 1$). Some values are as follows: on the surface of the sun $\phi_{sun} \approx 10^{-6}$, on the earth $\phi_{earth} \approx 10^{-9}$, on a white dwarf $\phi_{wd} \approx 10^{-4}$, on the surface of a proton $\phi_{proton} \approx 10^{-39}$. Note also that $|g_{00}| < 1$, which is related to gravitational time dilation to be discussed next.

Gravitational time dilation

A clock in a gravitational field runs slower than a clock in a flat space. To see why, we can use the principle of equivalence and imagine a locally inertial frame where gravity does not affect the clock. In this frame, we can choose coordinates ξ'^α where the clock is stationary and ticks at regular intervals $\Delta\tau$, as set by the maker. This measures the proper time of the clock and we have

$$\Delta\tau \equiv dt' = d\xi'^0 \; . \tag{29}$$

$\Delta\tau$ is the clock's basic unit. It could be, for example, the period of a wave arising from a specific atomic transition of an atom at rest and in the absence of gravity. In another inertial frame with coordinates ξ^α, in which the clock moves and travels an infinitesimal displacement $d\xi^\alpha$, the formula becomes

$$\Delta\tau = \sqrt{(d\xi'^0)^2} = \sqrt{-\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta} \tag{30}$$

as dictated by special relativity.

Now, in the frame with coordinates x^μ, where gravity is present and affects the clock, the space-time interval dx^μ between ticks is fixed by

$$\Delta\tau = \sqrt{-\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta} = \sqrt{-\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu} dx^\mu dx^\nu} = \sqrt{-g_{\mu\nu}(x) dx^\mu dx^\nu} \ . \tag{31}$$

If the clock moves with velocity $\frac{dx^\mu}{dt}$, were $t = x^0$ is the time coordinate in that frame, one may write

$$\Delta\tau = \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}\, dt \ . \tag{32}$$

In particular, for a clock at rest in the gravitational field, one may set $\frac{d\vec{x}}{dt} = 0$ to find

$$\Delta\tau = \sqrt{-g_{00}(x)} dt \ . \tag{33}$$

The lapse of time dt is time-dilated

$$dt = \frac{\Delta\tau}{\sqrt{-g_{00}(x)}} \quad > \quad \Delta\tau \tag{34}$$

(recall that in the weak field limit $g_{00} = -1 + \frac{2GM}{r}$ has modulus smaller than 1, i.e. $|g_{00}| \leq 1$).

To measure this gravitational time dilation, one has to compare time dilation at different points, otherwise the measuring device would suffer the same time delay. Thus, taking two different points x_1^μ and x_2^μ, one equates

$$\Delta\tau = \sqrt{-g_{00}(x_1)} dt_1 = \sqrt{-g_{00}(x_2)} dt_2 \tag{35}$$

to obtain

$$\frac{dt_1}{dt_2} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} \ . \tag{36}$$

For the frequency $\nu = \frac{1}{dt}$ corresponding to the period dt, this formula becomes

$$\frac{\nu_2}{\nu_1} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} \ . \tag{37}$$

In the weak field limit $g_{00}(x) = -1 - 2\phi(x)$, then setting $\nu_1 = \nu$ and $\nu_2 = \nu + \Delta\nu$, we find a change $\Delta\nu$ in frequency given by

$$\frac{\nu_2}{\nu_1} = \frac{\nu + \Delta\nu}{\nu} = 1 + \frac{\Delta\nu}{\nu} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} = \sqrt{\frac{1 + 2\phi(x_2)}{1 + 2\phi(x_1)}} \approx 1 + \phi(x_2) - \phi(x_1) \ . \tag{38}$$

For an application, consider an atomic transition at the earth's surface with position x_1 and frequency ν_1. Observe the same transition at the sun's surface with position x_2 and frequency ν_2. The sun's potential $\phi(x_2)$ is much larger than the earth's potential $\phi(x_1)$, so that the latter can be ignored, and we get

$$\frac{\Delta\nu}{\nu} = \phi(x_2) - \phi(x_1) \approx \phi(x_2) = -\frac{GM_{sun}}{R_{sun}} \approx -2.12\ 10^{-6} \ . \tag{39}$$

The frequency on the sun is smaller than the frequency seen far away for the same atomic transition: a *gravitational red shift* of the frequency is expected, as verified experimentally.

3 The principle of general covariance and tensor analysis

This topic is described in Chapter 4 of [1].
The *principle of equivalence* can be replaced by a more efficient *principle of general covariance*, which allows us to find in an easier way the correct equations of motion that are valid in the presence of a gravitational field. The *principle of general covariance* states that:
"*A physical equation is valid in an arbitrary gravitational field if it satisfies two conditions:*
(i) the equation reduces to the special relativistic form in the absence of gravity, where the metric is the Minkowski metric ($g_{\mu\nu} \to \eta_{\mu\nu}$) and the affine connection vanishes ($\Gamma^{\lambda}_{\mu\nu} \to 0$).
(ii) the equation is generally covariant, i.e. it keeps the same form under an arbitrary change of coordinates $x^\mu \to x'^\mu(x)$.

We can see that the principle of general covariance implies the principle of equivalence by noting that: point *(ii)* ensures that the equations are valid in any coordinate system if they are valid in at least one frame; point *(i)* verifies that they are valid in a locally inertial frame, where gravity is canceled by inertial forces as dictated by the principle of equivalence, and reduce to the form known from special relativity. To apply the principle of general covariance we need to use *tensor analysis*, which deals with arbitrary coordinate transformations, In mathematics this topic is part of differential geometry.

Tensor analysis

Scalars, vectors, and tensors are quantities defined by their behavior under a change of the coordinate system. A change of coordinates is specified by functions

$$x^\mu \quad \to \quad x'^\mu = x'^\mu(x) \tag{40}$$

that are required to be invertible. One can return to the original frame by using the inverse functions

$$x^\mu = x^\mu(x') \; . \tag{41}$$

Invertibility requires that at any point

$$\det \frac{\partial x'^\mu}{\partial x^\nu} \neq 0 \; . \tag{42}$$

Notice that using the chain rule for differentiation and using the transformations in (40)–(41), one finds that

$$\delta^\mu_\nu = \frac{\partial x'^\mu}{\partial x'^\nu} = \frac{\partial x'^\mu}{\partial x^\lambda} \frac{\partial x^\lambda}{\partial x'^\nu} \tag{43}$$

which tells that at any given point in spacetime the matrix

$$\frac{\partial x'^\mu}{\partial x^\lambda} \tag{44}$$

is the inverse of the matrix

$$\frac{\partial x^\lambda}{\partial x'^\nu} \tag{45}$$

and vice versa.

The displacements dx^μ and dx'^μ are thus related by

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \tag{46}$$

which suggests the following definitions of "geometrical" objects, that turn out to be particularly useful. One defines scalar, vectors, and tensors by

$$
\begin{array}{lll}
\phi(x) & \to \phi'(x') = \phi(x) & \text{scalar} \\
V^\mu(x) & \to V'^\mu(x') = \dfrac{\partial x'^\mu}{\partial x^\nu} V^\nu(x) & \text{contravariant vector} \\
W_\mu(x) & \to W'_\mu(x') = \dfrac{\partial x^\nu}{\partial x'^\mu} W_\nu(x) & \text{covariant vector} \\
T^{\mu\nu}(x) & \to T'^{\mu\nu}(x') = \dfrac{\partial x'^\mu}{\partial x^\lambda}\dfrac{\partial x'^\nu}{\partial x^\rho} T^{\lambda\rho}(x) & \text{tensor of rank (2,0)} \\
S^\mu{}_\nu(x) & \to S'^\mu{}_\nu(x') = \dfrac{\partial x'^\mu}{\partial x^\lambda}\dfrac{\partial x^\rho}{\partial x'^\nu} S^\lambda{}_\rho(x) & \text{tensor of rank (1,1)} \\
\cdots & \to \quad \cdots & \quad \cdots
\end{array}
\tag{47}
$$

and so on for tensors of rank (m, n), where there are m matrices $\frac{\partial x'^\mu}{\partial x^\nu}$ that rotate the m upper indices, and n matrices $\frac{\partial x^\mu}{\partial x'^\nu}$ that rotate the n lower indices. For example, the $(2,1)$-rank tensor $F^{\mu\nu}{}_\lambda$ transforms as

$$
F^{\mu\nu}{}_\lambda(x) \quad \to \quad F'^{\mu\nu}{}_\lambda(x') = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}\frac{\partial x^\lambda}{\partial x'^\tau} F^{\rho\sigma}{}_\tau(x) \tag{48}
$$

Note that with these definitions, contraction of an upper index with a lower index produces a scalar

$$
V^\mu(x)W_\mu(x) \quad \to \quad V'^\mu(x')W'_\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu}V^\nu(x)\frac{\partial x^\lambda}{\partial x'^\mu}W_\lambda(x) = \underbrace{\frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\lambda}{\partial x'^\mu}}_{\delta^\lambda_\nu} V^\nu(x)W_\lambda(x) = V^\nu(x)W_\nu(x) \tag{49}
$$

i.e.

$$
V^\mu(x)W_\mu(x) \quad \to \quad V'^\mu(x')W'_\mu(x') = V^\mu(x)W_\mu(x) \tag{50}
$$

which is precisely the transformation law of a scalar.

Not all quantities that we have defined so far are tensors: the metric $g_{\mu\nu}(x)$ is a rank $(0,2)$ tensor

$$
g_{\mu\nu}(x) \quad \to \quad g'_{\mu\nu}(x') = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu} g_{\lambda\rho}(x) \tag{51}
$$

but the affine connection $\Gamma^\lambda_{\mu\nu}$ is not a tensor as it transforms in a more complicated way

$$
\Gamma^\lambda_{\mu\nu}(x) \quad \to \quad \Gamma'^\lambda_{\mu\nu}(x') = \frac{\partial x'^\lambda}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\mu}\frac{\partial x^\gamma}{\partial x'^\nu}\Gamma^\alpha_{\beta\gamma}(x) + \frac{\partial x'^\lambda}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \tag{52}
$$

where the second term breaks the tensorial behavior (this behavior can be checked in (7), recalling that the affine connection for the Minkowski metric $\eta_{\alpha\beta}$ vanishes). The second term can be written also as

$$
\frac{\partial x'^\lambda}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} = -\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial^2 x'^\lambda}{\partial x^\alpha \partial x^\beta} \,. \tag{53}
$$

The proof is obtained by taking a derivative with respect to x'^μ of the relation

$$
\delta^\lambda_\nu = \frac{\partial x'^\lambda}{\partial x^\rho}\frac{\partial x^\rho}{\partial x'^\nu} \tag{54}
$$

and using the chain rule for differentiation.

Thanks to these definitions, tensorial equations, which are defined as equations that relate tensors of the same rank, take the same form in all reference frames: e.g. if $A_{\mu\nu}$ and $B_{\mu\nu}$ are tensors then the tensorial equation

$$A_{\mu\nu}(x) = B_{\mu\nu}(x) \tag{55}$$

maintains the same form in all frames, i.e.

$$A_{\mu\nu}(x) = B_{\mu\nu}(x) \qquad \longleftrightarrow \qquad A'_{\mu\nu}(x') = B'_{\mu\nu}(x') \ . \tag{56}$$

The same equation can be written also as

$$A_{\mu\nu}(x) - B_{\mu\nu}(x) = 0 \tag{57}$$

where the right-hand side is understood as the zero tensor of appropriate rank, i.e. the tensor which has all components null (then, it is easily verified that the components vanish in all frames).

Tensors are elements of a vector space of appropriate dimension. One can verify the following algebraic properties of tensors (*tensor algebra*):

A) A linear combination of tensors of the same rank is a tensor of the same rank.
E.g., if $A_{\mu\nu}$ and $B_{\mu\nu}$ are tensors and a and b scalars then

$$T_{\mu\nu} = aA_{\mu\nu} + bB_{\mu\nu} \tag{58}$$

is a tensor.

B) Tensor product (or direct product) of tensors.
The multiplications of the components of the tensors give rise to a new tensor of appropriate rank. E.g., if $A_{\mu\nu}$ and B^{μ} are tensors, then

$$T_{\mu\nu}{}^{\lambda} = A_{\mu\nu}B^{\lambda} \tag{59}$$

is a tensor of rank $(1,2)$.

C) Contraction of a contravariant index with a covariant index of a tensor produces a tensor of lower rank. E.g., taking the tensor $T_{\mu\nu}{}^{\lambda}$, one obtains a vector A_{μ} by setting

$$T_{\mu\nu}{}^{\nu} = A_{\mu} \ , \tag{60}$$

and another vector by by setting

$$T_{\mu\nu}{}^{\mu} = B_{\nu} \ . \tag{61}$$

Indeed, we already realized in eq. (50) that index contraction gives rise to a scalar, as far as the transformation properties of those indices are concerned. These again are tensorial equations, valid in every frame.

