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Preliminaries on path integrals and classical mechanics

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1 Brief introduction to path integrals

Quantization can be introduced in two equivalent ways:
- operator formalism (canonical quantization, Hilbert space, linear operators, etc ..)
- path integrals (functional integrals).

Path integrals have been introduced in quantum mechanics by Feynman in 1948, but until about 1970 they did not meet with much success, and the operatorial methods of canonical quantization were still the most widespread. In 1970 the success of gauge theories in the development of the Standard Model of particle physics gave a strong impulse to path integral methods. In fact, quantization of gauge theories is much more clear and elegant if performed with path integrals. Furthermore, path integrals indicate a way of relating a quantum field theory in D spacetime dimensions ($D-1$ spaces and 1 time) to the statistical mechanics of a system in D space dimensions. This link has given rise to a way of thinking and defining field theories using statistical mechanics and renormalization group ideas introduced by Wilson and others (lattice theories). Nowadays, it is convenient to master both methods, as according to the problem at hand one may find one formalism more convenient then the other, even though they are supposed to be equivalent.

To introduce path integrals let us follow Feynman and consider the two-slit experiment for the electron. The standard treatment used to explain the behaviour of an electron which passes through the two slits of a barrier and creates a figure of interference on a screen employs the *wave* nature of the electron together with the Huygens principle for calculating the interference pattern from the elementary waves that originate from the slits.

Feynman proposes an alternative description. He suggests to keep thinking of the electron as to a *particle* that however can accomplish both trajectories, each one with a certain "amplitude". The total amplitude A_{tot} is defined as the sum of the single amplitudes, and its square is related to the probability that the electron is revealed at a given point of the screen. Moreover, the elementary amplitude for each possible trajectory is related in a simple way to the classical action evaluated on the trajectory itself: Feynman, inspired by previous considerations of Dirac, associates to each trajectory an amplitude of unit norm (so that all trajectories "weigh" democratically the same way) and with phase equal to the value of the action S in unit of $\hbar$. In formulas:

$$\begin{aligned} A_{tot} &= A(c_1) + A(c_2) + ... + A(c_n) \\ A(c_n) &= e^{\frac{i}{\hbar} S(c_n)} \\ P(\text{probability}) &\sim |A_{tot}|^2 \end{aligned} \tag{1}$$

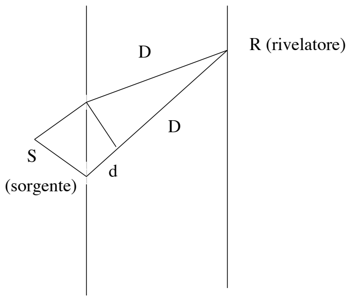
i.e.

$$A_{tot} = \sum_n e^{\frac{i}{\hbar}S(c_n)} \ . \tag{2}$$

An important part of this proposal is the identification of the phase of each elementary amplitude with the action of the system. Let us make a rough test of this proposal. Recall that the action of a free particle is given by the time integral of its kinetic energy

$$S[q] = \int_0^T dt \ \frac{1}{2} m\dot{q}^2 \ . \tag{3}$$

We simplify the problem by assuming that the velocity is constant along the two trajectories.



Using the quantities shown in the figure we estimate (considering $d \ll D$)

$$S(c_1) = \frac{m}{2}\frac{D^2}{T^2}T = \frac{m}{2}\frac{D^2}{T} \tag{4}$$

$$S(c_2) = \frac{m}{2}\frac{(D+d)^2}{T} = \frac{m}{2}\frac{D^2}{T} + \frac{mDd}{T} + O(d^2) \tag{5}$$

$$= S(c_1) + pd + O(d^2) \tag{6}$$

where $p = \frac{mD}{T}$ is the momentum of the electron. Therefore

$$\begin{aligned} A_{tot} &= A(c_1) + A(c_2) = \mathrm{e}^{\frac{i}{\hbar}S(c_1)} + \mathrm{e}^{\frac{i}{\hbar}S(c_2)} = A(c_1)\Big(1 + \mathrm{e}^{\frac{i}{\hbar}[S(c_2)-S(c_1)]}\Big) \\ &= A(c_1)\Big(1 + \mathrm{e}^{\frac{i}{\hbar}pd + O(d^2)}\Big) \ . \end{aligned} \tag{7}$$

We see that the maximum probability to reveal the electron on the screen is when

$$\mathrm{e}^{\frac{i}{\hbar}pd} = 1 \tag{8}$$

and thus

$$\frac{pd}{\hbar} = 2\pi n \quad (n \text{ integer}) \quad \rightarrow \quad \frac{p}{h}d = n \quad (n \text{ integer}). \tag{9}$$

One can interpret this condition as defining a wavelength $\lambda = \frac{h}{p}$, so that when d contains an integer number of times such wavelengths there is constructive interference. The de Broglie relation is obtained by this rudimentary "path integral", and suggests that it contains the essential elements of quantum mechanics. The number of slits can be increased, as well as the number of intermediate screens, to have the particle performing all possible paths from the initial point to the final point of observation, thus creating a path integral for the total amplitude.

The action is used in an essential way

$$S[q] = \int_{t_i}^{t_f} dt\ L(q, \dot{q})\ . \tag{10}$$

The classic path is the one that minimizes the action

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0\ . \tag{11}$$

In quantum mechanics, the transition amplitude is obtained by using the action $S[q]$ for any possible path

$$A = \sum_n e^{\frac{i}{\hbar}S(c_n)} \equiv \int Dq\ e^{\frac{i}{\hbar}S[q]}\ . \tag{12}$$

The final notation introduced here is that of the path integral or functional integral: $S[q]$ is a functional, i.e. a function of the functions $q(t)$, that indicate the possible "paths" of the system, and the symbol $\int Dq$ indicates the formal integration over the space of paths $\{q(t)\}$. Various mathematical subtleties on how to define exactly the path integration are still open, nevertheless path integrals have become one of the main tools to study quantum systems.

In this formulation the classic limit is intuitive: macroscopic systems have large values of the action S in units of $\hbar$, the quantum of action. Small variations of a path induce phase variations $\frac{i}{\hbar}\delta S[q]$ much bigger with respect to π, and the amplitudes of nearby paths cancel by destructive interference. This happens except at the point in which the action has a minimum, $\delta S = 0$, which identifies the classic trajectory. Trajectories close to the classical one have amplitudes that add up coherently since the phase does not vary: the functional integral is dominated by the classic path!

2 Action principle

Let us briefly review the action principle, in mechanics and in field theories, considering the case of a particle. The main purpose is to underline its relation to canonical quantization, and to stress the relevance of symmetries. As anticipated, the action is essential for the path integral quantization.