Finally, indices of a tensor may be raised and lowered by using the metric tensor $g_{\mu\nu}$, and its inverse $g^{\mu\nu}$ that satisfies

$$g_{\mu\nu}(x)g^{\nu\lambda}(x) = \delta^{\lambda}_{\mu}. \tag{62}$$

For example, from the contravariant vector V^{μ} one obtains the covariant vector V_{μ} as

$$V_{\mu} = g_{\mu\nu}V^{\nu} \tag{63}$$

and similarly, from the covariant vector V_{μ} one obtains the contravariant vector V^{μ} by

$$V^{\mu} = g^{\mu\nu}V_{\nu} \ . \tag{64}$$

Covariant derivatives

Derivatives of tensors are not tensors themselves. This causes a problem when defining tensorial equations that must contain derivatives. This problem is solved by using the concept of *covariant derivatives*, which are derivatives that when applied to tensors produce new tensors.

To expose the problem, let us verify that the derivative of a vector field $V^\nu(x)$

$$\partial_\mu V^\nu(x) = \frac{\partial V^\nu(x)}{\partial x^\mu} \tag{65}$$

is not a tensor. We compute

$$\begin{aligned} \partial_\mu V^\nu(x) \;\; \to \;\; \partial'_\mu V'^\nu(x') &= \frac{\partial V'^\nu(x')}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu}\Big(V^\beta(x)\frac{\partial x'^\nu}{\partial x^\beta}\Big) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha}\Big(V^\beta(x)\frac{\partial x'^\nu}{\partial x^\beta}\Big) \\ &= \frac{\partial x^\alpha}{\partial x'^\mu}\Big(\frac{\partial V^\beta(x)}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} + V^\beta(x)\frac{\partial^2 x'^\nu}{\partial x^\alpha \partial x^\beta}\Big) \\ &= \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\beta}\partial_\alpha V^\beta(x) + V^\beta(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x'^\nu}{\partial x^\alpha \partial x^\beta} \end{aligned} \tag{66}$$

where the first term would be the one expected for a tensorial transformation, but the second term breaks the tensorial character. A similar transformation appears in the transformation rule of the affine connection, see eqs. (52)–(53). This fact can be used to introduce the concept of the *covariant derivative* of the vector field, defined by

$$\nabla_\mu V^\nu \equiv \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{67}$$

which then transforms as a tensor

$$\nabla_\mu V^\nu(x) \;\; \to \;\; \nabla'_\mu V'^\nu(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\sigma}\nabla_\rho V^\sigma(x) \; . \tag{68}$$

Geometrically, the connection connects the tangent spaces of nearby points and allows to define the parallel transport of vectors, which are then compared in defining the covariant derivative.

The concept of a covariant derivative allows to consider tensorial equations with derivatives, and thus the identification of the correct differential equations describing physical systems under the force of gravity.

In general, covariant derivatives of contravariant and covariant vectors are defined by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \tag{69}$$

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\lambda_{\mu\nu} V_\lambda \tag{70}$$

and similarly for more general tensors, which are defined to have a connection for each index, e.g.

$$\nabla_\mu V^{\nu\lambda}{}_\rho = \partial_\mu V^{\nu\lambda}{}_\rho + \Gamma^\nu_{\mu\alpha} V^{\alpha\lambda}{}_\rho + \Gamma^\lambda_{\mu\alpha} V^{\nu\alpha}{}_\rho - \Gamma^\alpha_{\mu\rho} V^{\nu\lambda}{}_\alpha \; . \tag{71}$$

The covariant derivative satisfies the Leibniz rule for taking the derivative of products of tensors. For example, one verifies that on a scalar field the covariant derivative reduces to the usual partial derivative, consistently with the Leibniz rule: taking the scalar $V^\mu W_\mu$, its derivative can be expanded as

$$\partial_\nu (V^\mu W_\mu) = \partial_\nu V^\mu \, W_\mu + V^\mu \, \partial_\nu W_\mu \tag{72}$$

which of course is correct. On the other hand, one verifies that using covariant derivatives all terms with a connection cancel each other (this is easily recognized after renaming indices)

$$\begin{aligned}\partial_\nu(V^\mu W_\mu) &= \nabla_\nu(V^\mu W_\mu) = \nabla_\nu V^\mu\, W_\mu + V^\mu\, \nabla_\nu W_\mu \\ &= (\partial_\nu V^\mu + \Gamma^\mu_{\nu\lambda} V^\lambda) W_\mu + V^\mu(\partial_\nu W_\mu - \Gamma^\lambda_{\nu\mu} W_\lambda) \\ &= \partial_\nu V^\mu\, W_\mu + V^\mu\, \partial_\nu W_\mu\end{aligned} \tag{73}$$

The covariant derivative of the metric vanishes

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma} = 0\ , \tag{74}$$

a fact that is referred to by saying that the metric is covariantly constant. This can be verified using eq. (16), that relates the affine connection to the derivative of the metric. Basically, this statement reinterpret the content of eq. (13).

Conversely, eq. (74) can be used to derive relation (16). Let us show this point. Since the metric is covariantly constant we may write

$$\begin{aligned}0 &= \nabla_\lambda g_{\mu\nu} + \nabla_\mu g_{\lambda\nu} - \nabla_\nu g_{\lambda\mu} \\ &= \ \partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} g_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu} g_{\mu\sigma} \\ &\quad + \partial_\mu g_{\lambda\nu} - \Gamma^\sigma_{\mu\lambda} g_{\sigma\nu} - \Gamma^\sigma_{\mu\nu} g_{\lambda\sigma} \\ &\quad - \partial_\nu g_{\lambda\mu} + \Gamma^\sigma_{\nu\lambda} g_{\sigma\nu} + \Gamma^\sigma_{\nu\mu} g_{\lambda\sigma} \\ &= \partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\lambda\mu} - 2\Gamma^\sigma_{\lambda\mu} g_{\sigma\nu}\end{aligned} \tag{75}$$

where we have used the symmetry on the first two indices of the affine connection. Then, using the inverse metric we find

$$\Gamma^\sigma_{\lambda\mu} = \frac{1}{2} g^{\sigma\nu}(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\lambda\nu} - \partial_\nu g_{\lambda\mu}) \tag{76}$$

as expected.

Finally, let us consider a vector field defined along a curve, rather than all over space-time. The covariant derivative of a vector $V^\mu(\tau)$ defined along a curve parametrized by $x^\mu(\tau)$, where τ is the parameter, takes the form

$$\frac{DV^\mu}{d\tau} = \frac{dV^\mu}{d\tau} + \frac{dx^\rho}{d\tau}\Gamma^\mu_{\rho\sigma} V^\sigma\ . \tag{77}$$

As an example, consider a worldline parameterized by $x^\mu(\tau)$. Its derivative with respect to the parameter τ gives the four-velocity, i.e. the tangent vector to the curve

$$\frac{dx^\mu}{d\tau}\ . \tag{78}$$

This 4-velocity is easily verified to be a vector as it has the vector transformation law (just use the chain rule for differentiation). Its covariant derivative takes the form

$$\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} \tag{79}$$

which is again a vector defined along the curve. With the notation

$$\frac{dx^\mu}{d\tau} \equiv \dot{x}^\mu \tag{80}$$

we usually write it as

$$\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = \ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma \ . \tag{81}$$

The equation

$$\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = 0 \tag{82}$$

is known as the geodesic equation, already encountered in (6) to describe the motion of a particle under the force of gravity.

Parallel transport

The connection $\Gamma^\lambda_{\mu\nu}$ discussed above can be interpreted as providing a definition of the parallel transport of vectors. Let us elaborate more on this point. Vectors at a given point belong to the tangent space of the manifold at that point. The manifold in question refers to the spacetime in our applications. Tangent spaces at different points are in principle different spaces. Of course they are isomorphic, but there is no canonical isomorphism that relates vectors of one space to vectors of the other space in a unique way. Thus, one must define a rule how to relate vectors belonging to different spaces. One way to relate the tangent spaces at two different points is to choose a path that connects the two points, and then parallel transport the vectors of the first tangent space to the second tangent space. This definition in general depends on the chosen path. For nearby points, a vector $V^\mu(x)$ at point x^μ is parallel transported to a vector $V^\mu_{//}(x+\Delta x)$ at point $x^\mu + \Delta x^\mu$ as follows

$$V^\mu_{//}(x+\Delta x) = V^\mu - \Delta x^\nu\, \Gamma^\mu_{\nu\lambda} V^\lambda(x) \ . \tag{83}$$

This definition allows to introduce covariant derivatives geometrically. For a vector field $V^\mu(x)$ defined on the manifold, the covariant derivative measures how the vectors at nearby points differ from their parallel transported ones, i.e.

$$V^\mu(x+\Delta x) - V^\mu_{//}(x+\Delta x) = \Delta x^\nu(\partial_\nu V^\mu(x) + \Gamma^\mu_{\nu\lambda} V_\lambda(x) \equiv \Delta x^\nu\, \nabla_\nu V^\mu(x) \ . \tag{84}$$

This reproduces the same formula defined earlier. A vector V^μ parallel transported along the path parametrized by the functions $x^\mu(\tau)$ satisfies the equation

$$\frac{DV^\mu}{d\tau} = \frac{dV^\mu}{d\tau} + \frac{dx^\rho}{d\tau}\Gamma^\mu_{\rho\sigma}V^\sigma = 0 \ . \tag{85}$$

Gradient, curl, and divergence

The *gradient* of a scalar field $\phi(x)$ takes the usual form, as for this case the usual partial derivative and the covariant derivative coincide

$$\nabla_\mu \phi = \partial_\mu \phi \ . \tag{86}$$

The definition of the *curl* of a three-dimensional vector field $\vec{A}(x)$ is better given through its components

$$(\nabla \times \vec{A})^i = \epsilon^{ijk}\partial_j A_k = \frac{1}{2}\epsilon^{ijk}(\partial_j A_k - \partial_k A_j) \tag{87}$$

with the last expression that can be used to extend the definition to arbitrary dimensions. This is done by eliminating the antisymmetric ϵ tensor, i.e. defining

$$\partial_j A_k - \partial_k A_j \quad \to \quad \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{88}$$

This expression defines the curl in arbitrary dimensions and can be covariantized by substituting the covariant derivative with the usual ones. However, there is a simplification. One notices that all the terms with a connection cancel out. This shows that the original expression was already covariant without the need of introducing a metric $g_{\mu\nu}$ and related connection $\Gamma^\lambda_{\mu\nu}$.

This property extends to arbitrary antisymmetric tensors of rank $(0,p)$, i.e. totally antisymmetric tensors $A_{\mu_1\mu_2\cdots\mu_p}$ whose curl is defined by

$$(dA)_{\mu_1\mu_2\cdots\mu_{p+1}} = \partial_{\mu_1} A_{\mu_2\mu_3\cdots\mu_{p+1}} \pm \text{cyclic permutations} \tag{89}$$

where the sign depends on the permutation: the + sign for even permutations of $1, 2, \cdots, p+1$ and the $-$ sign for odd permutations. The totally antisymmetric tensors $A_{\mu_1\mu_2\cdots\mu_p}$ are often called *p-forms*, and the curl d is called *exterior derivative*. The exterior derivative is a covariant operation: one may check that all connections cancel out if one would have used covariant derivatives in its definition. The exterior derivative applied twice vanishes because of the antisymmetry, i.e. $d^2 = 0$.

Let us now comment on the *divergence* of a vector field $J^\mu(x)$. It is covariantized by substituting the partial derivative with the covariant one

$$\partial_\mu J^\mu \quad \to \quad \nabla_\mu J^\mu \ . \tag{90}$$

The connection remains, but it can be written in a simpler form

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\mu\nu} J^\nu \tag{91}$$

where

$$\Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\lambda} \partial_\nu g_{\mu\lambda} = \frac{1}{\sqrt{g}} \partial_\nu \sqrt{g} \tag{92}$$

with $g = |\det g_{\mu\nu}|$. At the end, one may write

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} J^\mu) \ . \tag{93}$$

The proof of (91) goes as follows

$$\frac{1}{\sqrt{g}} \partial_\nu \sqrt{g} = \partial_\nu \ln \sqrt{g} = \frac{1}{2} \partial_\nu \ln g = \frac{1}{2} \partial_\nu \ln \det g_{\mu\lambda} = \frac{1}{2} \partial_\nu \mathrm{tr} \ln \, g_{\mu\lambda} = \frac{1}{2} g^{\lambda\mu} \partial_\nu g_{\mu\lambda} \ . \tag{94}$$

4 Effects of gravitation

This topic is described in Chapter 5 of [1].
We now make use of the tensor calculus to study how gravity affects the equations of mechanics and electromagnetism studied previously in special relativity. We make use of the principle of general covariance. We take the equations of motion we know from special relativity and rewrite them in a form that makes them look generally covariant. This last step is achieved by identifying first the tensors that describe the physical quantities in the equations. Then, one substitutes the Minkowski metric $\eta_{\mu\nu}$ by $g_{\mu\nu}$, and derivatives of tensors by their covariant

derivatives. In this way, one finds equations of motion that are generally covariant. They include the force of gravity according to the principle of general covariance.