2.1 Lagrangian formalism

Consider a non-relativistic particle of mass m that moves in a single dimension with coordinate q, and subject to a conservative force $F = -\frac{\partial}{\partial q}V(q)$. Newton's equations of motion reads

$$F = m\ddot{q} \tag{13}$$

and can be derived from an action principle. The action is a functional of the trajectory of the particle $q(t)$ (the dynamical variable of the system) and associates a real number to each function $q(t)$. In general, physical systems are described by an action of the type

$$S[q] = \int_{t_i}^{t_f} dt\, L(q, \dot{q})\ , \qquad L(q, \dot{q}) = \frac{m}{2}\dot{q}^2 - V(q) \tag{14}$$

where $L(q, \dot{q})$ is the lagrangian. The principle of least action states that: *the classic trajectory that joins two points of configuration space is the one that minimizes the action* S.

To demonstrate this statement we study the condition for having a minimum. Varying the path $q(t)$ (with boundary conditions $q(t_i) = q_i$ and $q(t_f) = q_f$) in a new function $q(t) + \delta q(t)$, where $\delta q(t)$ is an arbitrary infinitesimal variation (with $\delta q(t_i) = \delta q(t_f) = 0$), and imposing that the action is minimized on the path $q(t)$ one finds

$$\begin{aligned} 0 &= \delta S[q] = S[q + \delta q] - S[q] \\ &= \int_{t_i}^{t_f} dt \left[m\dot{q}\delta\dot{q} - \frac{\partial V(q)}{\partial q}\delta q \right] = m\dot{q}\delta q\Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt \left[m\ddot{q} + \frac{\partial V(q)}{\partial q} \right] \delta q \\ &= -\int_{t_i}^{t_f} dt \left[m\ddot{q} + \frac{\partial V(q)}{\partial q} \right] \delta q \ . \end{aligned}$$

Since the variations $\delta q(t)$ are arbitrary functions, the minimum is reached when the function $q(t)$ satisfies the classical equations of motion

$$m\ddot{q} + \frac{\partial V(q)}{\partial q} = 0 \tag{15}$$

which reproduce (13). In general, one finds the Euler-Lagrange equations

$$\begin{aligned} 0 &= \delta S[q] = \delta \int_{t_i}^{t_f} dt\, L(q, \dot{q}) = \int_{t_i}^{t_f} dt \left[\frac{\partial L(q, \dot{q})}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L(q, \dot{q})}{\partial q}\delta q \right] \\ &= \frac{\partial L(q, \dot{q})}{\partial \dot{q}}\delta q\Big|_{t_i}^{t_f} - \int_{t_i}^{t_f} dt \left[\frac{d}{dt}\frac{\partial L(q, \dot{q})}{\partial \dot{q}} - \frac{\partial L(q, \dot{q})}{\partial q} \right] \delta q \\ &= -\int_{t_i}^{t_f} dt \left[\frac{d}{dt}\frac{\partial L(q, \dot{q})}{\partial \dot{q}} - \frac{\partial L(q, \dot{q})}{\partial q} \right] \delta q \end{aligned}$$

and thus

$$\frac{d}{dt}\frac{\partial L(q, \dot{q})}{\partial \dot{q}} - \frac{\partial L(q, \dot{q})}{\partial q} = 0 \ . \tag{16}$$

Observations:
1. The action has dimension $[S] = [\hbar] = [\text{energy} \times \text{time}] = ML^2/T$.
2. The lagrangian equations of motion are typically of second order in time, therefore one expects two "initial conditions" (or, more generally, boundary conditions), conveniently chosen by fixing the position at the initial and final times.
3. The equations of motion can be written as the functional derivative of the action

$$\frac{\delta S[q]}{\delta q(t)} = 0 \ , \tag{17}$$

with the functional derivative defined by the variation

$$\delta S[q] = \int dt\, \frac{\delta S[q]}{\delta q(t)} \delta q(t) \ .$$

4. The equations of motion do not change if one adds a total derivative to the lagrangian L, $L \to L' = L + \frac{d}{dt}\Lambda$.

5. The lagrangian formalism easily extends to systems with more than one degrees of freedom, and with a little more attention to field theories (systems with an infinite number of degrees of freedom).

To appreciate this last point, let us consider a set of dynamical fields $\phi_i(x) = \phi_i(t, \vec{x})$, where x indicates the spacetime point. By dynamics one means evolution over the time t. At fixed t the dynamical fields $\phi_i(t, \vec{x})$ are indexed by i (which labels a discrete set of fields) and by $\vec{x} \in R^3$, that tells us that at every point of space there is a dynamical variable: there are an infinite number of degrees of freedom. By discretizing the space, and considering a finite volume, one can approximate a field theory by a mechanical model with a finite number of degrees of freedom. Typically, when the latter are the true physical degrees of freedom (e.g. in the atomic structure of matter) but very large in number, the continuum approximation is very useful. The lagrangian L is often expressed as an integral of a lagrangian density $\mathcal{L}$

$$L(t) = \int d^3x \, \mathcal{L}(\phi_i, \partial_\mu \phi_i) \tag{18}$$

and the action takes the form

$$S[\phi] = \int dt \, L(t) = \int d^4x \, \mathcal{L}(\phi_i, \partial_\mu \phi_i) \; . \tag{19}$$

Imposing the extremality condition $\delta S = 0$ produces the Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0 \; . \tag{20}$$

2.2 Hamiltonian formalism

The basic idea of the hamiltonian formalism is to have equations of motion that are first order in time. To review it, we follow a simple example: a non-relativistic particle of coordinates q^i and configuration space lagrangian

$$L(q, \dot{q}) = \frac{m}{2} \dot{q}^i \dot{q}_i - V(q) \tag{21}$$

where indices are lowered with the metric δ_{ij} (and are thus equivalent to upper indices in our model, distinction of upper and lower indices is however useful in more general contexts). Transition to the hamiltonian formalism takes place as follows:
1) The dynamical variables are doubled by introducing conjugate momentum p_i to each coordinate q^i

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i} = m \dot{q}_i \; . \tag{22}$$

The set (q^i, p_i) constitutes the coordinates of phase space.
2) The hamiltonian $H(q, p)$ is defined as the Legendre transform of the lagrangian L

$$H(q^i, p_i) \equiv p_i \dot{q}^i - L(q, \dot{q}) = \frac{1}{2m} p^i p_i + V(q) \; . \tag{23}$$

It is a function on phase space.
3) The Poisson brackets are defined as follows. For any two functions A and B of phase space, their Poisson brackets is defined by

$$\{A, B\} = \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i} \tag{24}$$

where we have used the summation convention for repeated indices. In particular,

$$\{q^i, p_j\} = \delta^i_j \ , \quad \{q^i, q^j\} = 0 \ , \quad \{p_i, p_j\} = 0 \ . \tag{25}$$

4) The hamiltonian equations of motions can be written as

$$\dot{q}^i = \{q^i, H\} \ , \qquad \dot{p}_i = \{p_i, H\} \tag{26}$$

and are of first order in time. In our example they become

$$\dot{q}^i = \frac{\partial H}{\partial p_i} = \frac{p^i}{m} \ , \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} = -\frac{\partial V}{\partial q^i} \tag{27}$$

and are evidently equivalent to the lagrangian equations $m\ddot{q}^i = -\frac{\partial V}{\partial q^i}$. The hamiltonian H is interpreted as the generator of time translations, and moves the initial conditions (a point in phase space) over time by an infinitesimal amount dt. The generator of these canonical transformations is given by Hdt, and acts through the Poisson brackets ($\delta q = \{q, Hdt\}$, $\delta p = \{p, Hdt\}$).