Particle mechanics

We know that the motion of a free particle is described in special relativity by eqs. (2),(3). Using the particle coordinates x^μ, the equations read

$$\frac{d^2x^\mu}{d\tau^2} = 0 \, , \qquad d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu \, . \tag{95}$$

They must be covariantized. The 4-velocity $\frac{dx^\mu}{d\tau}$ is easily verified to be a vector

$$\frac{dx^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{dx'^\nu}{d\tau} \tag{96}$$

where the term $\frac{\partial x'^\mu}{\partial x^\nu}$ that determines the vectorial character is evaluated on the worldline at time τ. Then, we use the covariant derivative to preserve its vectorial character

$$\frac{D}{d\tau} \frac{dx^\mu}{d\tau} = 0 \, . \tag{97}$$

This is the correct equation that includes the gravitational force. In standard notations, it takes the form

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0 \, . \tag{98}$$

Similarly, the proper time is covariantized by substituting $\eta_{\mu\nu}$ with the metric $g_{\mu\nu}$

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu \, . \tag{99}$$

To summarize, the covariant extensions of eqs. (95) are

$$\frac{D}{d\tau} \frac{dx^\mu}{d\tau} = 0 \, , \qquad d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu \tag{100}$$

and they include the force of gravity acting on the particle. As we see, this is a much faster procedure than the one used in sect. 2.

Klein-Gordon equation

The Klein-Gordon equation is a relativistic equation for a scalar field $\phi(x)$. In natural units $(c = \hbar = 1)$ it reads

$$(\partial^\mu \partial_\mu - m^2)\phi(x) = 0 \tag{101}$$

The first derivative of a scalar is already covariant

$$\nabla_\mu \phi = \partial_\mu \phi \, . \tag{102}$$

The second derivative requires a connection

$$\nabla_\mu \nabla_\nu \phi = \partial_\mu \nabla_\nu \phi - \Gamma^\lambda_{\mu\nu} \nabla_\lambda \phi = \partial_\mu \partial_\nu \phi - \Gamma^\lambda_{\mu\nu} \partial_\lambda \phi \, . \tag{103}$$

Finally, the scalar product is taken with the inverse metric $g^{\mu\nu}$ (i.e. substituting $\eta^{\mu\nu}$ with $g^{\mu\nu}(x)$) and the covariantized Klein-Gordon equation becomes

$$(g^{\mu\nu} \nabla_\mu \nabla_\nu - m^2)\phi = 0 \tag{104}$$

written equivalently as

$$(\nabla^\mu \nabla_\mu - m^2)\phi = 0 \ . \tag{105}$$

It is customary to denote the covariant d'Alambert operator by a *box*

$$\Box = \nabla^\mu \nabla_\mu \tag{106}$$

and write the covariant equation as

$$(\Box - m^2)\phi = 0 \ . \tag{107}$$

For explicit calculations, it is often useful to rewrite the d'Alambertian operator $\Box$ acting on a scalar field the following way

$$\Box\phi = \frac{1}{\sqrt{g}}\partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \phi \ . \tag{108}$$

where $g = |\det g_{\mu\nu}|$. Here, the first derivatives ∂_μ acts all the way through.

Electrodynamics

Maxwell's equations can be written in special relativity as

$$\begin{aligned} &\partial_\mu F^{\mu\nu} = -J^\nu \\ &\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \end{aligned} \tag{109}$$

with the second one solved in terms of the 4-potential A_μ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{110}$$

Let us see how to covariantize them. The second equation in (109) and its solution (110) are covariantized by substituting derivatives with covariant derivatives

$$\begin{aligned} &\nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} + \nabla_\lambda F_{\mu\nu} = 0 \\ &F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \ . \end{aligned} \tag{111}$$

However, one verifies that the connection drops out in all of them, so that the original equations

$$\begin{aligned} &\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0 \\ &F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{112}$$

were nevertheless covariant. In a sense, these equations do not feel the force of gravity.

As for the first one in (109), one must first raise indices using the general metric $g_{\mu\nu}$

$$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} \tag{113}$$

and then write it with a covariant derivative

$$\nabla_\mu F^{\mu\nu} = -J^\nu \ . \tag{114}$$

This is a covariant equation if J^ν is a contravariant vector, which we assume to be the case.

Notice that the metric is covariantly constant, so that it commutes with the covariant derivative. This facts allows to present eq. (114) in the equivalent form

$$\nabla^\mu F_{\mu\nu} = -J_\nu \ . \tag{115}$$

The Lorentz force and gravity

Finally, let us include gravity in the Lorentz force equation (with $c = 1$)

$$m\frac{d^2x^\mu}{d\tau^2} = eF^{\mu\nu}\frac{dx_\nu}{d\tau} \tag{116}$$

by covariantization. We find

$$m\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = eF^{\mu\nu}g_{\nu\lambda}\frac{dx^\lambda}{d\tau} \tag{117}$$

recalling that indices are now lowered and raised with $g_{\mu\nu}$ and $g^{\mu\nu}$, as in eq. (113). To better expose the places where the metric sits, it is perhaps easier to rewrite the covariant equation in the form

$$m\frac{D}{d\tau}\frac{dx^\mu}{d\tau} = eg^{\mu\nu}F_{\nu\lambda}\frac{dx^\lambda}{d\tau} \tag{118}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, i.e.

$$m(\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu\dot{x}^\lambda) = eg^{\mu\nu}F_{\nu\lambda}\dot{x}^\lambda\ . \tag{119}$$

to recognize that the force of gravity acts through the terms with $g^{\mu\nu}$ and $\Gamma^\mu_{\nu\lambda}$.

5 Curvature

This topic is described in Chapter 6 of [1].

The equations of motion for the gravitational field describe the behavior of the metric tensor $g_{\mu\nu}(x)$, which contains the potential for the gravitational force. To ensure covariance, the equations must be constructed using tensors. It can be proven that no tensor can be constructed solely from the metric $g_{\mu\nu}$ and its first derivatives $\partial_\lambda g_{\mu\nu}$. This is so because the covariant derivatives of the metric vanish, $\nabla_\lambda g_{\mu\nu} = 0$, while the affine connection $\Gamma^\lambda_{\mu\nu}$, defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}), \tag{120}$$

is not a tensor. One may notice that $\Gamma^\lambda_{\mu\nu}$ and $\partial_\lambda g_{\mu\nu}$ contain both 40 independent components, so that the former is equivalent to the latter.

One can construct tensors by including second derivatives of the metric. To find them one can use the properties of covariant derivatives. This provides a quick way of identifying such tensors.

Covariant derivatives do not commute and they may be used to define implicitly the Riemann curvature tensor $R_{\mu\nu}{}^\lambda{}_\rho$ by the relation

$$[\nabla_\mu, \nabla_\nu]V^\lambda = R_{\mu\nu}{}^\lambda{}_\rho V^\rho\ . \tag{121}$$

The left-hand side is a tensor, so that also the right-hand side must be a tensor. In particular, the quantity $R_{\mu\nu}{}^\lambda{}_\rho$ must be a tensor.

The Riemann tensor $R_{\mu\nu}{}^\lambda{}_\rho$ is manifestly antisymmetric under the exchange of the first two indices μ, ν as a consequence of its definition. A direct calculation shows that

$$R_{\mu\nu}{}^\lambda{}_\rho = \partial_\mu\Gamma^\lambda_{\nu\rho} - \partial_\nu\Gamma^\lambda_{\mu\rho} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}\ . \tag{122}$$

A mnemonic for remembering this structure is to write

$$R_{\mu\nu}{}^\lambda{}_\rho = \overline{\nabla}_\mu\Gamma^\lambda_{\nu\rho} - (\mu \leftrightarrow \nu) \tag{123}$$

where $\overline{\nabla}_\mu$ contains a connection for the upper index only (in general, covariant derivatives are defined only for tensors).

Algebraic properties of the Riemann tensor are best written by lowering the upper index with the metric, $R_{\mu\nu\lambda\rho} = g_{\lambda\sigma} R_{\mu\nu}{}^{\sigma}{}_{\rho}$. They are the following ones

$$
\begin{aligned}
&R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu} && (\textit{symmetry}) && (124)\\
&R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho} = -R_{\mu\nu\rho\lambda} && (\textit{antisymmetry}) && (125)\\
&R_{\mu\nu\lambda\rho} + R_{\lambda\mu\nu\rho} + R_{\nu\lambda\mu\rho} = 0 && (\textit{cyclicity})\ . && (126)
\end{aligned}
$$

A way of proving these relations is to write them down in terms of the metric using (122) and (120). This is very laborious, but correct. A faster way is given in the exercises.

Additional tensors can be constructed by index contraction: they are the Ricci tensor $R_{\mu\nu}$ and the curvature scalar R

$$
\begin{aligned}
&R_{\mu\nu} = R_{\lambda\mu}{}^{\lambda}{}_{\nu} && \text{(Ricci tensor)} && (127)\\
&R = g^{\mu\nu} R_{\mu\nu} && \text{(Ricci scalar or curvature scalar)}\ . && (128)
\end{aligned}
$$

Other contractions do not give rise to independent tensors. From (124) it follows that the Ricci tensor is symmetric

$$R_{\mu\nu} = R_{\nu\mu}\ . \tag{129}$$

At this stage, it is useful to compute the number of independent components $C_{\text{Riem}}(D)$ of the Riemann tensor in arbitrary dimensions D. To start with let us consider the metric tensor $g_{\mu\nu}$ and the Ricci tensor $R_{\mu\nu}$ that are both symmetric tensors. In arbitrary D dimension, they have the same number of independent components $C_{\text{metric}}(D)$ of a symmetric matrix, given by

$$C_{\text{metric}}(D) = \frac{1}{2} D(D+1)\ . \tag{130}$$

As for the Riemann tensor, the number of its independent components is given by

$$
\begin{aligned}
C_{\text{Riem}}(D) &= \frac{1}{2}\Big(\frac{1}{2}D(D-1)\Big)\Big(\frac{1}{2}D(D-1)+1\Big) - \frac{D(D-1)(D-2)(D-3)}{4!}\\
&= \frac{1}{12} D^2(D^2-1)\ .
\end{aligned}
\tag{131}
$$

This value is obtained by considering the Riemann tensors as a symmetric matrix R_{AB} where A and B stands for the ordered pair of indices $(\mu\nu)$ with $\mu < \nu$. The last term subtracts the independent relations in (126), which are completely antisymmetric under the exchange of indices. A few values are reported in the following table

D	D^4	$C_{\text{Riem}}(D) = \frac{D^2(D^2-1)}{12}$	$C_{\text{metric}}(D) = \frac{D(D+1)}{2}$
1	1	0	1
2	16	1	3
3	81	6	6
4	256	20	10
5	625	50	15
10	10000	825	55
11	14641	1210	66

Of course, we use $D = 4$ for our purposes: we see that the Riemann tensor has 20 components while the Ricci tensor has 10 components, just like the metric.

The componets of the Riemann tensor describe the curvature of a general D-dimensional space. However, they do not do so in an invariant manner, for their values depend also on the particular coordinate system chosen. The invariant characterization of the a curved space must be given in terms of scalars constructed from $R_{\mu\nu\lambda\rho}$ and $g_{\mu\nu}$. The simplest scalar is R. Then, one could consider scalars like $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$. We will not study here this interesting problem.

5.1 Bianchi identities

The Riemann tensor satisfies the following differential Bianchi identities

$$\nabla_\mu R_{\nu\lambda\alpha\beta} + \nabla_\nu R_{\lambda\mu\alpha\beta} + \nabla_\lambda R_{\mu\nu\alpha\beta} = 0 \ . \tag{132}$$

The sum over the cyclic permutations of the first three indices makes the total combination antisymmetric in those indices.

One may contract the Bianchi identities on the indices (ν, α) (multiplying by $g^{\nu\alpha}$) to find

$$\nabla_\mu R_{\lambda\beta} + \nabla^\alpha R_{\lambda\mu\alpha\beta} - \nabla_\lambda R_{\mu\beta} = 0 \tag{133}$$

and contracting once more the indices (λ, β) one finds

$$\nabla_\mu R - 2\nabla^\alpha R_{\mu\alpha} = 0 \qquad \to \qquad \nabla^\mu \Big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Big) = 0 \ . \tag{134}$$

It is customary to define the Einstein tensor $G_{\mu\nu}$ by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{135}$$

which is covariantly conserved $\nabla^\mu G_{\mu\nu} = 0$, as seen from eq. (134).

Exercizes

These exercises help in proving some of the symmetry properties of the Riemann tensor.

Ex.1 *Recalling that the metric is covariantly constant* ($\nabla_\mu g_{\alpha\beta} = 0$) *use* $[\nabla_\mu, \nabla_\nu] g_{\alpha\beta} = 0$ *to prove the antisymmetry in the last two indices of the Riemann tensor,* $R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}$.