These equations can be obtained from an action. In phase space, the action takes the form

$$S[q, p] = \int_{t_i}^{t_f} dt \left(p_i \dot{q}^i - H(q, p)\right) \tag{28}$$

and minimizing it one finds

$$\begin{aligned} 0 \ = \ \delta S &= \int_{t_i}^{t_f} dt \left(\delta p_i \dot{q}^i + p_i \delta \dot{q}^i - \frac{\partial H}{\partial p_i}\delta p_i - \frac{\partial H}{\partial q^i}\delta q^i\right) \\ &= p_i \delta q^i \Big|_{t_i}^{t_f} + \int_{t_i}^{t_f} dt \left[\delta p_i\Big(\dot{q}^i - \frac{\partial H}{\partial p_i}\Big) - \delta q^i \Big(\dot{p}_i + \frac{\partial H}{\partial q^i}\Big)\right] \end{aligned}$$

from which one recognizes Hamilton's equations of motion. Note that in this formulation one needs $2n$ integration constants, which are given by specifying the coordinates q^i at initial and final times.

The hamiltonian structure is the starting point of canonical quantization:

$$z^a = (q^i, p_i) \quad \to \quad \hat{z}^a = (\hat{q}^i, \hat{p}_i) \quad \text{such that} \quad [\hat{z}^a, \hat{z}^b] = i\hbar \{z^a, z^b\} \ . \tag{29}$$

where the classical dynamical variables z^a are elevated to linear operators $\hat{z}^a$ acting on a Hilbert space. The vectors of the latter describe the possible quantum states of the system, whose evolution is governed by the Schrödinger equation.

2.3 Examples

Let us now describe some actions worth quantizing: they describe the motion of a particle in a scalar potential $V(x)$, vector potential $A_i(x)$, and tensor potential (metric) $g_{ij}(x)$.

2.3.1 Particle scalar potential

The motion of a particle in a flat euclidean space (with cartesian coordinates x^i and metric $ds^2 = \delta_{ij} dx^i dx^j = dx^i dx_i = dx^i dx^i$) is described by the coordinates $x^i(t)$ of the particle. We consider interactions given by a scalar potential $V(x)$.

Lagrangian formalism

The lagrangian takes the form

$$L(x, \dot{x}) = \frac{m}{2} \dot{x}^i \dot{x}_i - V(x) \tag{30}$$

and varying the action $S[x(t)] = \int dt \, L(x, \dot{x})$ one finds the lagrangian equations of motion

$$m \ddot{x}_i = -\frac{\partial V}{\partial x^i} \; . \tag{31}$$

Hamiltonian formalism

One defines the conjugate momanta

$$p_i \equiv \frac{\partial L}{\partial \dot{x}^i} = m \dot{x}_i \tag{32}$$

as independent variables, and construct the hamiltonian by the Legendre transform

$$H(x, p) = p_i \dot{x}^i - L(x, \dot{x}) = \frac{1}{2m} p_i p^i + V(x) \tag{33}$$

so that the equations of motion take the form

$$\begin{aligned} \dot{x}^i &= \{x^i, H\} = \frac{p^i}{m} \\ \dot{p}_i &= \{p_i, H\} = -\frac{\partial V}{\partial x^i} \; . \end{aligned} \tag{34}$$

These equations are first order in time, and equivalent to the lagrangian ones in (31).

One may interpret phase space with coordinates (x^i, p_i) as the set of all possible initial conditions of the system, and the hamiltonian as generating the dynamics by moving the initial point in time through the Poisson brackets.

2.3.2 Particle in a vector potential (magnetic field)

Let us now consider a particle of mass m and charge q in interaction with a vector potential $A_i(x)$, that identifies a (time-independent) magnetic field $B^i = \epsilon^{ijk} \partial_j A_k$ (i.e. $\vec{B} = \vec{\nabla} \times \vec{A}$).

Lagrangian formalism

The correct lagrangian (that produces the Lorentz force) is given by

$$L(x, \dot{x}) = \frac{m}{2} \dot{x}^i \dot{x}_i + q A_i(x) \dot{x}^i \; . \tag{35}$$

Varying the action $S[x(t)] = \int dt \, L(x, \dot{x})$ one finds the lagrangian equations of motion

$$m \ddot{x}_i = q(\partial_i A_j - \partial_j A_i) \dot{x}^j \tag{36}$$

that are written in terms of the magnetic field as

$$m\ddot{x}_i = q\epsilon_{ijk}\dot{x}^j B^k \qquad (\text{i.e. } m\ddot{\vec{x}} = q\,\dot{\vec{x}} \times \vec{B})\ . \tag{37}$$

Note that the relation $B^i = \epsilon^{ijk}\partial_j A_k$ can be inverted by $(\partial_i A_j - \partial_j A_i) = \epsilon_{ijk} B^k$. The notation $F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B^k$ is often used.

Hamiltonian formalism

Calculating the conjugate momenta

$$p_i \equiv \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}_i + qA_i(x) \tag{38}$$

and the hamiltonian

$$H(x,p) = p_i\dot{x}^i - L(x,\dot{x}) = \frac{1}{2m}\big(p_i - qA_i(x)\big)^2 \tag{39}$$

one finds the equations of motion in hamiltonian form

$$\begin{aligned} \dot{x}^i &= \{x^i, H\} = \frac{1}{m}(p^i - qA^i) \\ \dot{p}_i &= \{p_i, H\} = \frac{q}{m}(\partial_i A_j)(p^j - qA^j)\ . \end{aligned} \tag{40}$$

These differential equations are first order in time, and equivalent to the lagrangian ones.

Note that at the hamiltonian level, the interaction is introduced by making the *minimal substitution* $p_i \to p_i - qA_i(x)$ in the free hamiltonian. Thus, it is useful to introduce the definition of covariant momentum π_i as

$$\pi_i = p_i - qA_i(x)$$

which satisfies the Poisson bracket

$$\{\pi_i, \pi_j\} = qF_{ij}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B^k$ is the magnetic field. Note that in quantization the covariant momentum becomes proportional to the gauge covariant derivative. In terms of the covariant moment the equations of motion take the form

$$\dot{x}^i = \{x^i, H\} = \frac{\pi^i}{m} \qquad \dot{\pi}_i = \{\pi_i, H\} = \frac{q}{m}F_{ij}\pi^j\ . \tag{41}$$

For simplicity, we have considered a static magnetic field, but the treatment can be extended to a time dependent magnetic field, and more generally to a time dependent electromagnetic field, which we leave as an exercize.

Gauge invariance

The same magnetic field can be obtained by different vector potentials, related by a gauge transformation. Indeed, potentials related by

$$A_i(x) \ \to\ A'_i(x) = A_i(x) + \partial_i\Lambda(x)\ , \tag{42}$$

where $\Lambda(x)$ is an arbitrary function, identify the same magnetic field ($B'^i = B^i$).

The equations of motion with the Lorentz force (37) depend only on the magnetic field, and are therefore gauge invariant. This is visible also from the lagrangian: under a gauge transformation the lagrangian changes by a total derivative which, as we know, does not modify the equations of motion

$$L(x,\dot{x};A') = L(x,\dot{x};A) + q\,\dot{x}^i\partial_i\Lambda = L(x,\dot{x};A) + \frac{d(q\Lambda)}{dt}$$

where the notation $L(x,\dot{x};A)$ indicates the lagrangian (35) depending on the potential A_i.

In Hamiltonian formalism, a gauge transformation acts on conjugated momenta

$$p_i \;\rightarrow\; p'_i = p_i + q\partial_i\Lambda$$

but leaves unchanged the covariant moments π_i. Of course, by covariance here one refers to the covariance under gauge transformations.

2.3.3 Particle in a tensor potential (curved space): the nonlinear sigma model

Let us finally consider the motion of a particle in a curved space with metric

$$ds^2 = g_{ij}(x)dx^i dx^j \;. \tag{43}$$

Lagrangian formalism

The lagrangian contains only a kinetic term

$$L(x,\dot{x}) = \frac{m}{2}g_{ij}(x)\dot{x}^i\dot{x}^j \tag{44}$$

and varying the action $S[x(t)] = \int dt\; L(x,\dot{x})$ one finds

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \tag{45}$$

where the Christoffel symbols (the components of the metric connection), defined by

$$\Gamma^i_{jk} = \frac{1}{2}g^{im}(\partial_j g_{km} + \partial_k g_{jm} - \partial_m g_{jk}) \tag{46}$$

where g^{ij} is the inverse of the metric ($g^{ij}g_{jk} = \delta^i_k$), come out naturally from the variational principle. These are the geodesic equation in a curved space (often written as $\frac{D\dot{x}^i}{dt} = 0$).

Hamiltonian formalism

Defining the conjugate momenta

$$p_i \equiv \frac{\partial L}{\partial \dot{x}^i} = m g_{ij}(x)\dot{x}^j \tag{47}$$

and the hamiltonian

$$H(x,p) = p_i\dot{x}^i - L(x,\dot{x}) = \frac{1}{2m}g^{ij}(x)p_i p_j \tag{48}$$

one finds the equations of motion in phase space

$$\begin{aligned} \dot{x}^i &= \{x^i, H\} = \frac{1}{m} g^{ij} p_j \\ \dot{p}_i &= \{p_i, H\} = -\frac{1}{2m} (\partial_i g^{kl}) p_k p_l \ . \end{aligned} \tag{49}$$

Invariance under an arbitrary change of coordinates

The motion of the particle should not depend on the coordinates chosen. By changing coordinates, the metric tensor is transformed in such a way to keep unchanged the length element $ds^2 = g_{ij}(x) dx^i dx^j = g'_{ij}(x') dx'^i dx'^j$. Thus, under a change (diffeomorphism)

$$x^i \ \to \ x'^i = x'^i(x) \tag{50}$$

the metric must change by

$$g_{ij}(x) \ \to \ g'_{ij}(x') = g_{kl}(x) \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} \ . \tag{51}$$

For an infinitesimal change of coordinates

$$x^i \ \to \ x'^i = x^i - \xi^i(x) \qquad (\text{i.e.} \ \ \delta x^i \equiv x^{i'} - x^i = -\xi^i) \tag{52}$$

where $\xi^i(x)$ is an infinitesimal vector field, one finds for the metric, using (50) and (51),

$$\delta g_{ij} \equiv g'_{ij}(x) - g_{ij}(x) = \xi^k \partial_k g_{ij} + \frac{\partial \xi^k}{\partial x^i} g_{kj} + \frac{\partial \xi^k}{\partial x^j} g_{ik} \tag{53}$$

where we have calculated the difference between the new function g'_{ij} and the old function g_{ij} evaluated at the same point x (there is a so-called transport term from the point x' to the point x proportional to the derivative of the metric). One may check that that under these infinitesimal transformations the lagrangian remains invariant

$$\delta L(x, \dot{x}) = m g_{ij} \delta \dot{x}^i \dot{x}^j + \frac{m}{2} (\delta x^k \partial_k g_{ij}) \dot{x}^i \dot{x}^j + \frac{m}{2} \delta g_{ij} \dot{x}^i \dot{x}^j = 0 \ . \tag{54}$$

This symmetry is often called "background" symmetry because also the functions $g_{ij}(x)$ are transformed on top of the the dynamical variables $x^i(t)$.

3 Symmetries and Noether's theorem

The study of the symmetries of a physical system is very useful for identifying the equations of motion governing the system and for solving them. We define the concept of *symmetry* by:
A symmetry is a transformation of the dynamical variables $q(t)$, induced possibly by a transformation of the time t,

$$\begin{aligned} t \qquad &\longrightarrow \qquad t' = f(t) \\ q(t) \qquad &\longrightarrow \qquad q'(t') = F(q(t), t) \end{aligned} \tag{55}$$

that leaves the equations of motion invariant in form.
Since the equations of motion are invariant in form, they admit the same kind of solutions, and one cannot determine if the motion takes place in the "old frame of reference" or the "new frame of reference". These reference frames are to be treated on the same footing, without one of them being identified as privileged. A check to test if a transformation is a symmetry makes use of the action. If the action is invariant under the transformation (55), up to boundary terms (which emerge as integrals of total derivatives and do not modify the equations of motion)

$$S[q'] = S[q] \ + \ \text{boundary terms} \tag{56}$$

then the transformation is a symmetry: the equations obtained from the least action principle must be of the same form, being obtainable from identical actions.