Solution: One may calculate

$$0 = [\nabla_\mu, \nabla_\nu] g_{\alpha\beta} = R_{\mu\nu\alpha}{}^\gamma g_{\gamma\beta} + R_{\mu\nu\beta}{}^\gamma g_{\alpha\gamma} = R_{\mu\nu\alpha\beta} + R_{\mu\nu\beta\alpha} \ .$$

Ex. 2 *Rewriting the Bianchi identities for electromagnetism using covariant derivatives, show the cyclic property of the Riemann tensor, eq.* (126).

Ex. 3 *Use the cyclicity property* (126) *and the antisymmetries in eq.* (124) *to prove the symmetry property* $R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$.

Ex. 4 *From the Jacobi identity valid for arbitrary operators* A, B, C

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

which is a consequence of the associativity of the multiplication of operators, consider the case with $(A, B, C) \equiv (\nabla_\mu, \nabla_\nu, \nabla_\lambda)$ *acting on a vector field* V^ρ, *i.e.*

$$\Big([\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] \Big) V^\rho = 0$$

and prove the Bianchi identities in (132).

6 Einstein's equations of general relativity

This topic is described in Chapter 7 of [1].

We now come to Einstein's field equations. They are the dynamical equations for the metric $g_{\mu\nu}$ (the equivalent for the metric of the Maxwell's equations for the potential A_μ) and can be identified by using the principle of general covariance applied to Newton's theory of universal gravitation, once the latter has been modified to make it consistent with special relativity.

A *weak* and *static* field due to non-relativistic matter with mass density $\rho(x)$ is described by the Newtonian potential ϕ that satisfies the equation

$$\nabla^2 \phi = 4\pi G \rho \tag{136}$$

where $G = 6.67\ 10^{-11} Nm^2/Kg^2$ is the Newton gravitational constant. By now, we know that this potential is embedded in the component g_{00} of the metric as follows

$$g_{00} \approx -(1 + 2\phi) \ . \tag{137}$$

For example, a point-like particle of mass M at rest has a mass density

$$\rho(x) = M \delta^3(\vec{x}) \tag{138}$$

and it gives rise to a potential that satisfies the equation

$$\nabla^2 \phi = 4\pi G M \delta^3(\vec{x}) \qquad \to \qquad \phi(x) = -\frac{GM}{r} \qquad \to \qquad g_{00}(x) = -1 + \frac{2GM}{r} \ . \tag{139}$$

In special relativity, mass and energy are equivalent, so one can interpret $\rho(x)$ as describing the total energy density of the matter that gravitates. This density appears as the T_{00} component of the energy-momentum tensor of the matter system. Thus, we can rewrite the equation for the gravitational potential (136) in the form

$$\nabla^2 g_{00} = -8\pi G T_{00} \ . \tag{140}$$

Special relativity implies this equation must be the component of a tensorial equation: in different Lorentz frames the different components of the tensors are mixed by the Lorentz transformations. We know already that $T_{00} = \rho = M\delta^3(\vec{x})$ is the component of the tensor $T_{\alpha\beta}$, the stress energy-momentum tensor. Therefore, one deduces that there must exist a tensor $G_{\alpha\beta}$ (tensor under Lorentz transformations) with the component $G_{00} = -\nabla^2 g_{00}$ (the minus sign is conventional) that can be constructed with second derivatives of the metric so that the Lorentz invariant extension of (140) becomes

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta} \tag{141}$$

where the complete energy-momentum tensor $T_{\alpha\beta}$ appears on the right-hand side. Moreover, since the stress tensor $T_{\alpha\beta}$ satisfies the conservation law

$$\partial^\alpha T_{\alpha\beta} = 0 \tag{142}$$

also the Lorentz tensor $G_{\alpha\beta}$ should satisfy for consistency

$$\partial^\alpha G_{\alpha\beta} = 0 \ . \tag{143}$$

So far, this is just a consequence of special relativity.

Finally, general relativity is obtained by searching for a general covariant extension of the equation in (141), that must take the general covariant form

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{144}$$

The conservation of $T_{\alpha\beta}$, namely $\partial^\alpha T_{\alpha\beta} = 0$ is covariantized to $\nabla^\mu T_{\mu\nu} = 0$. By consistency, also $G_{\mu\nu}$ must be covariantly conserved, i.e. $\nabla^\mu G_{\mu\nu}$. The weak and static limit identifies it uniquely with the Einstein tensor that we discussed earlier on.

These considerations lead to the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{145}$$

which are generally covariant field equations for the metric $g_{\mu\nu}$. The tensor $T_{\mu\nu}$ is the energy-momentum tensor of the matter that gravitates.

An equivalent way of writing these equations is to first take the trace (multiplying by $g^{\mu\nu}$) to find (in four spacetime dimensions)

$$R - 2R = 8\pi G T^\mu{}_\nu \qquad \to \qquad R = -8\pi G T^\mu{}_\mu$$

so that Einstein's equations take the equivalent form

$$R_{\mu\nu} = 8\pi G \Big(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda \Big) \ . \tag{146}$$

In vacuum, these equations reduce to

$$R_{\mu\nu} = 0 \ . \tag{147}$$

An additional term with a dimensionful coupling constant Λ with positive mass dimensions, the so-called cosmological constant, can be added to the equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{148}$$

Originally introduced by Einstein to describe a static universe, nowadays it allows us to parametrize the presence of dark energy in the universe.

Finally, reintroducing by dimensional analysis the speed of light c, Einstein's equations take the form

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}} \ . \tag{149}$$

However, we will continue to use units with $c = 1$.

After the construction of these equations, Einstein proposed three tests to verify its validity. These classical tests were: the perihelion precession of the orbits of planets, the bending of light around stars, and the gravitational redshift. They are well described in standard textbooks. Here we will focus on more modern tests: the existence of gravitational waves and black holes.

7 Structure of Einstein's equations, gauge symmetry, harmonic gauge

The Einstein's equations (149) are differential equations for the metric $g_{\mu\nu}$. The form of the solution $\bar{g}_{\mu\nu}$ depends on the energy-momentum distribution of the matter content of spacetime, as encoded by the stress tensor $T_{\mu\nu}$. In the words of Wheeler, "*matter tells spacetime how* *to curve*", a sentence that describes Einstein equations, and "*curved spacetime tells matter how to move*", which refers to the equations of motion of matter once extended to include the gravitational force (as described in section 4).

There are 10 equations and one might think that this is the right number of equations needed to fix all of the 10 components of the metric. However, it is not so: there are gauge symmetries related to the arbitrary change of coordinates that invalidate this counting. In particular, given a solution $\bar{g}_{\mu\nu}$, one can obtain another solution $\bar{g}'_{\mu\nu}$ by performing a change of coordinates. This fact can be recognized also in a different way: the Einstein equations are not all independent since they satisfy the following differential constraints

$$\nabla^\mu\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} - 8\pi G T_{\mu\nu}\Big) = 0 \ . \tag{150}$$

We have used this property in the construction of Einstein's equations. More generally, the existence of constraints is guaranteed by a theorem of Emmy Noether concerning theories with gauge symmetries. The gauge symmetry in question is the one associated with an arbitrary change of coordinates. It depends on the four arbitrary functions $x'^\mu = x'^\mu(x)$.

The gauge symmetry implies that given a solution $g_{\mu\nu}(x)$, also $g'_{\mu\nu}(x)$ will be a solution if $g'_{\mu\nu}$ is obtained from $g_{\mu\nu}$ by a change of coordinates

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x)\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} \ . \tag{151}$$

Under the change of coordinates $x'^\mu = x^\mu - \xi^\mu(x)$, parametrized by the infinitesimal vector field $\xi^\mu(x)$, the metric varies infinitesimally as

$$\begin{aligned} \delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) &= \xi^\alpha \partial_\alpha g_{\mu\nu} + (\partial_\mu \xi^\alpha) g_{\alpha\nu} + (\partial_\nu \xi^\alpha) g_{\mu\alpha} \\ &= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \ . \end{aligned} \tag{152}$$

This equation describes the infinitesimal form of the gauge symmetries of gravity

$$\delta g_{\mu\nu}(x) = \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x) \tag{153}$$

which may be compared with the gauge symmetry of the Maxwell equations

$$\delta A_\mu(x) = \partial_\mu \alpha(x) \ . \tag{154}$$

Gauge-fixing

The gauge symmetries allows to select gauge-fixing conditions that may be used to simplify the study of solutions of the field equations, just like the Lorenz gauge $\partial^\mu A_\mu = 0$ used in the study of electromagnetism. In gravity, the gauge symmetries can be fixed by requiring the harmonic gauge (or De Donder gauge) conditions

$$\Gamma^\mu \equiv g^{\nu\lambda}\Gamma^\mu_{\nu\lambda} = 0 \qquad \leftrightarrow \qquad \partial_\nu(\sqrt{g}g^{\nu\mu}) = 0 \ . \tag{155}$$

These four conditions specify a class of reference frames in which the coordinates are harmonic functions, just like the cartesian coordinates of flat spacetime. For this reason, the coordinates of these reference frames are sometimes called quasi-cartesian coordinates.

To better appreciate the meaning of the harmonic gauge, let us remind that a scalar function ϕ is called harmonic if it satisfies the equation

$$\Box\phi = 0 \ . \tag{156}$$

This equation can be written more explicitly as

$$0 = \Box\phi = \nabla^\mu \partial_\mu \phi = g^{\mu\nu}(\partial_\mu \partial_\nu \phi + \Gamma^\lambda_{\mu\nu} \partial_\lambda \phi) = g^{\mu\nu} \partial_\mu \partial_\nu \phi + \Gamma^\lambda \partial_\lambda \phi \tag{157}$$

and in the harmonic gauge it simplifies to

$$g^{\mu\nu} \partial_\mu \partial_\nu \phi = 0 \tag{158}$$

In such a frame, the coordinates x^μ are themselves harmonic functions

$$\Box x^\mu = g^{\nu\lambda} \partial_\nu \partial_\lambda x^\mu = 0 \tag{159}$$

just like the cartesian coordinates of Minkowski space.

Finally, let us mention that one can rewrite the harmonic gauge condition in an alternative way. One can calculate

$$\begin{aligned}
\Gamma^\mu &= g^{\alpha\beta} \Gamma^\mu_{\alpha\beta} = g^{\alpha\beta} \frac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}) \\
&= \frac{1}{2} g^{\alpha\beta} (-\partial_\alpha g^{\mu\nu} g_{\beta\nu} - \partial_\beta g^{\mu\nu} g_{\alpha\nu} - g^{\mu\nu} \partial_\nu g_{\alpha\beta}) \\
&= -\partial_\nu g^{\mu\nu} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \partial_\nu g_{\alpha\beta} \\
&= -\partial_\nu g^{\mu\nu} - g^{\mu\nu} \frac{1}{\sqrt{g}} \partial_\nu \sqrt{g} \\
&= -\frac{1}{\sqrt{g}} \partial_\nu (\sqrt{g} g^{\nu\mu})
\end{aligned} \tag{160}$$

which shows that the harmonic gauge condition can be written also as

$$\partial_\nu (\sqrt{g} g^{\nu\mu}) = 0 \; . \tag{161}$$

Killing vectors and isometries

There may exist spacetimes that admit special vector fields $\xi^\mu(x)$ satisfying the so-called Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \; . \tag{162}$$

The solutions $\xi^\mu(x)$ are called *Killing vectors*. They define infinitesimal change of coordinates that leave the metric invariant, as recognized from eq. (153). They are symmetries of the spacetimes and are called *isometries*.

For the case of flat spacetime, the Minkowski space, the Killing vector generate the Poincaré group of isometries. In cartesian coordinates the Killing equations reduces to

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0 \tag{163}$$

and are solved by the following vector fields

$$\begin{array}{lll}
\xi^\mu(x) = a^\mu & & a^\mu \text{ constant (spacetime translations)} \\
\xi^\mu(x) = \omega^\mu{}_\nu x^\nu & \omega_{\mu\nu} = -\omega_{\nu\mu} & \text{constant (Lorentz transformations) .}
\end{array} \tag{164}$$

Flat spacetime is a maximally symmetric space in that it can be shown to admit the maximal number of independent Killing vectors fields possible. Other maximally symmetric spacetimes are the *de Sitter space*, which has a positive cosmological constant, and the *anti de Sitter space*, which has a negative cosmological constant.

8 Linearized Einstein's equations

To study Einstein's equations in a linearized approximation around flat spacetime, one sets the metric as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{165}$$

and considers $|h_{\mu\nu}(x)| \ll 1$. Then, in the linearized approximation one may raise and lower indices with the Minkowski metric

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta} \tag{166}$$

and define the "trace" of $h_{\mu\nu}$ as

$$h = \eta^{\mu\nu}h_{\mu\nu} \ . \tag{167}$$

Then, one may compute at the linear order in $h_{\mu\nu}$

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x) \ , \quad g = |\det g_{\mu\nu}| = 1 + h \ , \quad \sqrt{g} = 1 + \frac{1}{2}h \tag{168}$$

The Christoffel symbols linearize as

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) = \frac{1}{2}(\partial_\mu h_\nu{}^\rho + \partial_\nu h_\mu{}^\rho - \partial^\rho h_{\mu\nu}) \ , \tag{169}$$

the Riemann tensor as

$$R_{\mu\nu}{}^\rho{}_\sigma = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + ... = \frac{1}{2}\partial_\sigma(\partial_\mu h_\nu{}^\rho - \partial_\nu h_\mu{}^\rho) - \frac{1}{2}\partial^\rho(\partial_\mu h_{\nu\sigma} - \partial_\nu h_{\mu\sigma}) \tag{170}$$

and the Ricci tensor

$$R_{\nu\sigma} = R_{\mu\nu}{}^\mu{}_\sigma = \frac{1}{2}(\partial_\nu\partial^\mu h_{\sigma\mu} + \partial_\sigma\partial^\mu h_{\nu\mu} - \partial_\nu\partial_\sigma h - \Box h_{\nu\sigma}) \tag{171}$$

where we now use the symbol $\Box = \partial^\mu\partial_\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu$ for the d'Alembertian in flat spacetime.

Then, Einstein's equations in vacuum, see eq. (147), take the linearized form

$$\Box h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial_\mu\partial^\sigma h_{\sigma\nu} - \partial_\nu\partial^\sigma h_{\sigma\mu} = 0 \ . \tag{172}$$

One can verify that they are gauge invariant under a gauge symmetry: the gauge symmetry (153) simplifies at lowest order in $h_{\mu\nu}$ to

$$\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \tag{173}$$

where the four components of ξ_μ are arbitrary functions, and one may verify that eqs. (172) are invariant under them. These symmetries can be used to set four gauge-fixing conditions, that may be taken to be the linearized harmonic (De Donder) gauge

$$\partial^\sigma h_{\sigma\mu} = \frac{1}{2}\partial_\mu h \tag{174}$$

which simplifies Einstein's equations to

$$\Box h_{\mu\nu} = 0 \tag{175}$$

and support plane wave solutions (gravitational waves). It can be shown that only two independent polarizations of the gravitational waves exist, just like the electromagnetic waves.

Having fixed a gauge to remove the gauge redundancy, one may wonder if there are left-over gauge symmetries that do not modify the harmonic gauge condition (174). These *residual gauge symmetries* do exist and correspond to transformations with functions ξ_μ that satisfy $\Box\xi_\mu = 0$. Let us verify this statement. If we have a configuration $h_{\mu\nu}$ that satisfies (174), let us perform a gauge transformation to obtain the configuration

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{176}$$

and verify that it satisfies

$$\begin{aligned} \partial^\mu h'_{\mu\nu} &= \partial^\mu (h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu) \\ &= \partial^\mu h_{\mu\nu} + \Box\xi_\nu + \partial_\nu \partial^\mu \xi_\mu = \frac{1}{2}\partial_\mu h + \Box\xi_\nu + \partial_\nu \partial^\mu \xi_\mu \\ &= \frac{1}{2}\partial_\mu h' + \Box\xi_\nu \ . \end{aligned} \tag{177}$$

Thus, also $h'_{\mu\nu}$ satisfies the harmonic gauge if $\Box\xi_\nu = 0$.

Keeping the source term in the equations, as in (146), the wave equation in the De Donder gauge takes the form

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu} \tag{178}$$

where $S_{\mu\nu} = \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda{}_\lambda\right)|_{g_{\mu\nu}=\eta_{\mu\nu}}$ is the source term obtained from the "matter" energy-momentum tensor in flat space.

8.1 Electromagnetic waves and physical polarizations

First, let us review the case of electromagnetic waves and show that they have only two degrees of freedom, i.e. the two possible independent polarizations of the waves. We know that the introduction of the four-potential A_μ solves half of the Maxwell equations. The remaining ones in vacuum take the form

$$\partial^\mu F_{\mu\nu} = \partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) = \Box A_\nu - \partial_\nu (\partial^\mu A_\mu) = 0 \tag{179}$$

and are gauge invariant under

$$\delta A_\mu = \partial_\mu \theta \tag{180}$$

with θ an arbitrary function of spacetime. The gauge freedom allows to select the Lorenz gauge $\partial^\mu A_\mu = 0$. In this gauge, the equations simplify to

$$\begin{aligned} \Box A_\mu &= 0 \\ \partial^\mu A_\mu &= 0 \ . \end{aligned} \tag{181}$$

Plane-wave solutions are found using a plane-wave ansatz of the form

$$A_\mu(x) = \epsilon_\mu(k)\, e^{ik\cdot x} + c.c. \tag{182}$$

with an arbitrary wave vector k^μ and arbitrary polarization $\epsilon_\mu(k)$ depending on the wave vector k^μ. The exponent contains the Lorentz invariant phase $k \cdot x = k_\mu x^\mu = \eta_{\mu\nu} k^\mu x^\nu = -k^0 x^0 + \vec{k} \cdot \vec{x}$. The notation *c.c.* stands for complex conjugation and makes the solution real. Plugging this ansatz into the equations (181), one finds a solution when

$$k^\mu k_\mu = 0 \ , \qquad k^\mu \epsilon_\mu(k) = 0 \ . \tag{183}$$

Thus, only three polarizations $\epsilon_\mu(k)$ are possible. However, one of these polarizations is not physical. It is the one with $\epsilon_\mu(k) \sim k_\mu$. It does not carry any electric and magnetic fields. It corresponds to a residual gauge transformation of the empty state $A_\mu = 0$, and therefore it is gauge equivalent to the empty state. This residual gauge transformation has the form given in (180) with

$$\theta(x) \sim e^{ik\cdot x} \tag{184}$$

that satisfies $\Box\theta(x) = 0$, so to maintain the Lorenz gauge condition. For such a reason it is called residual gauge transformation. This gauge transformation gives

$$\delta A_\mu = \partial_\mu \theta \sim i k_\mu\, e^{ik\cdot x} \tag{185}$$

and shows that the polarization $\epsilon_\mu(k) \sim k_\mu$ is not physical: it is gauge equivalent to zero. Only two physical polarizations remain.

Let us exemplify this by considering the motion along the z axis. We can take

$$k^\mu = (k^0, \vec{k}) = (\omega, 0, 0, \omega) \tag{186}$$

which solves $k^\mu k_\mu = 0$ and produces the phase $e^{ik\cdot x} = e^{i\omega(z-t)}$. The two expected polarizations can be taken as

$$\begin{aligned} \epsilon^1_\mu &= (0,1,0,0) \\ \epsilon^2_\mu &= (0,0,1,0) \end{aligned} \tag{187}$$

which satisfy

$$k^\mu \epsilon^i_\mu = 0 \;, \qquad \epsilon^i_\mu \neq \alpha k_\mu \;. \tag{188}$$

The third independent polarization can be taken as the longitudinal polarization given by

$$\epsilon^{long}_\mu = (1,0,0,1) = \frac{1}{\omega} k^\mu \;. \tag{189}$$

This polarization is not physical as it corresponds to a gauge transformation of the vacuum state $A_\mu = 0$, i.e.

$$A'_\mu = \partial_\mu \theta = \epsilon^{long}_\mu e^{ik\cdot x} \qquad\qquad \theta = -\frac{i}{\omega} e^{ik\cdot x} \tag{190}$$

where the function θ satisfies $\Box\theta = 0$ because $k^2 = 0$. Being gauge equivalent to $A_\mu = 0$, it has a vanishing field strength $F_{\mu\nu} = 0$. In particular, it does not carry energy and momentum as the energy-momentum tensor vanishes for vanishing field strength, $T_{\mu\nu} = 0$.

Considering for example the solution with ϵ^1_μ, plugging it into (182), and multiplying with an arbitrary amplitude A_0 one finds

$$\begin{aligned} \vec{A} &= A_0 \cos(\omega z - \omega t)\, \hat{x} \\ \vec{E} &= -\frac{\partial \vec{A}}{\partial t} = E_0 \sin(\omega z - \omega t)\, \hat{x} \\ \vec{B} &= \vec{\nabla} \times \vec{A} = B_0 \sin(\omega z - \omega t)\, \hat{y} \end{aligned} \tag{191}$$

where $E_0 = B_0 = \omega A_0$, and $\hat{x}, \hat{y}, \hat{z}$ the usual unit vectors.

The above plane waves do not carry angular momentum. Plane waves carrying angular momentum are obtained using the circular polarization defined by

$$\epsilon^\pm_\mu = \epsilon^1_\mu \pm i\epsilon^2_\mu \;. \tag{192}$$

They are also said to correspond to the helicity $h = \pm 1$, as in a quantum interpretation they are related to photons carrying angular momentum $\pm\hbar$ along the direction of motion (helicity), and with a wavefunction of the form

$$A_\mu(x) = \epsilon^\pm_\mu(k) e^{ik_\nu x^\nu} = \epsilon^\pm_\mu(k) e^{\frac{i}{\hbar} p_\nu x^\nu} \tag{193}$$

where $p^\mu = \hbar k^\mu$ is the 4-momentum of the photon.

8.2 Gravitational waves and physical polarizations

We can now consider in a similar way the gravitational waves. We have seen that they satisfy the equations

$$\Box h_{\mu\nu} = 0 \tag{194}$$

$$\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h \tag{195}$$

where the second one corresponds to the harmonic gauge. Plane-wave solutions can be found using the ansatz

$$h_{\mu\nu}(x) = \epsilon_{\mu\nu}(k)\, e^{ik\cdot x} + c.c. \tag{196}$$

with k^μ an arbitrary wave vector and $\epsilon_{\mu\nu}$ an arbitrary polarization tensor which we take to depend on the wave vector k^μ. The exponent contains the Lorentz invariant phase $k \cdot x = k_\mu x^\mu = \eta_{\mu\nu} k^\mu x^\nu = -k^0 x^0 + \vec{k} \cdot \vec{x}$. The notation *c.c.* stands for complex conjugation and makes the solution real. Plugging this ansatz into the equations (194) and (195), one finds a solution if

$$k^\mu k_\mu = 0 \;, \qquad k^\mu \epsilon_{\mu\nu}(k) = \frac{1}{2} k_\nu \epsilon^\sigma{}_\sigma \;. \tag{197}$$

The second condition amounts to four linear relations between the 10 components of the polarization tensor. Thus, only six independent polarizations $\epsilon_{\mu\nu}(k)$ are possible. Four of these six polarizations are not physical, namely the ones with $\epsilon_{\mu\nu}(k) \sim k_\mu \epsilon_\nu(k) + k_\nu \epsilon_\mu(k)$ for some $\epsilon_\mu(k)$. They correspond to gauge transformations of the vanishing configuration $h_{\mu\nu}(x) = 0$. It has the form given in (173), but with ξ_μ of the form

$$\xi_\mu(x) \sim \epsilon_\mu(k) e^{ik\cdot x} \tag{198}$$

so that it satisfies $\Box \xi_\mu(x) = 0$. Thus, it does not ruin the harmonic gauge condition (195). It reads

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \sim i(k_\mu \epsilon_\nu(k) + k_\nu \epsilon_\mu(k))\, e^{ik\cdot x} \tag{199}$$

and shows that these types of polarizations are not physical. Note that these unphysical polarizations satisfy the second equation in (197). They are gauge equivalent to zero and therefore can be removed by appropriate gauge transformations. The conclusion is that only two physical polarizations remain.

Let us exemplify this by considering the motion along the z axis. We take

$$k^\mu = (k^0, \vec{k}) = (\omega, 0, 0, \omega) \tag{200}$$

which solves $k^\mu k_\mu = 0$ and produces the phase $e^{ik\cdot x} = e^{i\omega(z-t)}$. The two expected polarizations can be chosen as (using the previous em polarizations)

$$\begin{aligned} \epsilon^{\oplus}_{\mu\nu} &= \epsilon^1_\mu \epsilon^1_\nu - \epsilon^2_\mu \epsilon^2_\nu \\ \epsilon^{\otimes}_{\mu\nu} &= \epsilon^1_\mu \epsilon^2_\nu + \epsilon^2_\mu \epsilon^1_\nu \end{aligned} \tag{201}$$

which indeed satisfy

$$k^\mu \epsilon^i_{\mu\nu} = 0 \;, \qquad \epsilon^i_{\mu\nu} \neq \alpha(k_\mu \epsilon_\nu + k_\nu \epsilon_\mu) \tag{202}$$

for $i = (\oplus, \otimes)$, i.e.

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_{11} & \epsilon_{12} & 0 \\ 0 & \epsilon_{12} & -\epsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;. \tag{203}$$

Considering for example the solution with $\epsilon^{\oplus}_{\mu\nu}$, plugging it into (196), and multiplying with an arbitrary amplitude h_0 one finds

$$h_{\mu\nu}(z-t) = h_0 \cos(\omega z - \omega t)\, \epsilon^{\oplus}_{\mu\nu} \tag{204}$$

that inserted into the linearized metric $g_{\mu\nu}(x)$ gives the line element

$$\begin{aligned} ds^2 &= (\eta_{\mu\nu} + h_{\mu\nu}(z-t))dx^\mu dx^\nu \\ &= -dt^2 + (1 + h_{11}(z-t))dx^2 + (1 - h_{11}(z-t))dy^2 + dz^2 \end{aligned} \tag{205}$$

which is interpretable as deforming periodically invariant lengths as in the figure 1 (from [2]).