A physical system can present different types of symmetry: discrete symmetries, continuous symmetries (associated with a Lie group), local symmetries (called also gauge symmetries). An even more general concept is that of "background symmetry", described by generalized transformations that modify also the parameters of the theory (e.g. the coupling constants or those contained in external potentials). They are not true symmetries in the technical sense defined above, but relate solutions of a given theory with certain parameters to the solutions of another theory with different parameters.

For Lie symmetries, symmetries that depend continuously on some parameters, one can prove Noether's theorem. It states that:
For each continuous parameter of the symmetry group there exists a conserved charge (in field theories this conservation is expressed by an equation of continuity).

A proof is the following one. A transformation of symmetry which depends on a parameter α can be presented in a general way as

$$\begin{aligned} t \quad &\longrightarrow \quad t' = f(t,\alpha) \\ q(t) \quad &\longrightarrow \quad q'(t') = F(q(t),t,\alpha) \end{aligned} \tag{57}$$

where by definition the identity transformation is achieved for $\alpha = 0$. Infinitesimal transformations (with parameter $\alpha \ll 1$) they can be written as

$$\delta_\alpha q(t) \equiv q'(t) - q(t) = \alpha G(q(t),t) \tag{58}$$

with an appropriate function G obtainable from the F and f in (57). To prove that there is a conserved quantity associated with the symmetry, we extend the symmetry transformation to a more general transformation with parameter $\alpha(t)$, no longer constant but depending arbitrarily on the time t (i.e. an arbitrary function of time)

$$\delta_{\alpha(t)} q(t) = \alpha(t) G(q(t),t) \ . \tag{59}$$

Generically, this transformation will not be a symmetry, but one can certainly state that the action transforms as

$$\delta_{\alpha(t)} S[q] = \int dt \ \dot\alpha(t) Q(q(t),t) \tag{60}$$

up to boundary terms (integrals of total derivatives). In fact, if we take the case of α constant the action must be invariant by hypothesis (as we have a symmetry). So, for an arbitrary function $\alpha(t)$, the variation cannot depend directly on α, but only on its derivatives. Now the quantity Q that appears in (60) is conserved. To show that, one uses the equations of motion in the form of the "least action principle", which make the variation vanish for any transformation and in particular for the transformations with local parameter in (59)

$$0 = \delta_{\alpha(t)} S[q]\bigg|_{q_0} = \int dt\ \dot{\alpha}(t) Q\bigg|_{q_0} = -\int dt\ \alpha(t)\dot{Q}\bigg|_{q_0} \quad \Longrightarrow \quad \dot{Q}(q_0(t), t) = 0$$

where we have integrated by parts and used the arbitrariness of the function $\alpha(t)$ to deduce conservation. Note that we must evaluate the variation of the action at the point of minimum, indicated by q_0, which solves the Euler-Lagrange equations. The conserved charge Q is be evaluated on the solution of the equations of motion and is conserved. This type of Lie symmetries are called *rigid symmetries* or *global symmetries*. To each parameter of the Lie group there is an associated conserved charge Q.

Those types of Lie symmetries where the parameter is an arbitrary function of time are called *local symmetries* or *gauge symmetries*. The previous method does not produce any non-trivial conserved quantity, because the variation of the action is always zero, for any local parameter and without using the equations of motion. Local symmetries tell us that the dynamical variables are redundant: with a gauge transformation one can modify arbitrarily the time evolution of certain combinations of the dynamical variables, combinations whose evolution is evidently not fixed by the equations of motion.

These two types of symmetry are exemplified in the following examples: the non-relativistic particle and the relativistic particle.

3.1 Non-relativistic particle and symmetries (the Galilean group)

Let us consider a non-relativistic free particle. We wish to study the invariance under transformations generated by the Galilean group, obtaining the conserved charges that are guaranteed to exist by Noether's theorem.

As dynamical variables we use the cartesian coordinates of the particle position $x^i(t) \in R^3$ The action is given by the time integral of the lagrangian, which for a free particle coincides with the kinetic energy

$$S[x] = \int dt\ \frac{m}{2}\dot{x}^i\dot{x}^i\ . \tag{61}$$

The equations of motion are obtained by minimizing it

$$\frac{\delta S[x]}{\delta x^i(t)} \equiv -m\ddot{x}^i = 0\ . \tag{62}$$

We now study the infinitesimal symmetry transformations that make up the Galilean group and obtain the corresponding conserved charges.

<u>*Space translations*</u>: A space translation acts by

$$\delta x^i(t) \equiv x'^i(t) - x^i(t) = a^i \tag{63}$$

with a^i a constant infinitesimal vector. Under this transformation the action is invariant

$$\delta S[x] = 0 \tag{64}$$

and thus the transformation (63) is a symmetry. We now use Noether's method to find the conserved charges. We extend the symmetry in (63) to a more general transformation depending on a time dependent vector $a^i(t)$

$$\delta x^i(t) = a^i(t) \ . \tag{65}$$

The action is no more invariant, and a quick calculation gives

$$\delta S[x] = \int dt \ \underbrace{m\dot{x}^i}_{p^i} \dot{a}^i \ . \tag{66}$$

The term multiplying $\dot{a}^i$ is the conserved charge: the momentum p^i. To prove conservation we use the equations of motion, which imply that $\delta S = 0$ for any variation, and in particular for a variation of the form (65). Let us indicate by $x^i(t)$ the solution of the equations of motion: integrating by parts and using the arbitrariness of the functions $a^i(t)$ we deduce that $p^i = m\dot{x}^i$ is conserved

$$0 = \delta S[x^i(t)] = \int dt \ p^i(t)\dot{a}^i(t) = -\int dt \ \dot{p}^i(t) a^i(t) \qquad \Longrightarrow \qquad \dot{p}^i(t) = 0 \ . \tag{67}$$

Thus, the momentum $p^i = m\dot{x}^i$ is conserved in the evolution of the system as consequence of translational invariance in space.
<u>*Time translation*</u>: Time translation is also an invariance of the system. If we shift time by an infinitesimal quantity ϵ

$$t \to t' = t - \epsilon \tag{68}$$

and require that the functions $x^i(t)$ are scalar functions [1]

$$x^i(t) \quad \to \quad x'^i(t') = x^i(t) \tag{69}$$

then the action is invariant. We express the infinitesimal transformation by $\delta x^i(t) \equiv x'^i(t) - x^i(t)$, with the functions evaluated at the same time t,

$$\begin{aligned} \delta x^i(t) &= x'^i(t) - x^i(t) = x'^i(t) - x^i(t) + x'^i(t') - x'^i(t') \\ &= x'^i(t) - x'^i(t') + \underbrace{x'^i(t') - x^i(t)}_{=0} = x'^i(t) - x'^i(t+\epsilon) = \epsilon\, \dot{x}'^i(t) = \epsilon\, \dot{x}^i(t) \end{aligned} \tag{70}$$

which is valid up to ϵ^2 terms that are neglected. Using directly an arbitrary function $\epsilon(t)$

$$\delta x^i(t) = \epsilon(t)\dot{x}^i(t) \tag{71}$$

we compute

$$\delta S[x] = \int dt \ m\dot{x}^i \partial_t(\epsilon \dot{x}^i) = \int dt \ \Big[\partial_t\Big(\frac{\epsilon m}{2}\dot{x}^i\dot{x}^i\Big) + \dot{\epsilon}\frac{m}{2}\dot{x}^i\dot{x}^i\Big] = \int dt \ \dot{\epsilon}\, \underbrace{\Big(\frac{m}{2}\dot{x}^i\dot{x}^i\Big)}_{E} \tag{72}$$

where total derivatives are dropped, as usual. From this result we deduce two things: *(i)* if ϵ is constant $\dot{\epsilon} = 0$ and $\delta S[x] = 0$, so the corresponding transformation is a symmetry; *(ii)* using the equations of motion (which make $\delta S[x]|_{x(t)=x_{cl}(t)} = 0$ for any variation) and integrating

[1] The position of the particle does not change, it only changes the way to measure time.

by parts we deduce that $\dot{E} = 0$, i.e. the kinetic energy $E = \frac{m}{2}\dot{x}^i\dot{x}^i$ is conserved. This is the consequence of time translation invariance.
Space rotations: For spatial rotations, the coordinates are transformed as

$$\delta x^i(t) = \epsilon^{ijk}\omega^j x^k(t) \tag{73}$$

where the vector ω^i describes an infinitesimal rotation. Considering directly ω^i as arbitrary functions of time we obtain

$$\delta S[x] = \int dt\ \dot{\omega}^i \underbrace{\epsilon^{ijk}x^j m\dot{x}^k}_{(\vec{r}\times\vec{p})^i \equiv L^i}\ . \tag{74}$$

Again, there is symmetry for ω^i constant. The corresponding conserved charges are the three components of the angular momentum $L^i = \epsilon^{ijk}x^j p^k$.
Galilean transformations: We call proper galilean transformations those transformations that bring us to a new inertial frame moving with relative constant velocity v^i. The transformation on the dynamical variables is therefore given by

$$\delta x^i(t) = v^i t \tag{75}$$

and proceeding as before, extending the parameters of the transformation to arbitrary functions of time, we calculate up to boundary terms

$$\delta S[x] = \int dt\ \dot{v}^i \underbrace{\left(m\dot{x}^i t - m x^i\right)}_{G^i} \tag{76}$$

from which we deduce that there is a symmetry for v^i constant, and that the vector $G^i = m\dot{x}^i t - m x^i$ is conserved, as verified explicitly.

We have seen how the invariance of the non-relativistic free particle under the transformations of the Galilean group, a 10-parameter Lie group, produces 10 conserved quantities.

3.2 Relativistic particle

Let us now study the action for a relativistic particle. By definition it must be consistent with Lorentz invariance and, more generally, with transformations of the Poincaré group. We shall study four equivalent descriptions, focusing on the role of local symmetries (gauge symmetries). As we shall see, the latter arises by requiring that the Lorentz invariance (a global symmetry) be manifest.

(I) Consider the particle in the inertial frame with coordinates $x^\mu = (x^0, x^i) = (t, x^i)$. For simplicity we use $c = 1$. We consider the position $x^i(t)$ at time t as the dynamical variables. Imposing the invariance of the action under Lorentz transformations guarantees relativistic invariance. This request is implemented using the proper time T_0, which infinitesimally is given by

$$dT_0 = \sqrt{-ds^2} = \sqrt{-dx^\mu dx_\mu} = \sqrt{dt^2 - dx^i dx^i} = dt\sqrt{1 - \dot{x}^i\dot{x}^i}\ . \tag{77}$$

From special relativity we know that the proper time is a relativistic invariant, as it measures the invariant length of the worldline. Thus, for a particle of mass m an action proportional to the proper time

$$S_I[x^i(t)] = -m\int dT_0 = -m\int dt\ \sqrt{1 - \dot{x}^i(t)\dot{x}^i(t)} \tag{78}$$

is automatically invariant under Lorentz (and Poincarè) transformations. Moreover, it reproduces the action of the non-relativistic particle in the non-relativistic limit $(\dot{x}^i)^2 \ll 1$. The equations of motion are obtained from the principle of least action

$$\delta S_I[x^i] = 0 \qquad \Longrightarrow \qquad \frac{d}{dt}\left(\frac{m\dot{x}^i}{\sqrt{1-\dot{\vec{x}}\dot{\vec{x}}}}\right) = 0 \; . \tag{79}$$

The rigid symmetries are those generated by the Poincaré group, which must be present by construction, but the way these symmetries are realized in not manifest (i.e. one cannot use the tensor formalism to verify the Lorentz invariance on the dynamical variables $x^i(t)$, as time and space are treated differently.) There are no gauge symmetries, and the three dynamical variables are all "physical": the system has three degrees of freedom. Note also the geometric interpretation: the action is proportional to the length of the worldline traveled by the particle, a time-like length proportional to the proper time (indeed they measure the same thing).

(II) The previous formulation is correct, but it would be preferable to treat the space coordinates x^i and the time coordinate $x^0 \equiv t$ in a more symmetrical way to keep relativistic invariance more easily under control (i.e. manifest). One could use the four dynamical variables x^μ, which form a four-vector, but one of them (or more generally one combination of them) will have to be redundant so to guarantee equivalence with the previous action: this is possible if there are local symmetries (gauge symmetries). This is achieved in the following way: we indicate by $x^\mu(\tau)$ the dynamical variables which describe the worldline traveled by the particle in terms of an *arbitrary parameter* τ. The action is geometrically the same as before, proportional to the proper time, but now it takes the form of a functional of four variables

$$S_{II}[x^\mu(\tau)] = -m \int d\tau \, \sqrt{-\dot{x}^\mu \dot{x}_\mu} \tag{80}$$

where we now define $\dot{x}^\mu \equiv \frac{d}{d\tau}x^\mu$. The equations of motion read

$$\delta S_{II}[x^\mu] = 0 \qquad \Longrightarrow \qquad \frac{d}{d\tau}\left(\frac{m\dot{x}^\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}}\right) = 0 \; . \tag{81}$$

The rigid symmetries are *manifestly* those of the Poincarè group

$$x^\mu(\tau) \to x'^\mu(\tau) = \Lambda^\mu{}_\nu x^\nu(\tau) + a^\mu . \tag{82}$$

At the infinitesimal level they read

$$\delta x^\mu(\tau) = \omega^\mu{}_\nu x^\nu(\tau) + a^\mu \tag{83}$$

where $\delta x^\mu(\tau) = x'^\mu(\tau) - x^\mu(\tau)$ and $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ (both $\omega^\mu{}_\nu$ and a^μ are to be considered as infinitesimal parameters). This symmetry ensures that the model is relativistic. The corresponding conserved charges can be obtained by applying Noether's theorem.