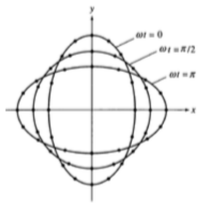

Figure 1: Polarization $\epsilon^{\oplus}_{\mu\nu}$

The polarization $\epsilon^{\otimes}_{\mu\nu}$, does much of the same, but rotated by 45 degrees, see Fig. 2

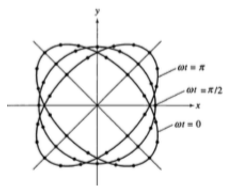

Figure 2: Polarization $\epsilon^{\otimes}_{\mu\nu}$

9 The Schwarzschild solution

Finding exact solutions to Einstein's field equations is very difficult. One strategy is to use conjectured symmetries of possible solutions and use the symmetries to restrict the functional form of the metric that is expected to solve the equations. This simplifies Einstein's equations, which become more tractable and hopefully solvable.

This strategy can be adopted to find the Schwarzschild solution. The Schwarzschild metric is obtained by asking for a *static* and *isotropic* solution of the Einstein equations in vacuum. This situation is realized outside a source that is supposed to be of spherical symmetry and static, like a non-rotating planet. To implement the required symmetries, time translation and rotational invariances, one assumes the existence of coordinates $x^\mu = (t, \vec{x})$ such that the metric takes the form

$$ds^2 = -F(r)\, dt^2 + 2E(r)\, dt\, \vec{x} \cdot \vec{dx} + D(r)\, (\vec{x} \cdot d\vec{x})^2 + C(r)\, d\vec{x} \cdot d\vec{x} \tag{206}$$

where $r = \sqrt{\vec{x} \cdot \vec{x}}$. This is the most general ansatz consistent with the symmetries. The form of the metric can be further simplified by making changes of coordinates. First of all, one may pass to spherical coordinates (r, θ, ϕ) for $\vec{x}$, and using $\vec{x} \cdot \vec{dx} = rdr$ one rewrites

$$ds^2 = -F(r)\, dt^2 + 2E(r) r\, dtdr + D(r) r^2 dr^2 + C(r) \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right], \tag{207}$$

where we have used that the three-dimensional flat metric in radial coordinates takes the form

$$ds^2_{(3)} \equiv d\vec{x} \cdot d\vec{x} = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 = dr^2 + r^2 d\Omega_2 \tag{208}$$

with $d\Omega_2$ the metric on the two-dimensional sphere S^2 with unit radius

$$d\Omega_2 = d\theta^2 + \sin^2\theta \, d\phi^2 \, . \tag{209}$$

Then, one may redefine the time by

$$t \to t' = t + \Phi(r) \tag{210}$$

so that

$$dt' = dt + \frac{d\Phi(r)}{dr} dr \tag{211}$$

and the first two terms inside ds^2 become

$$ds^2 = -F(r)\left(dt' - \frac{d\Phi(r)}{dr} dr\right)^2 + 2E(r) r \left(dt' - \frac{d\Phi(r)}{dr} dr\right) dr + \dots \tag{212}$$

that rearranges to

$$ds^2 = -F(r)\, dt'^2 + 2\left[rE(r) + F(r)\frac{d\Phi(r)}{dr}\right] dt' dr - \left[F(r)\left(\frac{d\Phi(r)}{dr}\right)^2 + 2rE(r)\frac{d\Phi(r)}{dr}\right] dr^2 + \dots \tag{213}$$

Now one can fix the function $\Phi(r)$ to satisfy

$$\frac{d\Phi(r)}{dr} = -\frac{rE(r)}{F(r)} \tag{214}$$

so that the mixed term $dt'dr$ vanishes. Then, the remaining part takes the form

$$ds^2 = -F(r)\, dt'^2 + G(r) dr^2 + C(r)\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right] \tag{215}$$

where

$$G(r) = r^2 \left(D(r) + \frac{E^2(r)}{F(r)}\right) . \tag{216}$$

Now one redefines the radius $r \to r'$ by requiring that

$$r'^2 = C(r) r^2 \tag{217}$$

so that r'^2 is identified with the radius of a sphere S^2 and leads to the relation

$$2r'dr' = \Big(2rC(r) + r^2 C'(r)\Big) dr = 2rC(r)\left(1 + \frac{rC'(r)}{2C(r)}\right) dr \tag{218}$$

implying

$$dr^2 = \frac{dr'^2}{C(r)\left(1 + \frac{rC'(r)}{2C(r)}\right)^2} \tag{219}$$

In terms of the new radius, one gets the so-called *standard form* of the metric

$$ds^2 = -B(r')\, dt'^2 + A(r') dr'^2 + r'^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \tag{220}$$

with

$$
\begin{aligned}
B(r') &= F(r) \\
A(r') &= \Big(1+\frac{G(r)}{C(r)}\Big)\Big(1+\frac{rC'(r)}{2C(r)}\Big)^{-2} .
\end{aligned}
\tag{221}
$$

Dropping the primes, one has found the static and isotropic metric in the standard form

$$
\boxed{ds^2 = -B(r)\,dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)} . \tag{222}
$$

<u>Einstein's equations</u>

The metric in the standard form can be inserted into the Einstein's equations which simplify dramatically. From (222), we see that the metric tensor $g_{\mu\nu}$ can be written as

$$
g_{\mu\nu} = \begin{pmatrix} -B(r) & 0 & 0 & 0 \\ 0 & A(r) & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{223}
$$

and its inverse $g^{\mu\nu}$

$$
g^{\mu\nu} = \begin{pmatrix} -\frac{1}{B(r)} & 0 & 0 & 0 \\ 0 & \frac{1}{A(r)} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} . \tag{224}
$$

We can now compute the components of the connection $\Gamma^\lambda_{\mu\nu}$.

Starting with the ones in $\Gamma^r_{\mu\nu}$, we compute

$$
\begin{aligned}
\Gamma^r_{rr} &= \frac{1}{2} g^{r\mu}(2\partial_r g_{r\mu} - \partial_\mu g_{rr}) = \frac{1}{2} g^{rr}\partial_r g_{rr} \\
&= \frac{1}{2A(r)}\partial_r A(r) = \frac{A'(r)}{2A(r)}
\end{aligned}
\tag{225}
$$

where the prime indicates the derivative with respect to r. Note that, in the calculation, the index r is fixed, while μ runs over the values (t, r, θ, ϕ). Similarly, one finds the other nonvanishing components and obtains the following list

$$
\Gamma^r_{rr} = \frac{A'(r)}{2A(r)} , \qquad \Gamma^r_{\theta\theta} = -\frac{r}{A(r)} , \qquad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{A(r)} , \qquad \Gamma^r_{tt} = \frac{B'(r)}{2A(r)} . \tag{226}
$$

Continuing this way, one finds

$$
\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} , \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \tag{227}
$$

$$
\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} , \qquad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta \tag{228}
$$

$$
\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{B'(r)}{2B(r)} . \tag{229}
$$

At this stage, one must compute the non-vanishing components of the Ricci tensor $R_{\mu\nu}$ given by

$$
\begin{aligned}
R_{\mu\nu} &= R_{\lambda\mu}{}^{\lambda}{}_{\nu} = \overline{\nabla}_\lambda \Gamma^\lambda_{\mu\nu} - \overline{\nabla}_\mu \Gamma^\lambda_{\lambda\nu} \\
&= \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\lambda\nu} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} .
\end{aligned}
\tag{230}
$$

A direct calculation delivers

$$
\begin{aligned}
R_{rr} &= -\frac{B''}{2B} + \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{r}\frac{A'}{A} \\
R_{\theta\theta} &= 1 - \frac{r}{2A}\left(\frac{B'}{B} - \frac{A'}{A}\right) - \frac{1}{A} \\
R_{\phi\phi} &= \sin^2\theta \; R_{\theta\theta} \\
R_{tt} &= \frac{B''}{2A} - \frac{1}{4}\frac{B'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{r}\frac{B'}{A} \; .
\end{aligned}
\tag{231}
$$

Thus, the independent equations to be solved are

$$
R_{rr} = R_{\theta\theta} = R_{tt} = 0 \; . \tag{232}
$$

To solve them, one notices that

$$
\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = \frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \; . \tag{233}
$$

This implies

$$
\frac{A'}{A} + \frac{B'}{B} = 0 \quad \to \quad A'B + B'A = 0 \quad \to \quad AB = constant \tag{234}
$$

and assuming that $A(r)$ and $B(r)$ both tend to 1 for $r \to \infty$, we get

$$
A = \frac{1}{B} \; . \tag{235}
$$

At this stage, from $R_{\theta\theta} = 0$, one finds

$$
R_{\theta\theta} = 1 - B'r - B = 0 \quad \to \quad \partial_r(rB) = 1 \quad \to \quad rB = r + constant \tag{236}
$$

and thus

$$
B(r) = 1 + \frac{constant}{r} \; . \tag{237}
$$

From the Newtonian limit $-g_{00} = 1 - \frac{2GM}{r} = B(r)$, one finds the constant to be $-2GM$, so that

$$
B(r) = 1 - \frac{2GM}{r} \; . \tag{238}
$$

This, leads to the Schwarzschild solution

$$
\boxed{ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)} \; . \tag{239}
$$

The same solution can be obtained by relaxing the hypothesis of time invariance (staticity). This is captured by *Birkhoff's theorem*, that states that any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat. This theorem guarantees that the assumption of staticity may be dropped, and still the exterior solution for the spacetime metric outside a spherical, nonrotating, gravitating body must be given by the Schwarzschild metric.

A striking feature of the Schwarzschild solution is that the metric presents singularities in the strong field regime at $r = 2GM$ and $r = 0$. It is in general not obvious to determine the nature of the singular behavior: it could be due to *i)* a breakdown of the coordinate system used to describe the solution, which would be otherwise nonsingular, *ii)* a true singularity of the spacetime as captured by some scalar observable in the curvature. We note that the singular behavior at $r_S = 2GM$ numerically takes the value

$$r_S = \frac{2GM}{c^2} = 3\frac{M}{M_\odot}\text{km} \tag{240}$$

which for ordinary bodies, like planets or stars, is well inside their physical radius (e.g. $r_\odot \sim 7\ 10^5\,\text{km}$). The Einstein equations in vacuum are no longer valid inside the physical radius, as one must modify the Einstein equations by taking considering the nonvanishing energy-momentum tensor of the constituents of the body so that the solution will be modified inside the physical radius.

For a proton, the Schwarzschild radius equals $r_S = 2Gm_p \sim 10^{-50}$cm, much smaller than the proton radius $r_p \sim 10^{-13}$cm $= 1$ fm.

However, a spherical body might go under a gravitational collapse, that may expose the singular points which must therefore be interpreted physically. It turns out that the point $r = r_S$, the Schwarzschild radius, is not a singular point of the geometry. For example, the scalar quantity

$$R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} = \frac{48(GM)^2}{r^6} \tag{241}$$

does not show any singular behavior at $r = r_S$. This point $r = r_S$ indicates the location of the so-called *event horizon*, a two-dimensional null surface. It is a surface of infinite red-shift for an observer at rest in these coordinates.

On the other hand, the above curvature scalar diverges at $r = 0$, which is therefore a true singularity of the Schwarzschild spacetimes.

Finally, let us mention that, in the presence of a cosmological constant Λ, the Schwarzschild solution is modified to

$$ds^2 = -\Big(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\Big)dt^2 + \Big(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\Big)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\ . \tag{242}$$

Now, on top of the length scale of the event horizon $r_S = 2GM$, there appears a cosmological length scale given by

$$r_C = \frac{1}{\sqrt{\Lambda}}\ . \tag{243}$$

The cosmological constant Λ parameterizes the effects of the vacuum energy of the universe, related to the concept of dark energy. Its smallness makes the cosmological length scale r_C very big, of the order of 10^{26} m, the radius of the visible universe. There is a range $r_S < r < r_C$ for the coordinate r that makes the extended solution approximately flat. In practice, this additional length scale is neglected, and much studies are devoted to the original Schwarzschild solution that asymptotically becomes a flat spacetime. That is, the radius r_C is effectively sent to infinity, equivalent to setting $\Lambda = 0$.

10 Black holes

The Schwarzschild solution indicates the existence of an event horizon and leads to the concept of a black hole. The recommended treatment is the one presented in [2], see chapter 8, in particular pages 324–342.

Let us present here a few remarks. We first wish to discuss in more details the singularities at $r = 2GM$ and $r = 0$. In general, it is not an easy task to determine the true nature of eventual singularities in the metric. As already mentioned, they may be artifacts of the coordinate systems or else could indicate a true singularity of spacetime.

Let us present two examples that help in building intuition.