In addition there is a local symmetry (gauge symmetry)

$$\delta x^\mu = \xi(\tau)\dot{x}^\mu(\tau) \tag{84}$$

where the infinitesimal parameter $\xi(\tau)$ depends arbitrarily on the time parameter τ. We can interpret it as a local time translation (recall the transformation rule (70)). Under this transformation the action is invariant (up to boundary terms)

$$\delta S_{II}[x^\mu] = \int d\tau \, \frac{d}{d\tau}\Big(\xi L_{II}\Big) \sim 0 \tag{85}$$

where L_{II} is the lagrangian in S_{II}. This local symmetry corresponds geometrically to a reparameterization of the worldline

$$\begin{aligned} \tau &\longrightarrow \tau' = f(\tau) \\ x^\mu(\tau) &\longrightarrow x'^\mu(\tau') = x^\mu(\tau) \end{aligned} \tag{86}$$

which for infinitesimal transformations $\tau' = \tau - \xi(\tau)$ reduces to (84) (mathematicians call this symmetry a diffeomorphism of the worldline). This gauge symmetry is needed to prove equivalence with formulation I. Equivalence is obtained by operating a gauge transformation (a reparameterization of the worldline) to fix one of the dynamical variables by requiring a gauge-fixing choice. One can impose the gauge-fixing condition

$$x^0(\tau) = t(\tau) = \tau \tag{87}$$

so that the variable $x^0(\tau)$ is not anymore dynamical: its time evolution is fixed by the gauge condition, and corresponds to use x^0 as the parameter to label points of the particle worldline. This reproduces the action S_I.

(III) A third formulation is obtained by introducing a new dynamical variable, a gauge variable (also called "gauge field"). By that one means a variable whose gauge transformation contains the derivative of the parameter of the local symmetry (the gauge parameter). In this specific case, the so-called einbein $e(\tau)$ is used as gauge field (from the german "einbein" meaning "one leg"). Geometrically the einbein defines an intrisic metric on the wordline by the formula $ds^2 = e^2(\tau)d\tau d\tau$. The action is given by

$$S_{III}[x^\mu(\tau), e(\tau)] = \int d\tau\ \frac{1}{2}(e^{-1}\dot{x}^\mu \dot{x}_\mu - em^2) \tag{88}$$

where it is assumed that the einbein e is non-vanishing, and therefore invertible. The local symmetry takes the form

$$\begin{aligned} \delta x^\mu &= \xi \dot{x}^\mu \\ \delta e &= \frac{d}{d\tau}(\xi e) \end{aligned} \tag{89}$$

and leads to $\delta S_{III} = \int d\tau\ \frac{d}{d\tau}(\xi L_{III}) \sim 0$. Note that the transformation rule of e contains the derivative of the local parameter ξ. The global symmetries are again given by transformations of the Poincarè group

$$\begin{aligned} \delta x^\mu(\tau) &= \omega^\mu{}_\nu x^\nu(\tau) + a^\mu \\ \delta e(\tau) &= 0\ . \end{aligned}$$

The equations of motion are

$$\frac{\delta S[x,e]}{\delta e(\tau)} = 0 \qquad \longrightarrow \qquad e^{-2}\dot{x}^\mu \dot{x}_\mu + m^2 = 0 \tag{90}$$

$$\frac{\delta S[x,e]}{\delta x^\mu(\tau)} = 0 \qquad \longrightarrow \qquad \frac{d}{d\tau}(e^{-1}\dot{x}^\mu) = 0\ . \tag{91}$$

To show equivalence with formulation II, we solve the algebraic equation (90)

$$e = \pm\frac{1}{m}\sqrt{-\dot{x}^\mu \dot{x}_\mu}\ . \tag{92}$$

Substituting this relation back in S_{III} one finds

$$S_{III}\left[x^\mu(\tau), e(\tau) = \pm\frac{1}{m}\sqrt{-\dot{x}^\mu\dot{x}_\mu}\right] = \mp m\int d\tau\ \sqrt{-\dot{x}^\mu\dot{x}_\mu}\ . \tag{93}$$

Choosing the solution with $e > 0$ brings back to formulation II. The other solution may be interpreted as a sign of the existence of antiparticles. Furthermore, action III is superior to the previous ones, as it includes the case of massless particles, obtained by setting $m = 0$ in the action.

(III') Gauge invariance can be used to fix a condition (gauge-fixing condition). Choosing to set a gauge-fixing condition on the einbein, for example with the choice $e = 1$ (it is possible ignoring topological complications, which we will do). Then the action (88) simplifies to

$$S_{III'}[x^\mu(\tau)] = \int d\tau\ \frac{1}{2}\dot{x}^\mu\dot{x}_\mu\ . \tag{94}$$

However, one must remember the equations of motion of e, which in the gauge become $\dot{x}^\mu\dot{x}_\mu + m^2 = 0$. This simplified action with associated constraint is equivalent to the original gauge invariant action: all the information on the dynamics is contained in the set

$$\begin{aligned} &\text{action} \qquad S_{III'}[x^\mu(\tau)] = \int d\tau\ \frac{1}{2}\dot{x}^\mu\dot{x}_\mu \\ &\text{constraint} \qquad \dot{x}^\mu\dot{x}_\mu + m^2 = 0\ . \end{aligned} \tag{95}$$

This is perhaps the simplest formulation possible for the dynamics of a relativistic particle.