Consider the two-dimensional metric

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2 \tag{244}$$

with coordinate ranges $x \in \mathbb{R}$ and $t > 0$. There is a singularity at $t = 0$. However, a change of the coordinate $t \to t' = \frac{1}{t}$ leads to $dt' = -\frac{1}{t^2}dt$ and brings the metric into the form

$$ds^2 = -dt'^2 + dx^2 \tag{245}$$

that is recognized as the flat metric of Minkowski spacetime. Thus, the singularities are eliminated. The original coordinates are seen to correspond to only a portion of the Minkowski space and the singular points at $t = 0$ (the x axis of the coordinate frame) are seen as corresponding to the $t' \to \infty$ part of Minkowski space. In the new (t', x) coordinates, one may extend the spacetime by letting $-\infty < t' < \infty$. This makes the space *geodesically complete*, that is a space in which all geodesics extend to arbitrarily large values of their affine parameter, see fig. 3.

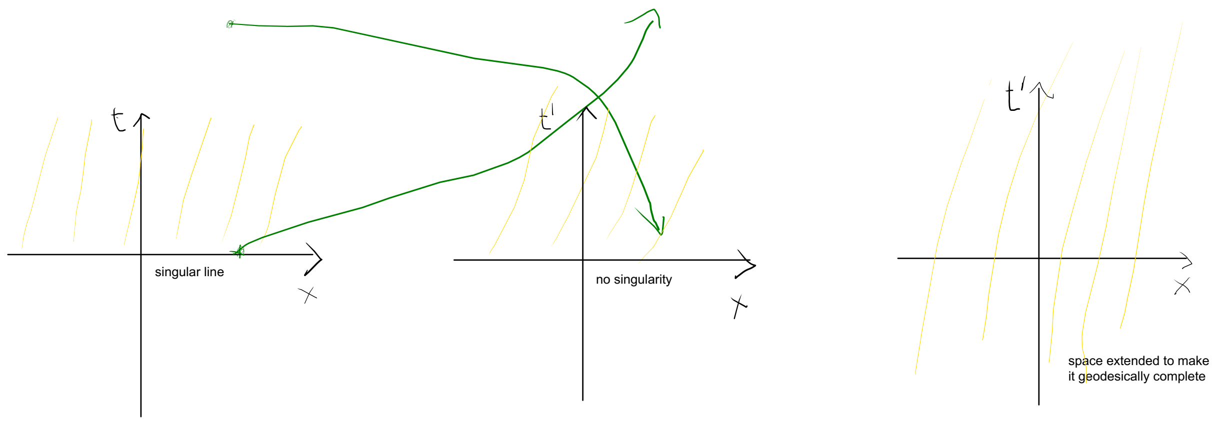

Figure 3: *Flat space in different coordinates described in example 1.*

A second example is the so-called Rindler spacetime. It is defined by the metric

$$ds^2 = -x^2dt^2 + dx^2 \tag{246}$$

with coordinate ranges $t \in \mathbb{R}$ and $x > 0$. There is a singularity at $x = 0$. More precisely, the determinant of the metric vanishes and the inverse metric is singular at $x = 0$. One may check that geodesics terminate with a finite length at $x = 0$, so that the space is not geodesically complete. On the other hand, the calculation of curvature scalars does not signal any bad behavior at $x = 0$. In fact, all components of the Riemann tensor vanish, suggesting that one is dealing with a portion of Minkowski space.

Let us look for a change of coordinates that makes the true nature of the Rindler spacetime manifest. One could proceed with a trial-and-error method. A more satisfactory geometrical approach is to consider geodesics that head toward the singularity, and use the affine parameters of the geodesics as coordinates (like the Cartesian coordinates, that are built from geodesics).

In two dimensions, it is useful to use null coordinates. For the two-dimensional Minkowski spacetime with Cartesian coordinates (t, x) and metric

$$ds^2 = -dt^2 + dx^2 \tag{247}$$

the null coordinates (u, v) are defined by

$$\begin{cases} u = t - x \\ v = t + x \end{cases} \tag{248}$$

with inverse transformation given by

$$\begin{cases} t = \frac{1}{2}(v + u) \\ x = \frac{1}{2}(v - u) \ . \end{cases} \tag{249}$$

In these new coordinates, the metric takes the form

$$ds^2 = -du\, dv \ . \tag{250}$$

The null coordinates are also called light-cone coordinates, see fig. 4.

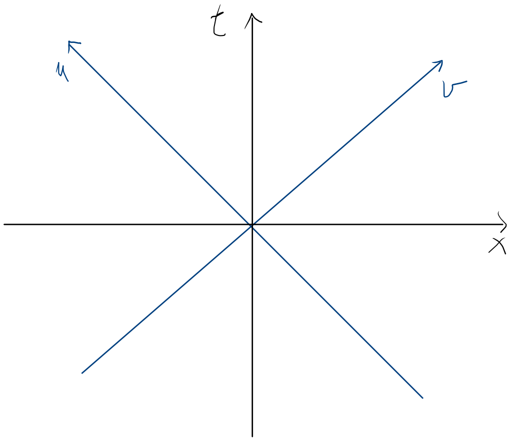

Figure 4: *Minkowski space with Cartesian (t, x) and null (u, v) coordinate frames*

Similarly, let us look for null geodesics in Rindler space. A geodesic is described the the functions $x^\mu(\lambda)$ where λ is an affine parameter. We denote by $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ the tangent vector to the geodesic. The geodesic satisfies the equation $\frac{D}{d\lambda}\dot{x}^\mu = 0$. Considering the Rindler metric (246), we impose that the tangent vector be lightlike so that the corresponding geodesic is a null geodesic. This leads to the equation

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -x^2\dot{t}^2 + \dot{x}^2 = 0 \ . \tag{251}$$

Eliminating the affine parameter λ from the equation gives

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2} \qquad \rightarrow \qquad dt = \pm\frac{dx}{x} \tag{252}$$

which integrates to

$$t = \pm \ln x + constant \ . \tag{253}$$

This suggests the introduction of "null coordinates" as the constants in (253) that parametrize the different null geodesics, namely

$$\begin{cases} u = t - \ln x \\ v = t + \ln x \ . \end{cases} \tag{254}$$

In the (u, v) coordinates, the metric becomes

$$ds^2 = -e^{v-u}\, du\, dv \ . \tag{255}$$

This transformation does not achieve the goal of analyzing the singularity at $x = 0$, as the coordinates u and v diverge at $x = 0$, as seen from eq. (254). However, now one can further reparametrize the coordinates to obtain new coordinates (U, V) defined by

$$\begin{cases} U = -e^{-u} \\ V = e^{v} \end{cases} \tag{256}$$

which puts the metric in the form

$$ds^2 = -dUdV \ . \tag{257}$$

This metric may be directly compared with (250) and recognized to be the metric of Minkowski spacetime in null coordinates. Thus, setting

$$\begin{cases} U = T - X \\ V = T + X \end{cases} \quad \text{with inverse} \quad \begin{cases} T = \frac{1}{2}(V + U) \\ X = \frac{1}{2}(V - U) \end{cases} \tag{258}$$

leads to the standard Minkowski metric

$$ds^2 = -dT^2 + dX^2 \ . \tag{259}$$

The original coordinates (t, x) are given in terms of the final coordinates (T, X) by

$$\begin{cases} x = (X^2 - T^2)^{1/2} \\ t = \tanh^{-1} \frac{T}{X} \end{cases} \tag{260}$$

with inverse transformation

$$\begin{cases} T = x \sinh t \\ X = x \cosh t \ . \end{cases} \tag{261}$$

We see that a fixed x we have a branch of a hyperbola in the $T - X$ plane described by $X^2 - T^2 = x^2$. Let us verify these expressions once more by following the various changes of variables. Going backward, we compute starting from (258)

$$T = \frac{1}{2}(V + U) = \frac{1}{2}(e^v - e^{-u}) = \frac{1}{2}(e^{t+\ln x} - e^{-t+\ln x}) = \frac{1}{2}x(e^t - e^{-t}) = x \sinh t \tag{262}$$

and similarly

$$X = \frac{1}{2}(V - U) = \frac{1}{2}(e^v + e^{-u}) = \frac{1}{2}(e^{t+\ln x} + e^{-t+\ln x}) = \frac{1}{2}x(e^t + e^{-t}) = x \cosh t \ . \tag{263}$$

The (T, X) coordinates are shown in fig. 5, which is taken from ref. [3]

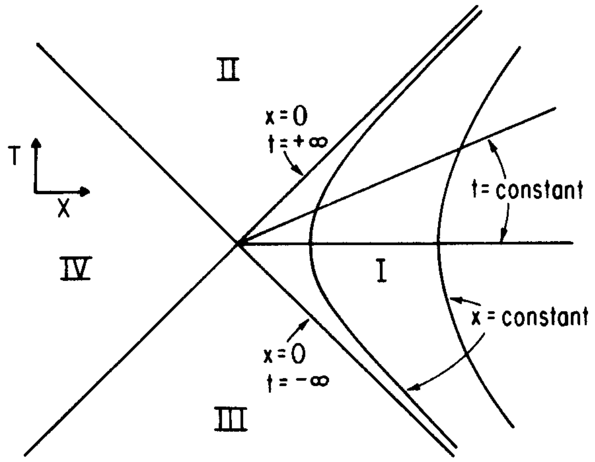

Fig. 6.8. Rindler spacetime, displayed as the "wedge," I, of two-dimensional Minkowski spacetime.

Figure 5: *Rindler space*

The Rindler space is the wedge $X > |T|$ of the Minkowski space, labeled as region I in fig. 5. The coordinate lines at $x = constant$ are interpreted as the worldlines of observers that are at rest in Rindler coordinates. These worldlines are those of accelerated observers once seen from the Cartesian coordinate frame, and they corresponds to branches of hyperbolae.

The singularities at $x = 0$ coincide with part of the light cone which is depicted in fig. 5, as for $x = 0$ the hyperbola degenerates to the light cone. It corresponds to the location of an event horizon for the Rindler observers, i.e. those observes that are at rest in Rindler coordinates. As said, these observers perform a hyperbolic accelerated motion in the Cartesian coordinate frame. They cannot access the region beyond the horizon. The event horizon is a surface of infinite red shift for Rindler observers. Note that:
i) Rindler observers cannot receive signals from events with $T \geq X$ (region II of fig. 5)
ii) Rindler observers cannot send signals to events with $T \leq -X$ (region III of fig. 5).

There is nothing singular in the Rindler geometry: Rindler space is that portion of Minkowski space accessible to observers that perform a hyperbolic accelerated motion in the Cartesian coordinate frame.

Of course, a Rindler observer could decide to stop accelerating (so it ceases to be a "Rindler observer") and thus be able to cross the horizon, see figure 6.

Schwarzschild metric and the black hole

We can now proceed to analyze the Schwarzschild metric that in Schwarzschild coordinates reads

$$ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (264)$$

We have already noted that the singularity at $r = r_S \equiv 2GM$ is not a true singularity of the Schwarzschild spacetime. It gives the location of a surface of infinite redshift. A static observer at $r > r_S$, for example, one that is sitting at very big values of r so that he/she is in a portion of space that is almost flat, will not be able to see any object crossing the surface at $r = r_S$: the static Schwarzschild observer measures an infinite amount of time for any object headed to $r = r_S$ to reach it. This surface is named *event horizon*.

Of course, moving objects can reach the event horizon and cross it, but the static Schwarzschild

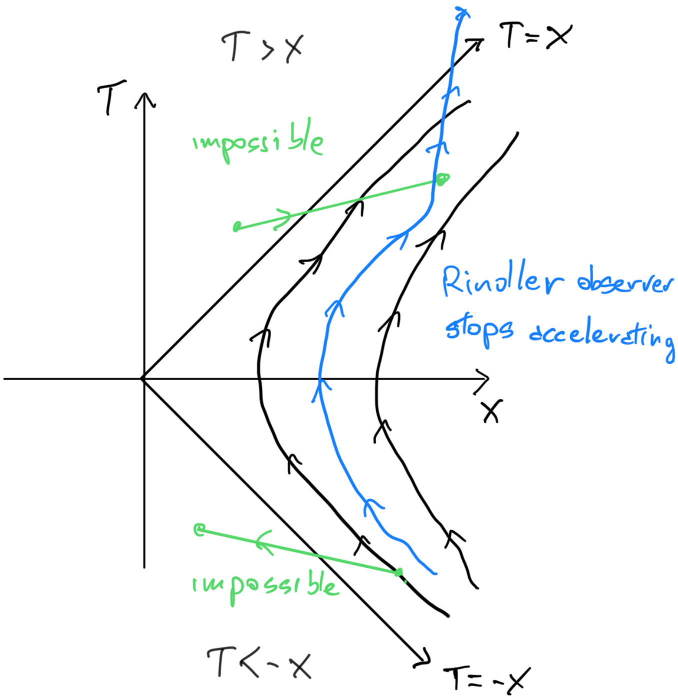

Figure 6: *Rindler space*

observer cannot see this fact happening. Moreover, once an object enters the event horizon, it cannot escape anymore and is bound to reach the singularity at $r = 0$ in a finite proper time. This defines a *black hole*, the region inside the event horizon from which nothing can escape because of the strong gravitational attraction. At the singularity, the worldline of any particle that crossed the horizon stops. The Schwarzschild spacetime is not geodesically complete. Note that inside the event horizon, the role of t and r are reversed: t turns into a spacelike coordinate as $g_{tt} = -(1 - \frac{r_S}{r})$ becomes positive, while r turns into a timelike coordinate as $g_{rr} = (1 - \frac{r_S}{r})^{-1}$ becomes negative.