(IV) At last, we pass to a fourth formulation, equivalent to the previous ones and useful for canonical quantization. It is the hamiltonian formulation. Starting from S_{III}, introducing the conjugate momenta $p_\mu = e^{-1}\dot{x}^\mu$ and the corresponding hamiltonian $H_E = \frac{e}{2}(p^\mu p_\mu + m^2) \equiv eC$ (where $C \equiv \frac{1}{2}(p^\mu p_\mu + m^2)$), one finds the action in phase space

$$S_{IV}[x^\mu(\tau), p_\mu(\tau), e(\tau)] = \int d\tau\ \Big(p_\mu\dot{x}^\mu - \frac{e}{2}(p^\mu p_\mu + m^2)\Big) \tag{96}$$

where x^μ are the coordinates in spacetime, p_μ their conjugate momenta, and e the einbein. All of them must be treated as independent dynamical variables. The gauge symmetry can be written in the form

$$\begin{aligned} \delta x^\mu &= \zeta p^\mu \\ \delta p_\mu &= 0 \\ \delta e &= \dot{\zeta} \end{aligned} \tag{97}$$

and one verifies that $\delta S_{IV} = \int d\tau\ \frac{d}{d\tau}[\frac{\zeta}{2}(p^2 - m^2)] \sim 0$. Eliminating p_μ by their algebraic equations of motion

$$\frac{\delta S_{IV}}{\delta p_\mu} = \dot{x}^\mu - ep^\mu = 0 \qquad \Longrightarrow \qquad p^\mu = e^{-1}\dot{x}^\mu \tag{98}$$

allows to recover the formulation *III* (also the form of the gauge symmetry is recovered by relating $\zeta = e\xi$).

Note the structure of the action S_{IV}. It depends on the phase space coordinates (x^μ, p_μ) and on the gauge field e which acts as a Lagrange multiplier. Its equations of motion impose a constraint in phase space (constrained hamiltonian mechanics)

$$C \equiv \frac{1}{2}(p^\mu p_\mu + m^2) = 0 \tag{99}$$

as depicted in figure 1.

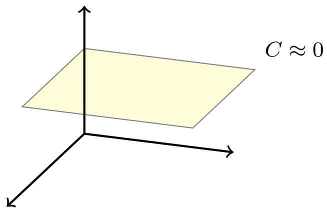


Figure 1: Phase space with the constraint surface identified by $C \approx 0$.

This constraint acts also as the generator of gauge transformations in phase space via Poisson brackets: using the local arbitrary infinitesimal parameter $\zeta(\tau)$ these transformations are generated by

$$\begin{aligned} \delta x^\mu &= \{x^\mu, \zeta C\} = \zeta p^\mu \\ \delta p_\mu &= \{p_\mu, \zeta C\} = 0 \end{aligned} \tag{100}$$

that reproduce the ones in (97). This is a general feature of the so-called constrained hamiltonian systems with first class constraints (those constraints that generate gauge symmetries). Points related by gauge transformations form an *orbit*, and the set of all orbits fill the constraint surface.

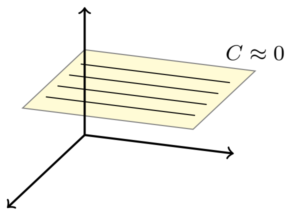


Figure 2: Constraint surface with the gauge orbits generated by gauge transformations.

Quantization

To quantize the relativistic particle, it is convenient to use canonical quantization and choose the formulation IV. Quantization is obtained by interpreting the dynamical classical variables (x^μ, p_μ) as linear operators $(\hat{x}^\mu, \hat{p}_\mu)$ acting on a Hilbert space $\mathcal{H}$. The operators are fixed by their commutation relations, required to be $i\hbar$ times the classical Poisson bracket

$$\{x^\mu, p_\nu\}_{PP} = \delta^\mu_\nu \qquad \longrightarrow \qquad [\hat{x}^\mu, \hat{p}_\nu] = i\hbar\delta^\mu_\nu \; . \tag{101}$$

An arbitrary vector of the Hilbert space $|\phi\rangle \in \mathcal{H}$ does not generally describe a physical state, because it is necessary to remember the equation of motion of the gauge field, $p^\mu p_\mu + m^2 = 0$.

The corresponding constraint function C is then elevated to an operator $\hat{C}$, which is then used as a constraint on the Hilbert space to select the physical states of the system

$$\hat{C}|\phi\rangle = 0 \qquad \to \qquad (\hat{p}^\mu \hat{p}_\mu + m^2)|\phi\rangle = 0\ . \tag{102}$$

The quantum hamiltonian $\hat{H}$ is proportional to the constraint $\hat{C} = \frac{1}{2}(\hat{p}^2 + m^2)$, and thus vanishes on physical states. The corresponding Schroedinger equation on physical states becomes

$$i\hbar \frac{\partial}{\partial \tau}|\phi\rangle = \hat{H}|\phi\rangle = 0 \tag{103}$$

and says that the physical states $|\phi\rangle$ are independent of τ.

The corresponding wave function $\phi(x) = \langle x^\mu|\phi\rangle$ is therefore independent of the parameter τ and satisfies (102), which we recognize as the Klein-Gordon equation

$$\big(- \hbar^2 \partial_\mu \partial^\mu + m^2 \big)\phi(x) = 0\ . \tag{104}$$

Thus, we conclude that the Klein-Gordon equation is obtained by quantizing canonically the relativistic particle. We refer to this as to the "first quantization" of the relativistic particle. Reintroducing the speed of light c with dimensional considerations, the Klein-Gordon equation is written in the usual way

$$\big(\partial_\mu \partial^\mu - \mu^2\big)\phi(x) = 0 \tag{105}$$

where $\mu = \frac{mc}{\hbar}$ is the inverse of the (reduced) Compton wavelength $\lambda = \frac{\hbar}{mc}$ associated with the relativistic particle.

4 Summary

We have seen that the free non-relativistic particle is fixed by the action

$$S[x^i] = \int dt\ \frac{m}{2}\delta_{ij}\dot{x}^i\dot{x}^j \tag{106}$$

which enjoys the global symmetries of the galilean group. It can be coupled to the external potentials g_{ij}, A_i, V to have the particle in a curved space, interacting with a magnetic field and a scalar potential

$$S[x^i] = \int dt\ \Big(\frac{m}{2}g_{ij}(x)\dot{x}^i\dot{x}^j + qA_i(x)\dot{x}^i - V(x)\Big)\ . \tag{107}$$

Similarly, the relativistic free particle is described by an action with gauge invariance, which in formulation III takes the form

$$S[x^\mu, e] = \int d\tau\ \frac{1}{2}(e^{-1}\dot{x}^\mu\dot{x}_\mu - em^2)\ . \tag{108}$$

It has a gauge symmetry and describes three physical degrees of freedom (the position of the relativistic particle in space). It can also be coupled to a spacetime metric $g_{\mu\nu}$, an electromagnetic field A_μ, and a scalar potential V by

$$S[x^\mu, e] = \int d\tau\Big(\frac{1}{2}e^{-1}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + qA_\mu(x)\dot{x}^\mu - e(\frac{1}{2}m^2 + V(x))\Big)\ . \tag{109}$$

The similiarities of the actions (109) and (107) is of great help in the study of the relativistic model and its quantization.