There is *classical analogy* for the black hole. In Newtonian gravity, a non-relativistic particle of mass m near a planet of mass M has a potential energy given by $V = -\frac{GMm}{r}$, with r the distance of the particle from the planet. The kinetic energy of the particle has the usual form $T = \frac{1}{2}mv^2$, so that the particle can escape to $r = \infty$ if it has enough kinetic energy to overcome the negative potential energy. There is an escape velocity which is independent of the particle's mass

$$\frac{1}{2}mv^2 = \frac{GMm}{r} \qquad \to \qquad v^2 = \frac{2GM}{r} \ . \tag{265}$$

Applying this to a light particle with velocity $v = c = 1$ (in our units), we get a minimal radius from which light can escape

$$r = 2GM \ . \tag{266}$$

Planets with a smaller radius would appear black because light cannot escape the planet's gravitational attraction (Michell 1784, Laplace 1796). This classical analogy gives the correct Schwarzschild radius, but the physical picture of how the light ray behaves is different in general relativity.

To appreciate better the structure of the Schwarzschild solution, let us study the null geodesics travelled by light rays. They define the light cones at the various points. We consider only radial light signals (i.e. those ones with fixed θ and ϕ coordinates). They must satisfy

The forward light cones near and inside a black hole. As $r \to \infty$, the light cone assumes its usual shape and direction; that is, $dr/dt = \pm 1$. The curve *ABBC* is the worldline of an ingoing light signal.

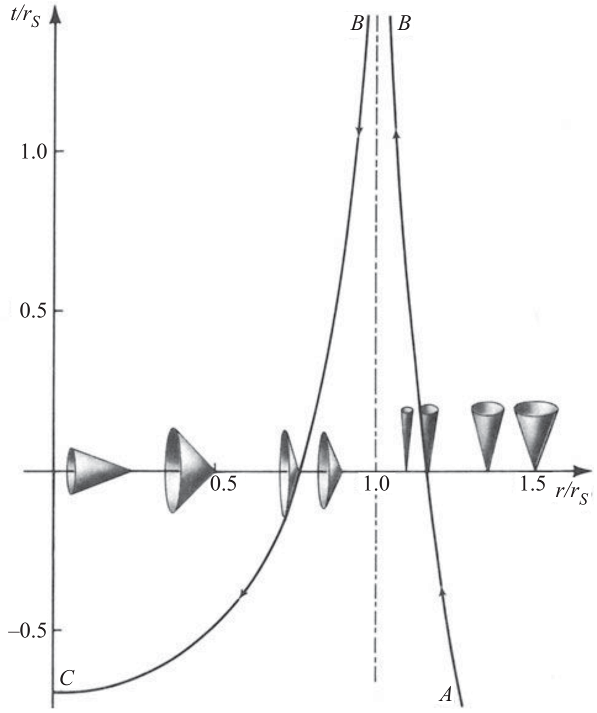

Figure 7: *Light cones in the Schwarzshild coordinates*

$ds^2 = 0$. Denoting $r_S = 2GM$ we have

$$ds^2 = -\left(1 - \frac{r_S}{r}\right) dt^2 + \left(1 - \frac{r_S}{r}\right)^{-1} dr^2 = 0 \tag{267}$$

that implies

$$\frac{dr}{dt} = \pm\left(1 - \frac{r_S}{r}\right) \tag{268}$$

which identifies the directions of the light cones. They are depicted in fig. 7, taken from [2].

Note that a photon near $r = r_S$, say at $r = r_S + \epsilon$, with infinitesimal $\epsilon > 0$, has a velocity that tends to zero

$$\frac{dr}{dt} = \pm\left(1 - \frac{r_S}{r}\right) = \pm\left(1 - \frac{r_S}{r_S + \epsilon}\right) \approx \pm\frac{\epsilon}{r_S} \quad \longrightarrow \quad 0\,. \tag{269}$$

In particular, a radially moving photon stands still on the surface horizon.

As mentioned, inside the horizon, r and t exchange their roles. All signals are lead inevitably to hit the singularity at $r = 0$, as indicated by the light cones depicted in fig. 7.

Kruskal extension

The Kruskal coordinates are a useful set of coordinates that are not singular at the event horizon. They can be used to find the maximal extension of the Schwarzschild geometry. In such coordinates, all geodesics either extend to infinity in both directions or initiate/end up at a singularity. The picture emerging is the one shown in fig. 8, taken from [3].

The Kruskal coordinates do not show explicitly the staticity of the external part of the black hole. They are useful to study the strong field region, while the asymptotically flat region is better appreciated in the original Schwarzshild coordinates.

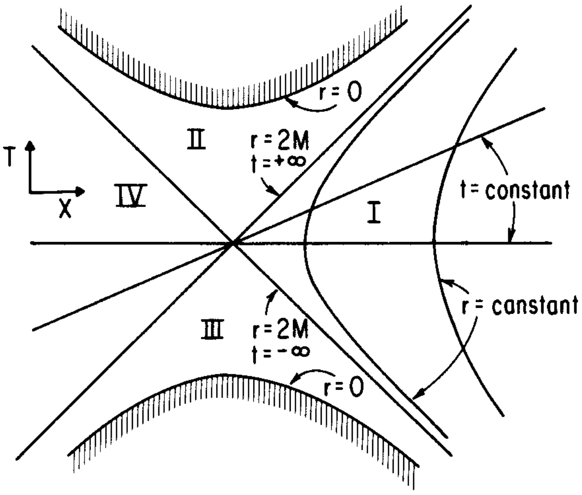

Figure 8: *The Kruskal extension of Schwarzschild spacetime*

Let us describe Kruskal coordinates , using the intuition developed in the treatment of the Rindler space. Let us discuss the relevant t, r part of the coordinates and look for radial null geodesics whose tangent vectors $\dot{x}^\mu(\lambda)$ satisfy (we now set $G = 1$)

$$g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 = 0 \tag{270}$$

in a way analogous to eq. (251) of the Rindler case. Eliminating the affine parameter λ from the equation gives

$$\left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2M}{r}\right)^{-2} \qquad \to \qquad dt = \pm\left(1 - \frac{2M}{r}\right)^{-1} dr \tag{271}$$

which integrates to

$$t = \pm\, r_* + constant \tag{272}$$

where the "Regge-Wheeler tortoise coordinate" r_* is defined by

$$r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right) \tag{273}$$

which indeed satisfies

$$\frac{dr_*}{dr} = \left(1 - \frac{2M}{r}\right)^{-1} . \tag{274}$$

We can thus define the null coordinates u, v by

$$\begin{cases} u = t - r_* \\ v = t + r_* \end{cases} \tag{275}$$

in analogy with eq. (248), and discover that the metric becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du\, dv . \tag{276}$$

where r must be viewed as a function of u and v, defined implicitly by

$$r + 2M \ln\left(\frac{r}{2M} - 1\right) = r_* = \frac{1}{2}(v - u) . \tag{277}$$

Using this last equation we rewrite the metric (276) as

$$ds^2 = -\frac{2Me^{-r/2M}}{r} e^{(v-u)/4M} du\, dv \; . \tag{278}$$

where we have factored the metric into a piece, $e^{-r/2M}/r$, which is non singular as $r \to 2M$ (i.e. when $u \to \infty$ and $v \to -\infty$) times a piece with a simple u, v dependence. Comparison with the Rindler case suggests the use of new coordinate U and V

$$\begin{cases} U = -e^{-u/4M} \\ V = e^{v/4M} \end{cases} \tag{279}$$

which puts the metric in the form

$$ds^2 = -\frac{32M^3 e^{-r/2M}}{r} dU\, dV \; . \tag{280}$$

Now, there is no longer a singularity at $r = 2M$ (i.e. at $U = 0$ and $V = 0$) and thus we can extend the Schwarzschild spacetime solution by allowing U and V to take all possible values compatible with $r > 0$. The final transformation

$$\begin{cases} T = \frac{1}{2}(U + V) \\ X = \frac{1}{2}(V - U) \end{cases} \tag{281}$$

leads to

$$ds^2 = -\frac{32M^3 e^{-r/2M}}{r} (-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \; . \tag{282}$$

The relation between the old and new coordinates is given by

$$\begin{cases} \left(\frac{r}{2M} - 1\right) e^{r/2M} = X^2 - T^2 \\ \frac{t}{2M} = \ln\left(\frac{X+T}{X-T}\right) = 2\tanh^{-1}\frac{T}{X} \; . \end{cases} \tag{283}$$

The metric in these Kruskal coordinates is depicted in fig. 8.

Note that the parts III and IV of the Kruskal extension of Schwarzshild spacetime are not expected to be physical.

Region III contains the so-called "*white hole*". Region IV is asymptotically flat and it may be interpreted as a different universe. Alternatively, it could be regarded as a region of our universe, but it is located outside the light-cone of region I, so communication between these two regions is impossible. Astronauts from these two regions could cross their respective horizons, meet briefly, and then end up in the singularity. These two regions are joined by the so-called *wormhole*, sometimes named Einsten-Rosen bridge, the geometry that can be observed by looking at the diagram in fig. 8. Nowaday, the main consensus is that regions III and IV of the Kruskal extension of Schwarzshild spacetime are not expected to be physical, but some researcher continue to analyze their possible implications.

Black holes can be formed by collapsing matter, and some picture of the spacetime diagrams are presented in figures 9 and 10, also taken from the textbook [3]. From this perspective, region III and IV are not relevant, they are "covered up" by the infalling matter and thus replaced by a normal spacetime region.

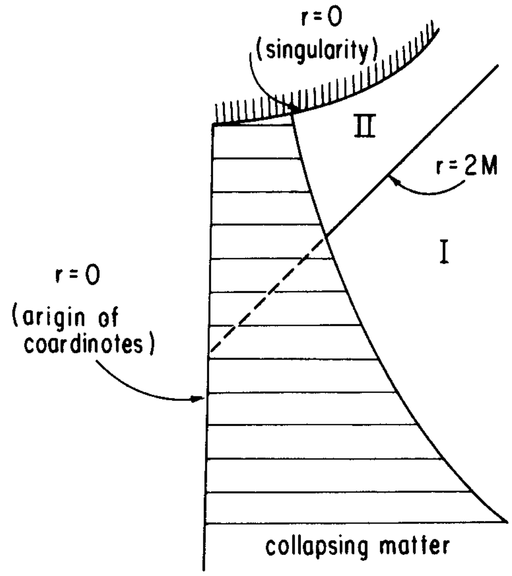

Fig. 6.11. The spacetime resulting from the complete gravitational collapse of a spherical body. All of regions III and IV of the extended Schwarzschild spacetime (Fig. 6.9) are "covered up" by the collapsing matter. However, (part of) the black hole region II is produced.

Figure 9:

References

[1] S. Weinberg, "Gravitation and Cosmology", John Wiley & Sons, 1972.

[2] H. Ohanian and R. Ruffini, "Gravitation and Spacetime", Cambridge University Press, 2013.

[3] R. Wald, "General Relativity", The University of Chicago Press, 1984.

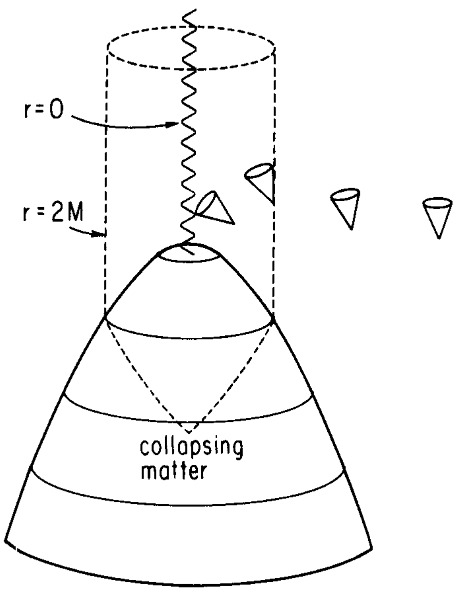

Fig. 6.12. Another representation of the spacetime of Figure 6.11. Here, one of the two suppressed spatial dimensions is restored, so each of the circles shown on the collapsing body corresponds to the 2-sphere surface of the body at an instant of time. However, the light cones no longer are represented by 45° lines. Indeed, the spacelike nature of the singularity and the inevitable capture by the singularity of any particle or light ray in the region $r < 2M$ is illustrated here by the "tipping over" of the future light cones in the strong field region.

Figure 10: