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Constrained hamiltonian systems and relativistic particles

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In this chapter we introduce worldline actions that can be used to describe relativistic particles with and without spin at the quantum level. Relativistic particles are particles whose dynamics is Lorentz invariant. Lorentz transformations mix time and space, and this may cause a conflict with the fact that dynamics singles out a time parameter along which the system evolves. A direct way to describe relativistic particles is to use the time as measured in the chosen inertial frame of reference, for example the lab frame. But then invariance under a Lorentz transformation is not manifest, as time gets mixed with the three space coordinates tracing the position of the particle. Moreover it is difficult to see how to introduce consistent interactions with other particles, or fields, in this set up. A useful alternative is to treat space and time democratically, but the price to pay is that one finds gauge symmetries compensating unphysical degrees of freedom. One may eliminate completely the gauge degrees of freedom by identifying a set of truly physical variables ("unitary gauge"), a procedure that is used to recover the previous formulation. However, it is often convenient to select other types of gauges to keep the Lorentz symmetry manifest ("Lorentz covariant gauges"). A covariant set up allows to introduce interactions in a simpler way.

When treating gauge systems with hamiltonian methods one finds "constrained hamiltonian systems", systems whose dynamics is restricted to a suitable submanifold of phase space. Below we give a concise review of the treatment and quantization of singular lagrangians, i.e. those lagrangians that give rise to constrained hamiltonian systems. This will provide us with a suitable language to describe actions for relativistic particles, with and without spin, and discuss their canonical quantization. Then we introduce various couplings, in particular to gauge fields and to gravitational backgrounds. Eventually, we discuss the corresponding path integral quantization.

1 Constrained hamiltonian systems

Constrained hamiltonian systems typically emerge when one tries to set up an hamiltonian formulation of singular lagrangians, that is lagrangians $L(q^i, \dot{q}^i)$ for which

$$\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = 0 \,. \tag{1}$$

In such a case, when one tries to pass to the hamiltonian formalism by introducing the momenta p_i as independent variables, one finds that the relations between momenta and velocities are not invertible. The momenta are related to the velocities $\dot{q}^i$ by the equations

$$p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i} \,. \tag{2}$$

These equations are not invertible in terms of the velocities precisely when eq. (1) holds, that is when $\det \frac{\partial p^i(\dot q)}{\partial \dot q^j} = 0$. This condition gives rise to constraints between the canonical coordinates (q^i, p_i) of phase space, and they define a hypersurface on which the dynamics takes place. Additional constraints may arise by requiring that the time evolution of the constraints themselves vanishes. Having found all constraints, one can show that they can be classified in two classes: first class constraints and second class constraints.

First class constraints are related to gauge symmetries. They define a constraint surface in phase space. In addition, they generate gauge transformations which relate different points of the surface describing the same physics. All points related this way make up a "gauge orbit", and different gauge orbits correspond to different physical configurations. A proper understanding of this situation is useful whenever one is dealing with gauge systems.

Second class constraints are not related to gauge symmetries, and arise essentially because one tries to set up a hamiltonian formulation of a system that is already in a hamiltonian form. They can be treated using the so-called Dirac brackets, that play the role of the Poisson brackets on the constraint surface. The Dirac brackets make it consistent to solve the second class constraints for a set of independent coordinates of the constraint surface: the latter is then considered as the appropriate phase space on which the hamiltonian dynamics takes place.

Let us spend a few more comments on the procedure how the constraints are obtained in passing form the lagrangian to the hamiltonian. As already said, for singular lagrangians the equations (2) that define the momenta are not invertible in terms of the velocities, and one finds a set of constraints (primary constraints) of the form

$$\phi_a(q,p) \approx 0 \tag{3}$$

where a runs over the number of constraints thus found. The symbol $\approx$ means that the constraints $\phi_a(q,p)$ as functions on the whole phase space do not vanish identically, and thus should not be set to zero before computing the Poisson brackets (the latter are defined for functions of the whole phase space). When the contraint functions are set to vanish, they identify an hypersurface of phase space on which the dynamics must take place, see Figure 1.

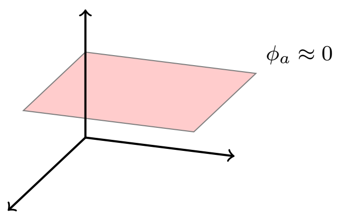

Figure 1: Phase space with the constraint surface identified by $\phi_a \approx 0$.

The canonical hamiltonian H_0 is defined by

$$H \equiv p_i \dot q^i - L \approx H_0 \tag{4}$$

which means that one can use the constraints to simplify the form of the hamiltonian and eliminate any eventual dependence on velocities, that should not appear in functions defined on phase space. As the constraints have been used to find H_0, one defines also the extended hamiltonian H_E, obtained by adding to the canonical hamiltonian H_0 the constraint functions ϕ_a times suitable Lagrange multipliers l^a

$$H_E = H_0 + l^a \phi_a \ . \tag{5}$$

This is the hamiltonian extended to the full phase space that should be used to generate the evolution of the system through Poisson brackets. Note that, using the above notation, one has $H_E \approx H_0$.

At this stage one requires that the dynamics generated by H_E should not take out of the constraint surface. In particular, one requires that

$$\dot{\phi}_a = \{\phi_a, H_E\} \approx 0 \tag{6}$$

where curly brackets denote Poisson brackets. This can be achieved in two possible ways: either (i) it is satisfied directly or by fixing the values of some Lagrange multipliers l^a, or (ii) one finds new constraint $\chi_\alpha \approx 0$ (secondary constraints) to be added to the previous ones. The procedure is repeated iteratively, till one finds all possible constraints. As anticipated, the latter may be classifies as first or second class constraints (of course one can have both at the same time, though for simplicity we consider each case separately).

1.1 Second class constraints

It is useful to discuss second class constraints first. We consider a phase space with canonical coordinates denoted collectively by z^A and a set of constraints $D_a(z) \approx 0$, with $a = 1, ..., n$, that identify an hypersurface in phase space on which the dynamics takes place. These constraints are called second class if they satisfy

$$\det\{D_a, D_b\}\Big|_{D_a=0} \neq 0 \tag{7}$$

i.e. $\det\{D_a, D_b\} \not\approx 0$ using the notation with the weak equality sign. Note that it is enough that the determinant be non vanishing on the constraint surface. In such a case the above condition is sufficient to guarantee that the restriction of the the symplectic structure of the original phase space to the constraint surface is still a symplectic structure. The latter is used to identify the Poisson brackets on the constraint surface, which is then called reduced phase space. Then one can simply work on the reduced phase space, defined by the constraints $D_a = 0$, using the reduced symplectic structure. The formula defining this structure in terms of the variables of the original phase space was found by Dirac. He devised the so-called Dirac brackets, given for any two functions A, B of phase space by

$$\{A, B\}_D = \{A, B\} - \{A, D_a\}(M^{-1})^{ab}\{D_b, B\} \ , \quad M_{ab} \equiv \{D_a, D_b\} \tag{8}$$

which is well defined on the constraint surface, since there $\det M_{ab} \neq 0$. One may check that the constraints D_a have vanishing Dirac bracket with everything else. Indeed one computes for any arbitrary function A

$$\{A, D_a\}_D = \{A, D_a\} - \{A, D_c\}(M^{-1})^{cb}\{D_b, D_a\} = \{A, D_a\} - \{A, D_a\} = 0 \ . \tag{9}$$

Thus it is consistent to implement the constraints strongly, and solve them to find a set of independent coordinates of the reduced phase space. Canonical quantization may then proceed as usual, setting up commutation relations defined by the Dirac brackets.

Typically these constraints arise when one is trying to introduce an hamiltonian formalism for dynamical variables that are already hamiltonian (i.e. with equations of motion that are already first order in time). A simple example may clarify this statement. Let us consider a model defined by the lagrangian

$$L(q,Q,\dot q,\dot Q) = Q\dot q - V(q,Q) \tag{10}$$

that is immediately verifed to be singular. Clearly this is a system with equations of motion that are first order in time, and if we had denoted Q by p and $V(q,Q)$ by $H(q,p)$, we would have recognized the standard form of a phase space lagrangian. Nevertheless we pretend to ignore this knowledge, and proceed with the construction of the hamiltonian formalism using the standard prescriptions. We introduce momenta p and P, conjugate to q and Q respectively, by

$$p = \frac{\partial L}{\partial \dot q} = Q\ , \qquad P = \frac{\partial L}{\partial \dot Q} = 0 \tag{11}$$

and find immediately two constraints

$$D_1 \equiv p - Q = 0\ , \qquad D_2 \equiv P = 0\ . \tag{12}$$

It is easily verified that there are no other constraints and that they are second class. Using Dirac brackets allows to solve the constraints by setting $Q = p$ and $P = 0$, that leaves one with (q,p) as independent coordinates of the reduced phase space. In this simple example one checks that Dirac brackets reproduce precisely the standard Poisson brackets of the (q,p) phase space.

1.2 First class constraints

One defines constraints $C_\alpha(z)$ to be first class if their Poisson brackets satisfy the algebra

$$\{C_\alpha, C_\beta\} = f_{\alpha\beta}{}^\gamma C_\gamma \tag{13}$$

with $f_{\alpha\beta}{}^\gamma$ called structure functions, as they may depend on the phase space coordinates. This implies that the Poisson brackets of the constraints vanish on the constaint surface

$$\{C_\alpha, C_\beta\} \approx 0\ . \tag{14}$$

Again, the weak equality symbol means that one first computes Poisson brackets in the full phase space, and then restricts to the constraint surface by setting $C_\alpha = 0$. Incidentally, one may notice that the concept of Dirac brackets cannot be implemented at this stage.

A key property of first class constraints is that, on top of defining the constraint surface, they also generate gauge transformations by acting with Poisson brackets. Given a generic function of phase space A, its infinitesimal gauge transformation is defined by

$$\delta A = \{A, \epsilon^\alpha C_\alpha\} \tag{15}$$

with infinitesimal local parameters $\epsilon^\alpha \equiv \epsilon^\alpha(t)$ that depend arbitrarily on the time t. In particular, the gauge transformations of the basic coordinates of phase space z^A read

$$\delta z^A = \{z^A, \epsilon^\alpha C_\alpha\}\ . \tag{16}$$

All points sitting on the constraint surface and related by gauge transformations form gauge orbits, equivalence class of points that describe the same physical configuration, see Figure 2

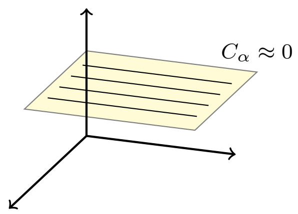

Figure 2: Constraint surface with the gauge orbits generated by gauge transformations.

Gauge invariant quantities are those whose gauge transformations vanish on the constraint surface. Thus B is gauge invariant if

$$\delta B = \{B, \epsilon^\alpha C_\alpha\} \approx 0 \tag{17}$$

which happens if

$$\{B, C_\alpha\} = b_\alpha{}^\beta C_\beta \tag{18}$$

for some functions $b_\alpha{}^\beta$ of phase space.

Time evolution is generated by a gauge invariant hamiltonian H, a hamiltonian that satisfies

$$\{H, C_\alpha\} = h_\alpha{}^\beta C_\beta \tag{19}$$

for suitable functions $h_\alpha{}^\beta$. Such a relation can be read as computing a gauge transformation on H, which is required to vanish on the constraint surface for the hamiltonian to be gauge invariant there ($\delta H = \{H, \epsilon^\alpha C_\alpha\} \approx 0$). Conversely, it can be read as computing the time evolution of the constraint C_α and requiring it to vanish on the constraint surface, so that no additional constraint must be imposed for consistency ($\dot{C}_\alpha \approx 0$). One may take as hamiltonian the canonical hamiltonian H_0 discussed earlier, though for notational simplicity we drop the subscript.

All this information is encoded in an action principle that makes use of the dynamical variables z^A and of Lagrange multipliers λ^α

$$S[z^A, \lambda^\alpha] = \int dt \left(\frac{1}{2} (\Omega^{-1})_{AB}\, z^A \dot{z}^B - H(z) - \lambda^\alpha C_\alpha(z) \right) \, . \tag{20}$$

Here above we have used the notation introduced in eq. (**??**) of chapter **??** to allow for dynamical systems with both commuting and anticommuting variables. We recall that the first term is the symplectic term that fixes the elementary Poisson brackets to be

$$\{z^A, z^B\} = \Omega^{AB} \tag{21}$$

(we consider canonical coordinates for which the matrix Ω^{AB} is constant, a situation that can always be achieved locally, thanks to a theorem by Darboux). The equation of motion of the lagrange multipliers λ^α reproduce the constraint equations $C_\alpha = 0$. To verify that the functions C_α play the additional role of generators of gauge transformations, one checks that the action is invariant under gauge transformations, with infinitesimal parameters $\epsilon^\alpha \equiv \epsilon^\alpha(t)$ depending arbitrarily on time, defined as

$$\begin{aligned} \delta z^A &= \{z^A, \epsilon^\alpha C_\alpha\} \\ \delta \lambda^\alpha &= \dot{\epsilon}^\alpha - \epsilon^\beta \lambda^\gamma f_{\gamma\beta}{}^\alpha - \epsilon^\beta h_\beta{}^\alpha \, . \end{aligned} \tag{22}$$

The gauge transformations of the Lagrange multipliers λ^α depend on the time derivative of the gauge parameters ϵ^α, and for this reason they are also called "gauge fields" (fields in one dimension!). In addition they contain the structure functions $f_{\alpha\beta}{}^\gamma$ of the first class algebra and the functions $h_\alpha{}^\beta$ in eq. (19).

The verification of the gauge invariance of the action proceeds by a direct calculation. Up to total derivatives one starts computing

$$\delta S = \int dt \left(-\dot{z}^A (\Omega^{-1})_{AB} \delta z^B - \delta H(z) - \lambda^\alpha \delta C_\alpha(z) - \delta\lambda^\alpha C_\alpha(z) \right) . \tag{23}$$

The first term, using $\delta z^A = \{z^A, \epsilon^\alpha C_\alpha\} = \Omega^{AB} \frac{\partial_L \epsilon^\alpha C_\alpha}{\partial z^B}$, gives rise by the chain rule to the term $-\epsilon^\alpha \dot{C}_\alpha$, which integrates by part to $\dot{\epsilon}^\alpha C_\alpha$ up to boundary terms. For the second term one gets

$$\delta H = \frac{\partial_R H}{\partial z^A} \delta z^A = \{H, \epsilon^\alpha C_\alpha\} = \epsilon^\alpha h_\alpha{}^\beta C_\beta \tag{24}$$

and similarly for the third term

$$\lambda^\alpha \delta C_\alpha = \lambda^\alpha \{C_\alpha, \epsilon^\beta C_\beta\} = \epsilon^\beta \lambda^\alpha f_{\alpha\beta}{}^\gamma C_\gamma \tag{25}$$

a result valid for both bosonic or fermionic constraints (fermionic constraints require gauge parameters ϵ and corresponding gauge fields λ to be Grassmann variables, of course). Collecting all terms and renaming indices, one finds that the transformation rule of the gauge field λ^α stated above is the correct one to achieve $\delta S = 0$.

The geometrical picture that emerges is that on the full phase space there is an hypersurface, defined by the constraints $C_\alpha = 0$, on which the dynamics takes place. Points of this hypersurface related to other points of the same surface by gauge transformations make up gauge orbits, equivalence class of points describing the same physical situation, as anticipated (see again Figure 2).

The original Poisson bracket structure is singular when restricted to the hypersurface defined by first class constraints. As already mentioned, one may check that Dirac brackets cannot be defined in the present case. How to deal with this situation, and how to quantize the model, can be done in several ways, which make use of the possibility of performing gauge transformations to satisfy suitable gauge fixing conditions. The methods for treating first class constraints and construct canonical quantization of gauge systems can be grouped into three main classes: *i)* the reduced phase space method, *ii)* the Dirac method, *iii)* the BRST method. In the following, we assume the first class constraints to be independent of each other, otherwise certain reducibility relations must be taken into account.

Before presenting a brief discussion of the three methods, it is useful to keep in mind a (trivial) example of a gauge system. It shows the essence of gauge symmetries and provides a simple testing ground for exemplifying the various methods. The model depends on two dynamical variables x and y, and is identified by the lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}\dot{x}^2 . \tag{26}$$

There is an obvious gauge symmetry which transforms nontrivially only the variable y by $\delta y(t) = \epsilon(t)$. It shows that the evolution of $y(t)$ is not fixed by any dynamical law and is arbitrary, as one can modify it by a gauge transformation. Obviously the variable y is unphysical, and could be dropped straight away. However we pretend to keep it into the game to exemplify the methods mentioned above. In passing to the hamiltonian formulation we obtain the momenta

$$p_x = \dot{x} \;, \quad p_y = 0 \tag{27}$$

and find the constraint $C \equiv p_y = 0$. It has vanishing Poisson bracket with the canonical hamiltonian $H_0 = \frac{1}{2}p_x^2$. There are no other constraints, and thus C is a first class constraint. This information is encoded in the phase space action

$$S[x, y, p_x, p_y, \lambda] = \int dt \left(p_x \dot{x} + p_y \dot{y} - \frac{1}{2} p_x^2 - \lambda p_y \right) \tag{28}$$

on which one can test the general statements made above.

Reduced phase space method

Given that the constraint surface is made up by gauge orbits generated by the first class constraints C_α, the idea is to pick a representative from each gauge orbit by using suitable gauge fixing functions $F^\alpha = 0$. The gauge is properly chosen if the set of constraints (C_α, F^α) form a system of second class constraints, which identify a "reduced phase space", see Figure 3.

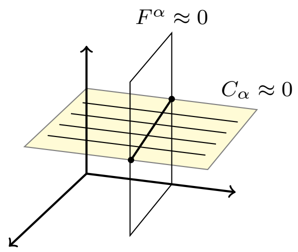

Figure 3: The reduced phase space is the intersection of the two sufaces.

Then one can use the corresponding Dirac brackets, and solve the constraints explicitly to find a set of independent coordinates of the reduced phase space. Canonical quantization now proceeds by finding linear operators with commutation relations specified by the Dirac brackets. This last step may be not so obvious, as the Dirac brackets in the chosen coordinates of the reduced phase space might be complicated. There is a theorem (Darboux's theorem) that guarantees that canonical coordinates exists locally, but it may be difficult to find them out and work with them.

In the example of eq. (26), one may choose as gauge fixing function $F \equiv y = 0$, a configuration that can always be reached using gauge transformations on the variable y (indeed we saw that its evolution is arbitrary, so that one may fix it just by setting $y(t) = 0$). The gauge is well fixed, and indeed the system $C \equiv p_y$ and $F \equiv y$ forms a set of second class constraints. We use them to eliminate the unphysical variables y and p_y. The reduced phase space has coordinates (x, p_x) and the Dirac bracket reduce just to the standard Poisson bracket. Canonical quantization proceeds now as usual.

Dirac method

In this method, one prefers to work in the full phase space to take advantage of the canonical sympletic structure. However one has to remember that physical configurations must lie on the constraints surface $C_\alpha = 0$. Poisson brackets are well-defined on the full phase space, so that one may proceed with canonical quantization. One constructs operators acting on a Hilbert space for all of the phase space variables. However, not all states of the Hilbert space are going to be physical. Classical constraints C_α turn into operators $\hat{C}_\alpha$ that generate gauge transformations at the quantum level. They are used to select the vectors $|\psi_{ph}\rangle$ of the Hilbert space that describe physical configurations (gauge invariant configurations). This is done by requiring that a physical state $|\psi_{ph}\rangle$ satisfies

$$\hat{C}_\alpha |\psi_{ph}\rangle = 0 \qquad \text{for all } \alpha \; . \tag{29}$$

It may happen that this requirement is too strong for certain theories, so that it might be necessary to step back and require the weaker condition

$$\langle \psi'_{ph} | \hat{C}_\alpha | \psi_{ph} \rangle = 0 \tag{30}$$

for arbitrary physical states $|\psi_{ph}\rangle$ and $|\psi'_{ph}\rangle$. In this form these subsidiary conditions had been used by Gupta and Bleuler in describing quantum electrodynamics in the Lorenz gauge. For this reason this method is sometimes called the Dirac-Gupta-Bleuler method.

Having found the subspace of physical states, one should be careful to define a proper scalar product between them, so that the subspace of physical states forms truly a Hilbert space with positive norm. This usually requires the use of some gauge fixing functions, but we do not wish to review here this important issue.

In our standard example, upon quantization one has the operators $(\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y)$ acting on the full Hilbert space, that can be taken as the set of functions $\psi(x, y)$ of both x and y. The condition that select physical states is

$$\hat{p}_y \psi_{ph}(x, y) = 0 \quad \to \quad \frac{\partial}{\partial y} \psi_{ph}(x, y) = 0 \tag{31}$$

so that only wave functions independent of y are physical, as it should be. Note that for these physical states the scalar product must contain some gauge fixing function to be well defined. Using a delta function for the gauge fixing condition $F \equiv y = 0$ employed earlier, one defines

$$\langle \psi_{ph,1} | \psi_{ph,2} \rangle = \int dx dy \, \delta(y) \psi^*_{ph,1}(x) \psi_{ph,2}(x) \tag{32}$$

which is the expected result.

BRST method

This is the most general method that allows for much flexibility in selecting gauge fixing conditions. It encodes the use of Faddeev-Popov ghosts and the ensuing BRST symmetry, originally found in the path integral quantization of lagrangian gauge theories. In this method one enlarges even further the phase space by introducing ghosts degrees of freedom. In the full phase space one finds a symmetry, the BRST symmetry, that encodes the complete information about the first class gauge algebra. Again we present the essentials to give a flavor of the subject. The key property of this construction is the nilpotency of the BRST charge Q, that generates the BRST symmetry, and the associated concept of cohomology, used to select physical states and physical operators.

To start with, we assume that the constraints C_α are all independent. Then one enlarges the original phase space by introducing ghosts c^α and ghost momenta P_α, associated to each constraint C_α, but with opposite Grassmann parity of the latter. That is anticommuting ghosts for bosonic constraints, and commuting ghosts for fermionic constraints. They are defined to have an elementary Poisson bracket of the form

$$\{P_\alpha, c^\beta\} = -\delta_\alpha^\beta \tag{33}$$

that corresponds to a term in the action of the form $S = \int dt\, (\dot{c}^\alpha P_\alpha + ..)$. These variables have assigned a ghost number: $+1$ for c^α and -1 for P_α. All other phase space variables have vanishing ghost number by definition.

In the enlarged phase space, called BRST phase space, one defines the BRST charge by three requirements:
i) the BRST charge Q is real, anticommuting, and of ghost number 1;
ii) Q acts on the original phase space variables as gauge transformations with the ghost variables c^α replacing the gauge parameters (plus eventual higher order terms in the ghosts);
iii) Q is nilpotent, i.e. it has vanishing Poisson bracket with itself, $\{Q, Q\} = 0$.

This is enough to identify the BRST charge uniquely. In fact, from conditions *i)* and *ii)* one finds $Q = c^\alpha C_\alpha + ...$, which can be thought as initial conditions to be fed into *iii)*. The reality properties of the ghosts are chosen to make Q real. Then imposing *iii)* one finds the complete BRST charge. It takes the form

$$Q = c^\alpha C_\alpha + (-1)^{c_\beta} \frac{1}{2} c^\beta c^\alpha f_{\alpha\beta}{}^\gamma P_\gamma + ... \tag{34}$$

where dots indicate higher order terms in the ghost moments P_α. Here $(-1)^{c_\beta}$ is the Grassmann parity of the ghost c_β, so that the formula holds for any type of constraints, bosonic or fermonic or a combination of both. This formula is actually exact if the structure functions $f_{\alpha\beta}{}^\gamma$ are constant. Higher order terms appear on the right hand side in more general cases, they depend on the precise form of the first class constraint algebra, and they are all fixed by the nilpotency condition. To prove eq. (34) one expands the BRST charge in terms of ghost momenta as $Q = Q_0 + Q_1 + Q_2 + ...$, where the subscript counts the number of ghost momenta, and with $Q_0 = c^\alpha C_\alpha$ needed to satisfy points *i)* and *ii)* above. Then one computes

$$\{Q, Q\} = \{Q_0, Q_0\} + 2\{Q_1, Q_0\} + ... \tag{35}$$

to find Q_1 as in (34) in order to cancel a term arising from $\{Q_0, Q_0\}$. For constant structure functions the cancellation is exact, $\{Q_0, Q_0\} + 2\{Q_1, Q_0\} = 0$, and the term $\{Q_1, Q_1\}$ vanishes by itself thanks to the Jacobi identities, satisfied by the structure constants $f_{\alpha\beta}{}^\gamma$. Thus setting $Q_n = 0$ for $n > 1$ one obtains the exact solution to $\{Q, Q\} = 0$.

The power of the BRST construction resides in the fact that the nilpotency of the BRST charge allows to define the concept of cohomology. The cohomology is a vector space made up of elements that are equivalence classes. Different cohomology classes are identified with different physical observables. For that purpose let us first make an aside and review the concept of cohomology.

Let us consider a vector space V and a linear operator $\delta : V \longrightarrow V$ such that $\delta^2 = 0$. Such an operator is called nilpotent. One defines the kernel of δ, to be indicated by Ker(δ), as all elements $\alpha \in V$ such that $\delta\alpha = 0$

$$\text{Ker}(\delta) = \{\alpha \in V \mid \delta\alpha = 0\}\ . \tag{36}$$

Its elements are vectors that are said to be "closed", and often called cocycles. Then, one defines the image of δ, to be indicated by $\mathrm{Im}(\delta)$, as all elements $\beta \in V$ such that there exists an element $\gamma \in V$ for which $\beta = \delta\gamma$

$$\mathrm{Im}(\delta) = \{\beta \in V \mid \exists \gamma \in V \text{ for which } \beta = \delta\gamma\} \ . \tag{37}$$

Its elements are are vectors that are said to be "exact", and often called coboundaries. Clearly, all exact elements are closed, $\mathrm{Im}(\delta) \subset \mathrm{Ker}(\delta)$, because of nilpotency. However not all closed elements may be exact. The cohomology measures the amount of non-exactness. It is defined as the set of equivalence classes $[\alpha]$ of closed elements that differ by exact elements

$$\alpha \sim \alpha' \qquad \text{if} \qquad \alpha' = \alpha + \delta\gamma \ . \tag{38}$$

The space of equivalent classes is denoted by

$$H(\delta) = \frac{\mathrm{Ker}(\delta)}{\mathrm{Im}(\delta)} \tag{39}$$

and is called group of cohomology, or simply cohomology, see Figure 4.

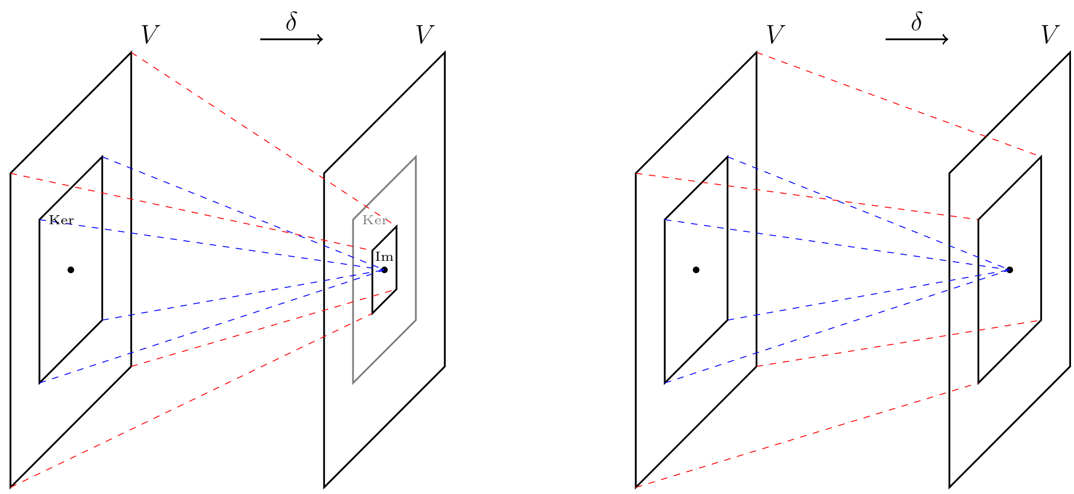

Figure 4: The cohomology measures the amount of non-exactness of closed elements. It is nontrivial when $\mathrm{Im}(\delta) \subset \mathrm{Ker}(\delta)$. The right hand side depicts the case of vanishing cohomology, $\mathrm{Im}(\delta) = \mathrm{Ker}(\delta)$ and all closed elements are exact.

Returning to the BRST construction, one finds that the operator $\{Q, .\}$, that acts on functions A of the BRST phase space as $\{Q, A\}$, is a nilpotent operator. Nilpotency can be verified by using the Jacobi identities of the Poisson brackets, and the properties that the BRST charge Q is anticommuting and satisfies $\{Q, Q\} = 0$, so that one finds $\{Q, \{Q, A\}\} = 0$ for any function A.

Physical observables are defined by the cohomology classes of the BRST operator $\{Q, .\}$ with vanishing ghost number. Thus a physical observable A can be thought of as a function of phase space (with vanishing ghost number) that is BRST invariant

$$\{Q, A\} = 0 \ , \tag{40}$$

keeping in mind that observables related by BRST exact functions describe the same physical situation (i.e. are gauge equivalent)

$$A' \sim A \qquad \text{if} \qquad A' = A + \{Q, B\} \ . \tag{41}$$

Thus A and A' sit in the same cohomology class and identify the same physical observable.

In particular one can show the existence of a BRST invariant hamiltonian H. It satisfies $\{Q, H\} = 0$, which tells at the same time that H is BRST invariant and that Q is conserved along the time evolution generated by H. Equivalent hamiltonians H' are obtained by adding BRST exact terms

$$H' \sim H \qquad \text{if} \qquad H' = H + \{Q, \Psi\} \ . \tag{42}$$

The freedom of choosing Ψ parametrizes the freedom of selecting different gauges to fix the dynamics of the unphysical degrees of freedom. For this reason the function Ψ is often called "gauge fermion" (Ψ must be Grassmann odd for H' be bosonic).

The cohomological structure dictated by the BRST symmetry is carried over to the quantum theory, which now contains the fundamental operators $\hat{Q}$ and $\hat{H}$. In particular the BRST operator $\hat{Q}$ is hermitian, with ghost number one, and satisfies $\hat{Q}^2 = 0$. Physical states are defined by the cohomology of $\hat{Q}$ on the full Hilbert space at vanishing ghost number. That is, physical states are given by vectors of the Hilbert space at vanishing ghost number and satisfying $\hat{Q}|\psi_{ph}\rangle = 0$. States $|\psi'_{ph}\rangle$ equivalent to the previous one are those of the form $|\psi'_{ph}\rangle = |\psi_{ph}\rangle + \hat{Q}|\chi\rangle$ for some $|\chi\rangle$. Similarly, BRST invariant operators are those commuting with the BRST charge $\hat{Q}$ in a graded sense, $[\hat{Q}, \hat{A}_{ph}\} = 0$, with an equivalence relation given by $\hat{A}_{ph} \sim \hat{A}'_{ph} = \hat{A}_{ph} + [\hat{Q}, \hat{B}\}$ for some $\hat{B}$.

One may check that matrix elements of physical operators between physical states do not depend on the representative chosen in the respective classes of equivalence, namely

$$\langle\psi_{ph}|\hat{A}_{ph}|\phi_{ph}\rangle = \langle\psi'_{ph}|\hat{A}'_{ph}|\phi'_{ph}\rangle \ . \tag{43}$$

For example, if $|\phi'_{ph}\rangle = |\phi_{ph}\rangle + \hat{Q}|\chi\rangle$, then

$$\langle\psi_{ph}|\hat{A}_{ph}|\phi'_{ph}\rangle = \langle\psi_{ph}|\hat{A}_{ph}|\phi_{ph}\rangle + \langle\psi_{ph}|\hat{A}_{ph}\hat{Q}|\chi\rangle \tag{44}$$

but the last term vanishes as $\langle\psi_{ph}|$ and $\hat{A}_{ph}$ are physical,

$$\langle\psi_{ph}|\hat{A}_{ph}\hat{Q}|\chi\rangle = \langle\psi_{ph}|[\hat{A}_{ph}, \hat{Q}]|\chi\rangle + \langle\psi_{ph}|\hat{Q}\hat{A}_{ph}|\chi\rangle = 0 \tag{45}$$

where we have taken $\hat{A}_{ph}$ bosonic for simplicity, and thus

$$\langle\psi_{ph}|\hat{A}_{ph}|\phi'_{ph}\rangle = \langle\psi_{ph}|\hat{A}_{ph}|\phi_{ph}\rangle \ . \tag{46}$$

In particular, one can use the freedom of adding BRST exact terms to observables to select suitable gauge fixing fermions $\hat{\Psi}$ for the hamiltonian $\hat{H}$, and express the transition amplitude as

$$\langle\psi_{ph}|e^{-\frac{i}{\hbar}t(\hat{H}+\{\hat{Q},\hat{\Psi}\})}|\phi_{ph}\rangle \tag{47}$$

which eventually may be cast as a (gauge fixed) path integral. Note that the full Hilbert space with ghost degrees of freedom, often called BRST Hilbert space, is not positive definite: the BRST operator is hermitian and nilpotent, but nonvanishing. This forces the BRST Hilbert space to have an indefinite norm (so, technically speaking, the BRST Hilbert space it is not a true Hilbert space). It is however important that the inner product be positive definite for the physical states only. Finally, we recall that the BRST method can be developed directly in the path integral context, and in particular at the lagrangian level, though we will not address its detailed construction.

In our standard simple example given by $L(x,y,\dot{x},\dot{y}) = \frac{1}{2}\dot{x}^2$, upon BRST quantization one finds the operators $(\hat{x},\hat{p}_x,\hat{y},\hat{p}_y,\hat{c},\hat{P})$ acting on the BRST Hilbert space, which may be taken as the set of functions $\psi(x,y,c) = \psi_0(x,y) + \psi_1(x,y)c$ of the coordinates x,y and c, the latter being a real Grassmann variable. This BRST Hilbert space is endowed with a suitable inner product, that however is not positive definite (we do not need to specify it here for our limited purposes). The momenta acts as $\hat{p}_x = -i\frac{\partial}{\partial x}$, $\hat{p}_y = -i\frac{\partial}{\partial y}$, and $\hat{P} = -i\frac{\partial}{\partial c}$. The BRST charge takes the form $\hat{Q} = \hat{c}\hat{p}_y = -ic\frac{\partial}{\partial y}$, and the condition that selects select physical states is

$$\hat{Q}|\psi\rangle = 0 \quad \to \quad \frac{\partial}{\partial y}\psi_0(x,y) = 0 \tag{48}$$

so that they are described by wave functions $\psi_0(x)$ depending only on the x coordinate, as expected. The $\psi_1(x,y)$ part remains arbitrary, but it does not contribute to physical amplitudes as it can be easily shown to be BRST exact (in addition it does not have zero ghost number, so it would have been excluded anyhow).

2 Relativistic particles

The description of relativistic particles is a basic example where gauge symmetries allow for a manifestly Lorentz covariant formulation.

2.1 Scalar particle

Let us consider the case of a massive particle with spin 0. The correct action must be Lorentz invariant to guarantee invariance under change of inertial frame. In the free case the action is proportional to the proper time, a well-known relativistic invariant. If one describes the motion in an inertial frame with cartesian coordinates $x^\mu = (x^0, x^i) = (t, x^i)$ (we employ units with $c=1$), then one may use as dynamical variables the position $x^i(t)$ of the particle at time t. An infinitesimal lapse of proper time can be written as[1] $\sqrt{-ds^2} = \sqrt{-dx^\mu dx_\mu} = \sqrt{dt^2 - dx^i dx^i} = dt\sqrt{1-\dot{x}^i\dot{x}^i}$, so that one finds an action of the form

$$S_I[x^i(t)] = -m\int\sqrt{-ds^2} = -m\int dt\ \sqrt{1-\dot{x}^i(t)\dot{x}^i(t)} \tag{49}$$

where m is the mass of the particle. The overall normalization is fixed by checking that in the non relativistic limit one finds the standard non relativistic kinetic energy (plus the famous constant potential energy $E = mc^2$ due to the rest mass of the particle). This relativistic description is correct, but Lorentz invariance is not manifest. Interactions must be introduced in a very careful way not to destroy the latter.

[1] We recall that we use the Minkowski metric $\eta_{\mu\nu}$ such that $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = -(dx^0)^2 + dx^i dx^i$. Lorentz transformations that leave the line element invariant take the form $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ with $\Lambda^\mu{}_\nu$ satisfying the relation $\Lambda^\mu{}_\lambda\Lambda^\nu{}_\rho\eta_{\mu\nu} = \eta_{\lambda\rho}$. The set of all such matrices form the Lorentz group $O(1,D-1)$ of a D dimensional Minkowski spacetime.

It would be useful to have a description in which Lorentz invariance is kept manifest. This can be done introducing additional dynamical variables supplemented by gauge symmetries, so that one may recover equivalence with the original formulation. In our case, one would like to treat the time coordinate x^0 on the same footing as the spatial coordinates x^i. This can be done as follows. One can use an arbitrary parameter τ to label positions on the worldline, which is embedded in space time by the functions $x^\mu(\tau)$. Using the latter as dynamical variables one finds that the action takes the form

$$S_{II}[x^\mu(\tau)] = -m \int d\tau \ \sqrt{-\dot{x}^\mu \dot{x}_\mu} \tag{50}$$

where now $\dot{x}^\mu \equiv \frac{d}{d\tau} x^\mu$. Lorentz invariance is manifest, as the action is evidently a Lorentz scalar. In addition, one notices that it depends on the velocities $\dot{x}^\mu$ only, so that constant space time translations are seen as additional symmetries that complete the Lorentz group to the full Poincaré group.

To compensate the manifest Lorentz symmetry the model acquires an invariance under a local symmetry, related to the reparametrizations of the worldline

$$\begin{aligned} \tau \quad &\longrightarrow \quad \tau' = \tau'(\tau) \\ x^\mu(\tau) \quad &\longrightarrow \quad x'^\mu(\tau') = x^\mu(\tau) \ . \end{aligned} \tag{51}$$

Infinitesimally, with $\tau' = \tau - \xi(\tau)$, one finds the variations

$$\delta x^\mu(\tau) \equiv x'^\mu(\tau) - x^\mu(\tau) = \xi(\tau) \dot{x}^\mu(\tau) \tag{52}$$

where the infinitesimal parameter $\xi(\tau)$ depends arbitrarily on time τ. Under the transformations (52) the action changes by a total derivative

$$\delta S_{II}[x^\mu] = \int d\tau \ \frac{d}{d\tau}\Big(\xi L_{II}\Big) \tag{53}$$

where $L_{II} = -m\sqrt{-\dot{x}^\mu \dot{x}_\mu}$ is the lagrangian. This proves that the action is invariant under arbitrary reparametrizations of the worldline. This invariance is in fact rather manifest in (50) for finite transformations as well.

Equivalence with formulation I is easily proven: one may choose x^0 as the parameter that labels points on the worldline (this is a gauge fixing choice)

$$x^0(\tau) = \tau \tag{54}$$

so that the variable $x^0(\tau)$ is not dynamical anymore: its time evolution is fixed by the gauge condition. This reproduces the action S_I.

However, one may wish to fix different gauges, so we keep for the moment the gauge freedom. The hamiltonian formulation shows that it is a contrained system. The canonical momenta p_μ are given by

$$p_\mu = \frac{\partial L_{II}}{\partial \dot{x}^\mu} = \frac{m \dot{x}_\mu}{\sqrt{-\dot{x}^2}} \tag{55}$$

and one recognizes that they satisfy the constraint

$$p_\mu p^\mu + m^2 = 0 \ . \tag{56}$$

It is a first class constraint. One may also check that the canonical hamiltonian vanishes.

The phase space action takes the form

$$S_{ph.sp.}[x^\mu(\tau), p_\mu(\tau), e(\tau)] = \int d\tau \left(p_\mu \dot{x}^\mu - \frac{e}{2}(p^\mu p_\mu + m^2)\right) \tag{57}$$

where the Lagrange multiplier e reproduces the first class constraint $H \equiv \frac{1}{2}(p^\mu p_\mu + m^2) = 0$ as its equations of motion: e is called the einbein, as its square defines an intrinsic metric on the worldline. The constraint is traditionally denoted by H as it plays the role of an hamiltonian once the einbein is gauge fixed to a constant value (the canonical hamiltonian vanishes instead, as just seen). Recalling the general structure of hamiltonian gauge systems, described in eqs. (20) and (22), one finds that the gauge symmetry can be written as

$$\begin{aligned} \delta x^\mu &= \{x^\mu, \zeta H\} = \zeta p^\mu \\ \delta p_\mu &= \{p_\mu, \zeta H\} = 0 \\ \delta e &= \dot{\zeta} \end{aligned} \tag{58}$$

where $\zeta(\tau)$ is an arbitrary gauge parameter. The hamiltonian form of the action permits to describe massless particles as well.

Dirac quantization of the model shows the connection to the Klein-Gordon equation. As usual, one extends the phase space variables (x^μ, p_μ) to linear operators $(\hat{x}^\mu, \hat{p}_\mu)$ with commutation relations fixed by the classical Poisson brackets (we use units with $\hbar = 1$)

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu \qquad \longrightarrow \qquad [\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu \ . \tag{59}$$

States $|\phi\rangle$ of the Hilbert space $\mathcal{H}$ evolve in the time parameter τ through the Schroedinger equation. However the canonical hamiltonian vanishes and the Schroedinger equation just tells that states are independent of τ

$$i\hbar \frac{\partial}{\partial \tau}|\phi\rangle = 0 \ . \tag{60}$$

In addition, generic states of the Hilbert space are not physical in general, as one must take into account the constraint $p^\mu p_\mu + m^2 = 0$. The latter is used à la Dirac to select the physical states of the system

$$(\hat{p}^\mu \hat{p}_\mu + m^2)|\phi\rangle = 0 \ . \tag{61}$$

In terms of the wave function $\phi(x) = \langle x^\mu|\phi\rangle$ it takes the form of the Klein-Gordon equation

$$\big(-\partial_\mu \partial^\mu + m^2\big)\phi(x) = 0 \ . \tag{62}$$

Thus we see how the Klein-Gordon equation is obtained by first quantizing a relativistic scalar particle.

Eliminating the momenta p_μ through their algebraic equations of motion

$$\frac{\delta S_{ph.sp.}}{\delta p_\mu} = \dot{x}^\mu - e p^\mu = 0 \qquad \Longrightarrow \qquad p^\mu = e^{-1}\dot{x}^\mu \tag{63}$$

produces the action in configuration space

$$S_{co.sp.}[x^\mu(\tau), e(\tau)] = \int d\tau \ \frac{1}{2}(e^{-1}\dot{x}^\mu \dot{x}_\mu - e m^2) \ . \tag{64}$$

The local symmetry takes now the geometric form

$$\delta x^\mu = \xi \dot{x}^\mu \ , \quad \delta e = \frac{d}{d\tau}(\xi e) \tag{65}$$

where the local parameter ξ is related to the previous one in (58) by $\zeta = e\xi$. Absorbing the einbein in ζ allows to present the gauge symmetry in an abelian form, which may be convenient for performing various algebraic manipulations. The configuration space action is useful for quantizing with path integrals. As in this book we mostly use euclidean path integrals, we perform a Wick rotation to euclidean time ($\tau \to -i\tau$) to obtain an euclidean action S_E ($iS_{co.sp.} \to -S_E$) that takes the form

$$S_E[x^\mu(\tau), e(\tau)] = \int d\tau \ \frac{1}{2}(e^{-1}\dot{x}^\mu \dot{x}_\mu + em^2) \tag{66}$$

with a corresponding Wick rotation in space time ($x^0 \to -ix^D$) used to achieve a fully positive definite euclidean action.

2.2 Spin 1/2 particles

A spin 1/2 particle is similarly described in a manifestly covariant way by a gauge model with one local supersymmetry on the worldline. For the massless case, the phase space action depends on the particle space time coordinates x^μ joined by the real Grassmann variables ψ^μ, supersymmetric partners of the former. The latter supply degrees of freedom associated to spin. In addition, there are Lagrange multipliers e and χ, with commuting and anticommuting character, respectively, that gauge suitable first class constraints. Eventually, their effect is to eliminate negative norm states from the physical spectrum, and make the particle model consistent with unitarity at the quantum level. The gauge fields (e, χ) are called einbein and gravitino, respectively, as they form the supergravity multiplet in one dimension.

Let us see how all this is realized explicitly. The action takes the form

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + \frac{i}{2}\psi_\mu \dot{\psi}^\mu - eH - i\chi Q \right) \tag{67}$$

where the first class constraints are given by

$$H = \frac{1}{2}p^2 \ , \qquad Q = p_\mu \psi^\mu \tag{68}$$

and generate through Poisson brackets the $N = 1$ susy algebra in one dimension

$$\{Q, Q\} = -2iH \ . \tag{69}$$

This algebra is computed by using the graded Poisson brackets of the phase space coordinates, $\{x^\mu, p_\nu\} = \delta^\mu_\nu$ and $\{\psi^\mu, \psi_\nu\} = -i\delta^\mu_\nu$, fixed by the sympletic term of the action.

The gauge transformations are generated on (x, p, ψ) through Poisson brackets with $V \equiv \zeta H + i\epsilon Q$ where ζ and ϵ are local parameters with appropriate Grassmann parity

$$\begin{aligned} \delta x^\mu &= \{x^\mu, V\} = \zeta p^\mu + i\epsilon\psi^\mu \\ \delta p_\mu &= \{p_\mu, V\} = 0 \\ \delta \psi^\mu &= \{\psi^\mu, V\} = -\epsilon p^\mu \end{aligned} \tag{70}$$

while on gauge fields they are obtained using the structure constants of the constraint algebra

$$\delta e = \dot{\zeta} + 2i\chi\epsilon \ , \quad \delta\chi = \dot{\epsilon} \ . \tag{71}$$

Let us now study canonical quantization to uncover the consequences of the constraints and see how the Dirac equation emerges. Elevanting the phase space variables to operators one find the following (anti) commutation relations (in the quantum case curly brackets denote anticommutators, as customary)

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu \ , \qquad \{\hat{\psi}^\mu, \hat{\psi}^\nu\} = \eta^{\mu\nu} \tag{72}$$

while other graded commutator vanish. The former relations are realized on the usual infinite dimensional Hilbert space of functions of the particle coordinates. The latter relations are seen to give rise to a Clifford algebra that may be identified with the usual Clifford algebra of the Dirac gamma matrices ($\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$). Thus, they can be realized on the finite dimensional Hilbert space of spinors (of dimensions $2^{[\frac{D}{2}]}$, where square brackets [..] indicate the integer part) as

$$\hat{\psi}^\mu \ \to \ \frac{1}{\sqrt{2}}\gamma^\mu \ . \tag{73}$$

The direct product of the two Hilbert spaces obtained above form the full Hilbert space of the model, identified with the space of spinor fields.

Again the full information of physical states resides in the constrains implemented à la Dirac. In particular, the constraint due to the susy charge $\hat{Q} = \hat{p}_\mu \hat{\psi}^\mu$ gives rise to the massless Dirac equations

$$\hat{p}_\mu \hat{\psi}^\mu |\Psi\rangle = 0 \quad \to \quad \gamma^\mu \partial_\mu \Psi(x) = 0 \tag{74}$$

where as usual spinorial indices are suppressed and matrix algebra is understood. The constraint $\hat{H}|\Psi\rangle = 0$ leads to the massless Klein Gordon equation for all components of the spinor Ψ, but is automatically satisfied as a consequence of the algebra $\hat{Q}^2 = \hat{H}$. Thus, we recognize how a first quantized description of a spin 1/2 particle emerges from canonical quantization.

To study the corresponding path integral quantization it is useful to eliminate the momenta p_μ by their algebraic equations of motion

$$\frac{\delta S}{\delta p_\mu} = \dot{x}^\mu - ep^\mu - i\chi\psi^\mu = 0 \qquad \Longrightarrow \qquad p^\mu = e^{-1}(\dot{x}^\mu - i\chi\psi^\mu) \tag{75}$$

to obtain the action in configuration space

$$S_{co.sp.}[x, \psi, e, \chi] = \int d\tau \left(\frac{1}{2} e^{-1} (\dot{x}^\mu - i\chi\psi^\mu)^2 + \frac{i}{2} \psi_\mu \dot{\psi}^\mu \right) \tag{76}$$

whose local symmetries may be recovered from the phase space ones.

Finally, a Wick rotation to euclidean time ($\tau \to -i\tau$) produces the euclidean action S_E ($iS_{co.sp.} \to -S_E$)

$$S_E[x, \psi, e, \chi] = \int d\tau \left(\frac{1}{2} e^{-1} (\dot{x}^\mu - \chi\psi^\mu)^2 + \frac{1}{2} \psi_\mu \dot{\psi}^\mu \right) \tag{77}$$

with a corresponding Wick rotation in space time ($x^0 \to -ix^D$ and $\psi^0 \to -i\psi^D$) used to achieve a formally positive definite euclidean action.

2.3 Massive spin 1/2 particles

The massive case is slightly more subtle. To obtain it we use a general method of introducing a mass term starting from the massless theory formulated in one dimension higher.

Let us exemplify the procedure for the scalar particle, and then apply it to the spin 1/2 case. We denote the extra dimension by x^5, and coordinates by $x^M = (x^\mu, x^5)$, so that indices split as $M = (\mu, 5)$. The massless scalar particle in one dimension higher is described by the phase space action

$$S = \int d\tau \left(p_M \dot{x}^M - \frac{e}{2} p_M p^M \right) . \tag{78}$$

Now one can impose the constraint $p_5 = m$, where m is a constant to be identified as the mass of the particle in one dimension lower. The action takes the form

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + m\dot{x}^5 - \frac{e}{2}(p_\mu p^\mu + m^2) \right) . \tag{79}$$

The term with the coordinate x^5 is a total derivative and can be dropped form the action. Indeed $p_5 - m = 0$ can be considered as a first class constraint, so that x^5 becomes a gauge degree of freedom that can be disregarded. Thus one obtains the massive case.

We can follow the same steps for the spinor case. Starting from an action of the form (67) in one dimension higher, and imposing the constraint $p_5 = m$ one finds

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + \frac{i}{2}\psi_\mu \dot{\psi}^\mu + \frac{i}{2}\psi^5 \dot{\psi}^5 - e\frac{1}{2}(p_\mu p^\mu + m^2) - i\chi(p_\mu \psi^\mu + m\psi^5) \right) \tag{80}$$

where again x^5 drops out, but ψ^5 is retained. Let us check that this indeed describes a free, massive spin 1/2 particle, at least in even dimensions. We focus directly in $D = 4$ dimensions, and note that on top of the operators in eq. (72) one find the extra fermionic operator $\hat{\psi}^5$ that can be identified with the chirality matrix $\gamma^5/\sqrt{2}$, and the susy constraint $p_\mu \psi^\mu + m\psi^5 = 0$ becomes at the quantum level

$$(-i\gamma^\mu \partial_\mu + m\gamma^5)\Psi = 0 . \tag{81}$$

One can multiply it by γ^5 and recognize that the set $\tilde{\gamma}^\mu = -i\gamma^5\gamma^\mu$ is an equivalent set of gamma matrices (they satisfy the same Clifford algebra). Dropping now the tilde one may recognize it as the massive Dirac equation written in the standard form

$$(\gamma^\mu \partial_\mu + m)\Psi = 0 . \tag{82}$$

2.4 Massless spin 1 particles

A spin 1 particle can similarly be described in a manifestly covariant way by a model with two local supersymmetries on the worldline. This model is often called the $N = 2$ spinning particle or, equivalently, the $SO(2)$ spinning particle. Its action is characterized by a $N = 2$ extended supergravity on the worldline. The gauge fields $(e, \chi, \bar{\chi}, a)$ of the $N = 2$ supergravity contain the einbein e which gauges worldline translations, complex conjugate gravitinos χ and $\bar{\chi}$ which gauge the $N = 2$ worldline supersymmetry, and the $U(1) \equiv SO(2)$ gauge field a for the symmetry which rotates by a phase the worldline fermions and gravitinos. The einbein and the gravitinos correspond to constraints that eliminate negative norm states and make the particle model consistent with unitarity at the quantum level. The constraints arising from the $U(1)$ gauge field a makes the model irreducible, eliminating some further degrees of freedom.

Let us see explicitly how all of this emerges. The action in flat target spacetime is written in terms of graded phase space variables given by real bosonic coordinates and momenta (x^μ, p_μ) and complex fermionic variables $(\psi^\mu, \bar\psi_\mu)$, with Poisson brackets $\{x^\mu, p_\nu\} = \delta^\mu_\nu$ and $\{\psi^\mu, \bar\psi_\nu\} = -i\delta^\mu_\nu$. The Grassmann variables are again used to generate suitable spin degrees of freedom. The constraints to be imposed to guarantee unitarity and irreducibility of the model are generated by the charges

$$H = \frac{1}{2} p_\mu p^\mu \ , \quad Q = p_\mu \psi^\mu \ , \quad \bar Q = p_\mu \bar\psi^\mu \ , \quad J = \bar\psi_\mu \psi^\mu \ . \tag{83}$$

This symmetry algebra can be gauged since the charges close under Poisson brackets and identify a set of first class constraints

$$\{Q, \bar Q\} = -2iH \ , \quad \{J, Q\} = iQ \ , \quad \{J, \bar Q\} = -i\bar Q \tag{84}$$

(other Poisson brackets vanish). Introducing the gauge fields $G = (e, \bar\chi, \chi, a)$ corresponding to the constraints $C_\alpha = (H, Q, \bar Q, J)$ produces the action

$$S = \int d\tau \Big[p_\mu \dot x^\mu + i\bar\psi_\mu \dot\psi^\mu - e \underbrace{\Big(\frac{1}{2} p_\mu p^\mu\Big)}_{H} - i\bar\chi \underbrace{\Big(p_\mu \psi^\mu\Big)}_{Q} - i\chi \underbrace{\Big(p_\mu \bar\psi^\mu\Big)}_{\bar Q} - a \underbrace{\Big(\bar\psi_\mu \psi^\mu\Big)}_{J} \Big] \tag{85}$$

which is manifestly Poincaré invariant in target space. It describes a relativistic model. The gauge transformations on the phase space variables are generated by the Poisson bracket with the generator $V \equiv \zeta H + i\bar\epsilon Q + i\epsilon \bar Q + \alpha J$, where $(\zeta, \bar\epsilon, \epsilon, \alpha)$ are local parameters with appropriate Grassmann parity,

$$\begin{aligned}
\delta x^\mu &= \{x^\mu, V\} = \zeta p^\mu + i\bar\epsilon \psi^\mu + i\epsilon \bar\psi^\mu \\
\delta p_\mu &= \{p_\mu, V\} = 0 \\
\delta \psi^\mu &= \{\psi^\mu, V\} = -\epsilon p^\mu - i\alpha \psi^\mu \\
\delta \bar\psi^\mu &= \{\bar\psi^\mu, V\} = -\bar\epsilon p^\mu + i\alpha \bar\psi^\mu
\end{aligned} \tag{86}$$

while the gauge transformations on gauge fields are obtained using the structure constants of the constraint algebra (84)

$$\begin{aligned}
\delta e &= \dot\zeta + 2i\bar\chi\epsilon + 2i\chi\bar\epsilon \\
\delta \chi &= \dot\epsilon + ia\epsilon - i\alpha\chi \\
\delta \bar\chi &= \dot{\bar\epsilon} - ia\bar\epsilon + i\alpha\bar\chi \\
\delta a &= \dot\alpha \ .
\end{aligned} \tag{87}$$

A peculiarity of this model is the possibility of adding a Chern-Simons term for the worldline gauge field a

$$S_{CS} = q \int d\tau \, a \tag{88}$$

which is obviously invariant under the gauge transformations (87). Absence of quantum anomalies requires quantization of the Chern-Simons coupling[2]

$$q = \frac{D}{2} - p - 1 \ , \qquad p \text{ integer.} \tag{89}$$

[2]The quantization of the Chern-Simons coupling is almost self-evident in canonical quantization, to be discussed shortly. It emerges to guarantee nontrivial solutions to the gauge constraints.

With this precise coupling the $N=2$ spinning particle describes an antisymmetric gauge field of rank p, and corresponding field strength of rank $p+1$, which for $p=1$ gives a massless spin 1 particle in first quantization.

Let us derive these statements by reviewing the canonical quantization of the model. The phase space variables are turned into operators satisfying the following (anti)commutation relations

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu \ , \qquad \{\hat{\psi}^\mu, \hat{\psi}^\dagger_\nu\} = \delta^\mu_\nu \ . \tag{90}$$

States of the full Hilbert space can be identified with functions of the coordinates x^μ and ψ^μ. By x^μ we denote the eigenvalues of the operator $\hat{x}^\mu$, while for the fermionic variables we use bra coherent states defined by

$$\langle\psi|\hat{\psi}^\mu = \langle\psi|\psi^\mu \ = \psi^\mu\langle\psi| \ . \tag{91}$$

A state $|\phi\rangle$ is then described by the wave function

$$\phi(x,\psi) \equiv (\langle x| \otimes \langle\psi|)|\phi\rangle \tag{92}$$

and the "coordinate" operators $\hat{x}^\mu$ and $\hat{\psi}^\mu$ act on it by multiplication with x^μ and ψ^μ, respectively, while the "momentum" operators $\hat{p}_\mu$ and $\hat{\psi}^\dagger_\mu$ act by derivation by $-i\partial_\mu$ and $\frac{\partial}{\partial\psi^\mu}$, respectively.

Since the ψ's are Grassmann variables the wave function has the following general expansion

$$\phi(x,\psi) = F(x) + F_\mu(x)\psi^\mu + \frac{1}{2}F_{\mu_1\mu_2}(x)\psi^{\mu_1}\psi^{\mu_2} + \ldots + \frac{1}{D!}F_{\mu_1\ldots\mu_D}(x)\psi^{\mu_1}\cdots\psi^{\mu_D} \ . \tag{93}$$

The classical constraints C_α now become operators $\hat{C}_\alpha$ which are used to select physical states through the requirement $\hat{C}_\alpha|\phi_{ph}\rangle = 0$. In the above representation they take the form of differential operators

$$\begin{aligned} \hat{H} &= -\frac{1}{2}\partial_\mu\partial^\mu \ , \quad \hat{Q} = -i\psi^\mu\partial_\mu \ , \quad \hat{Q}^\dagger = -i\partial_\mu\frac{\partial}{\partial\psi_\mu} \\ \hat{J} &= -\frac{1}{2}\Big[\psi^\mu, \frac{\partial}{\partial\psi^\mu}\Big] - q = p+1-\psi^\mu\frac{\partial}{\partial\psi^\mu} \end{aligned} \tag{94}$$

where we have redefined $\hat{J}$ to include the Chern-Simons coupling, and antisymmetrized $\hat{\psi}^\mu$ and $\hat{\psi}^\dagger_\mu$ to resolve an ordering ambiguity. The constraint $\hat{J}|\phi_{ph}\rangle = 0$ selects states with only $p+1$ ψ's, namely

$$\phi_{ph}(x,\psi) = \frac{1}{(p+1)!}F_{\mu_1\ldots\mu_{p+1}}(x)\psi^{\mu_1}\cdots\psi^{\mu_{p+1}} \ . \tag{95}$$

The constraints $\hat{Q}|\phi_{ph}\rangle = 0$ gives integrability conditions (Bianchi identities once solved for a gauge potential)

$$\partial_{[\mu}F_{\mu_1\ldots\mu_{p+1}]}(x) = 0 \tag{96}$$

and the constraint $\hat{Q}^\dagger|\phi_{ph}\rangle = 0$ produces the other Maxwell equations in vacuum

$$\partial^{\mu_1}F_{\mu_1\ldots\mu_{p+1}}(x) = 0 \ . \tag{97}$$

The constraint $\hat{H}|\phi_{ph}\rangle = 0$ leads to the massless Klein Gordon equation for all components of the tensor field, and is automatically satisfied as consequence of the algebra $\{\hat{Q},\hat{Q}^\dagger\} = 2\hat{H}$.

Thus we see that the $N=2$ spinning particle describes the propagation of a p-form gauge potential $A_{\mu_1\ldots\mu_p}$ in a gauge invariant way, namely through its $F_{\mu_1\ldots\mu_{p+1}}$ field strength. Setting $p=1$ one finds the Maxwell equations for electromagnetism in vacuum, interpreted as the wave equations describing a massless particle of spin 1 in first quantization.

Eliminating algebraically the momenta p_μ by using their equations of motion

$$p^\mu = \frac{1}{e}(\dot{x}^\mu - i\bar{\chi}\psi^\mu - i\chi\bar{\psi}^\mu) \tag{98}$$

gives the action in configuration space

$$S_{co.sp.} = \int d\tau \Big[\frac{1}{2}e^{-1}(\dot{x}^\mu - i\bar{\chi}\psi^\mu - i\chi\bar{\psi}^\mu)^2 + i\bar{\psi}_\mu\dot{\psi}^\mu - a(\bar{\psi}^\mu\psi_\mu - q)\Big] \ . \tag{99}$$

The corresponding gauge invariances can be deduced from the phase space ones using (98). A Wick rotation (where the gauge field a is also Wick-rotated as $a \to ia$ to keep the gauge group compact) brings it into the euclidean form

$$S_E = \int d\tau \Big[\frac{1}{2}e^{-1}(\dot{x}^\mu - \bar{\chi}\psi^\mu - \chi\bar{\psi}^\mu)^2 + \bar{\psi}_\mu(\partial_\tau + ia)\psi_\mu - iqa\Big] \ . \tag{100}$$

2.5 Massive spin 1 particles

The inclusion of a mass term can be obtained again by considering the massless case in one dimension higher. Thus, we consider $D+1$ dimensions, and eliminate one dimension (say x^5) by setting $p_5 = m$. The coordinate x^5 can be dropped from the action (it appears as a total derivative), while the corresponding $N=2$ fermionic partners are retained and denoted by θ and $\bar{\theta}$. This procedure gives the massive $N=2$ action, which in phase space reads

$$S \ = \ \int d\tau \Big[p_\mu \dot{x}^\mu + i\bar{\psi}_\mu\dot{\psi}^\mu + i\bar{\theta}\dot{\theta} - eH - i\bar{\chi}Q - i\chi\bar{Q} - aJ\Big] \tag{101}$$

where the constraints $C_\alpha \equiv (H, Q, \bar{Q}, J)$ are now given by

$$H = \frac{1}{2}(p_\mu p^\mu + m^2) \ , \quad Q = p_\mu\psi^\mu + m\theta \ , \quad \bar{Q} = p_\mu\bar{\psi}^\mu + m\bar{\theta} \ , \quad J = \bar{\psi}^\mu\psi_\mu + \bar{\theta}\theta - q \ . \tag{102}$$

Their Poisson brackets generate the $N=2$ susy algebra in one dimension

$$\{Q, \bar{Q}\} = -2iH \ , \quad \{J, Q\} = iQ \ , \quad \{J, \bar{Q}\} = -i\bar{Q} \tag{103}$$

and is gauged by the gauge fields $\lambda^\alpha \equiv (e, \bar{\chi}, \chi, a)$. The quantized Chern-Simons coupling $q \equiv \frac{D+1}{2} - p - 1$ has been inserted directly into the definition of J and allows to describe an antisymmetric tensor of rank p. This is seen in canonical quantization. The phase space variables are turned into operators satisfying the (anti)commutation relations

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu \ , \qquad \{\hat{\psi}^\mu, \hat{\psi}^\dagger_\nu\} = \delta^\mu_\nu \ , \qquad \{\hat{\theta}, \hat{\theta}^\dagger\} = 1 \ . \tag{104}$$

States of the Hilbert space can be described by functions of the coordinates $(x^\mu, \psi^\mu, \theta)$, and since ψ^μ and θ are Grassmann variables, a wave function has the following general expansion

$$\begin{aligned}\phi(x,\psi,\theta) \ &= \ F(x) + F_\mu(x)\psi^\mu + \frac{1}{2}F_{\mu_1\mu_2}(x)\psi^{\mu_1}\psi^{\mu_2} + \ldots + \frac{1}{D!}F_{\mu_1\ldots\mu_D}(x)\psi^{\mu_1}\cdots\psi^{\mu_D} \\ &+ \ im\Big(A(x)\theta + A_\mu(x)\theta\psi^\mu + \frac{1}{2}A_{\mu_1\mu_2}(x)\theta\psi^{\mu_1}\psi^{\mu_2} + \ldots \\ &+ \ \frac{1}{D!}A_{\mu_1\ldots\mu_D}(x)\theta\psi^{\mu_1}\cdots\psi^{\mu_D}\Big) \ . \end{aligned} \tag{105}$$

The imaginary unit i makes it possible to impose reality conditions on the fields F and A, and the factor m is introduced for obtaining a standard normalization of the A fields.

The classical constraints C become operators $\hat{C}$, which select the physical states by $\hat{C}|\phi_{ph}\rangle = 0$. In the above representation the constraints take the form of differential operators

$$\hat{H} = \frac{1}{2}(-\partial_\mu \partial^\mu + m^2) \ , \quad \hat{Q} = -i\psi^\mu \partial_\mu + m\theta \ , \quad \hat{Q}^\dagger = -i\partial_\mu \frac{\partial}{\partial \psi_\mu} + m\frac{\partial}{\partial \theta}$$
$$\hat{J} = -\frac{1}{2}\Big[\psi^\mu, \frac{\partial}{\partial \psi^\mu}\Big] - \frac{1}{2}\Big[\theta, \frac{\partial}{\partial \theta}\Big] - q = p + 1 - \psi^\mu \frac{\partial}{\partial \psi^\mu} - \theta\frac{\partial}{\partial \theta} \tag{106}$$

where in $\hat{J}$ we have antisymmetrized $\hat{\psi}^\mu$, $\hat{\psi}^\dagger_\mu$ and $\hat{\theta}$, $\hat{\theta}^\dagger$ to resolve an ordering ambiguity. The $\hat{J}|\phi_{ph}\rangle = 0$ constraint selects states with only $p+1$ Grassmann variables, namely

$$\phi_{ph}(x, \psi) = \frac{1}{(p+1)!} F_{\mu_1 \ldots \mu_{p+1}}(x) \psi^{\mu_1} \cdots \psi^{\mu_{p+1}} + \frac{im}{p!} A_{\mu_1 \ldots \mu_p}(x) \theta \psi^{\mu_1} \cdots \psi^{\mu_p} \ . \tag{107}$$

The constraint $\hat{Q}|\phi_{ph}\rangle = 0$ gives integrability conditions on F_{p+1} and solves them in terms of A_p (the former are then the Bianchi identities)

$$\partial_{[\mu} F_{\mu_1 \mu_2 \ldots \mu_{p+1}]} = 0 \ , \quad F_{\mu_1 \mu_2 \ldots \mu_{p+1}} = \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} \ . \tag{108}$$

The constraint $\hat{Q}^\dagger|\phi_{ph}\rangle = 0$ produces the Proca equations together with the familiar longitudinal constraint on A_p

$$\partial^{\mu_1} F_{\mu_1 \ldots \mu_{p+1}} = m^2 A_{\mu_2 \ldots \mu_{p+1}} \ , \quad \partial^{\mu_1} A_{\mu_1 \ldots \mu_p} = 0 \ . \tag{109}$$

The constraint $\hat{H}|\phi_{ph}\rangle = 0$ is identically satisfied as a consequence of the algebra $\{\hat{Q}, \hat{Q}^\dagger\} = 2\hat{H}$.

Thus, the model reproduces the Proca field equations for p-forms. For $p = 1$ it gives the standard description of a massive spin 1 particle.

2.6 Spin $N/2$ massless particles

The previous constructions can be generalized to describe spin $N/2$ particles in a manifestly covariant way. One obtains a model with $O(N)$ extended local supersymmetries on the worldline ($O(N)$ extended supergravity), which may be called $O(N)$ spinning particle. The gauge fields of the $O(N)$ supergravity contains: the einbein e which gauges worldline translations, N gravitinos χ_i with $i = 1, ...N$ which gauge N worldline supersymmetries, and a gauge field a_{ij} for gauging the $O(N)$ symmetry which rotates the worldline fermions and gravitinos. The einbein and the gravitinos correspond to contraints that eliminate negative norm states and make the particle model consistent with unitarity. The contraints arising from the gauge field a_{ij} makes the model irreducible, eliminating some further degrees of freedom.

In more details, the model is described by bosonic (x^μ, p_μ) and real fermionic ψ^μ_i phase space variables with graded Poisson brackets $\{x^\mu, p_\nu\} = \delta^\mu_\nu$ and $\{\psi^\mu_i, \psi^\nu_j\} = -i\eta^{\mu\nu}\delta_{ij}$. The indices $i, j = 1, \cdots, N$ are internal indices labeling the various worldline fermion species. The constraints

$$H = \frac{1}{2} p_\mu p^\mu \ , \quad Q_i = p_\mu \psi^\mu_i \ , \quad J_{ij} = i\psi^\mu_i \psi_{j\mu} \tag{110}$$

close under Poisson brackets and generate the O(N) extended supersymmetry algebra in one dimension, used as first class constraints

$$\begin{aligned} \{Q_i, Q_j\} &= -2i\delta_{ij} H \\ \{J_{ij}, Q_k\} &= \delta_{jk} Q_i - \delta_{ik} Q_j \\ \{J_{ij}, J_{kl}\} &= \delta_{jk} J_{il} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik} + \delta_{il} J_{jk} \ . \end{aligned} \tag{111}$$

Introducing corresponding gauge fields e, χ_i, a_{ij} one obtains the action

$$S = \int d\tau \left(p_\mu \dot{x}^\mu + \frac{i}{2}\psi_{i\mu}\dot{\psi}_i^\mu - e \underbrace{\left(\frac{1}{2}p_\mu p^\mu\right)}_{H} - i\chi_i \underbrace{\left(p_\mu \psi_i^\mu\right)}_{Q_i} - \frac{1}{2}a_{ij} \underbrace{\left(i\psi_i^\mu \psi_{j\mu}\right)}_{J_{ij}} \right) \tag{112}$$

with gauge symmetries generated by $V \equiv \zeta H + i\epsilon_i Q_i + \frac{1}{2}\beta_{ij}J_{ij}$ on the phase space variables

$$\begin{aligned} \delta x^\mu &= \{x^\mu, V\} = \zeta p^\mu + i\epsilon_i \psi_i^\mu \\ \delta p_\mu &= \{p_\mu, V\} = 0 \\ \delta \psi_i^\mu &= \{\psi_i^\mu, V\} = -\epsilon_i p^\mu + \beta_{ij}\psi_j^\mu \end{aligned} \tag{113}$$

and corresponding transformations on the gauge fields

$$\begin{aligned} \delta e &= \dot{\zeta} + 2i\chi_i \epsilon_i \\ \delta \chi_i &= \dot{\epsilon}_i - a_{ij}\epsilon_j + \beta_{ij}\chi_j \\ \delta a_{ij} &= \dot{\beta}_{ij} + \beta_{im}a_{mj} + \beta_{jm}a_{im} \ . \end{aligned} \tag{114}$$

Eliminating algebraically the momenta p_μ by using their equation of motion

$$p^\mu = \frac{1}{e}(\dot{x}^\mu - i\chi_i \psi_i^\mu) \tag{115}$$

one obtains the action in configuration space

$$S_{c.sp.} = \int d\tau \Big[\frac{1}{2}e^{-1}(\dot{x}^\mu - i\chi_i\psi_i^\mu)^2 + \frac{i}{2}\psi_{i\mu}\dot{\psi}_i^\mu - \frac{i}{2}a_{ij}\psi_i^\mu \psi_{j\mu} \Big] \ . \tag{116}$$

The corresponding gauge invariances can be deduced from the phace space one.

Canonical quantization à la Dirac in $D = 4$ shows that the model describes massless fields of spin $N/2$. Indeed, one can prove that the constraints generate the massless Bargmann Wigner equations for a multispinor wave function with N spinorial indices $\Psi_{\alpha_1 \dots \alpha_N}$. The equations take the form of a Dirac equation for each spinorial index

$$(\gamma^\mu \partial_\mu)_{\alpha_i}{}^{\tilde{\alpha}_i} \Psi_{\alpha_1 .. \tilde{\alpha}_i .. \alpha_N}(x) = 0 \qquad i = 1, ..., N \tag{117}$$

where, in addition, certain algebraic constraints are required to eliminate the lower spin components, present in the tensor product of N spin 1/2 fields. The algebraic constraints, that arise from the J_{ij} charges, make in particular the spinor completely symmetric, though further relations are present in general.

3 Coupling to background fields

In the previous section we have described the propagation of free relativistic particles. The next step is to introduce interactions. The simplest option is to couple the particles to background fields. The latter take into account either external configurations fixed by the experimental apparatus or the effect of other quantum particles. One may consider various type of backgrounds, like a scalar potentials generated by fields of spin 0, or vector potential as in the case of abelian gauge fields. Gravitational effects can be described by coupling to a curved background. The guiding principle is to use the symmetries to identify and constrain possible interaction terms. In particular the manifest Lorentz symmetry simplify enormously this task.

Let us start briefly describing the coupling to a scalar potential. For the spin 0 particle, it is enough to introduce it in the Hamiltonain constraint, shifting for example the mass term by $m^2 \to m^2 + V(x)$, where $V(x)$ represents the external scalar potential. Similarly, for the spin 1/2 particle, considering the massive case, one can shift the mass term in the susy constraint Q by $m \to m + \lambda\phi(x)$, where $\phi(x)$ represent an external scalar field and λ a corresponding coupling constant. Then one works out the $N = 1$ susy algebra to compute how the hamiltonian constraint looks like for consistency. The discussion of more complicated cases, like the coupling of the spin 1/2 particle to pseudoscalars fields are postponed to later chapters.

Let us now consider the coupling to an external abelian vector field A_μ. For the spin 0 particle, this can be done by using the minimal substitution

$$p_\mu \to p_\mu - qA_\mu(x) \equiv \pi_\mu \tag{118}$$

in the hamiltonian constraint H. Here q is the charge of the particle, and π_μ is called the covariant momentum. It has non trivial Poisson bracket proportional to the gauge invariant field strength

$$\{\pi_\mu, \pi_\nu\} = q(\partial_\mu A_\nu - \partial_\nu A_\mu) = qF_{\mu\nu} \ . \tag{119}$$

Upon quantization, the covariant momentum gives rise to the gauge covariant derivative $\hat{\pi}_\mu \to -iD_\mu = -i(\partial_\mu - iqA_\mu)$. One recognizes that the quantum hamiltonian constraint $\hat{H} \sim \hat{\pi}^\mu \hat{\pi}_\mu + m^2$ produces the correct minimally coupled Klein Gordon equation. Inserting this in the action (57), eliminating the momenta, and Wick rotating to euclidean time produces the euclidean action

$$S[x^\mu, e; A_\mu] = \int d\tau \ \left(\frac{1}{2} e^{-1} \dot{x}^\mu \dot{x}_\mu + \frac{1}{2} e m^2 - iqA_\mu(x)\dot{x}^\mu \right) \tag{120}$$

which contain the expected coupling to the gauge field. Note that inserted in the path integral it gives rise to the Wilson line $e^{iq\int A_\mu dx^\mu}$. Note also that after gauge fixing the einbein e to a constant, namely $e(\tau) = 2T$ where T is often called the Fock Schwinger proper time, the euclidean action takes the form

$$S[x^\mu; A_\mu] = \int d\tau \ \left(\frac{1}{4T} \dot{x}^\mu \dot{x}_\mu + Tm^2 - iqA_\mu(x)\dot{x}^\mu \right) \ . \tag{121}$$

As we shall see, when performing the path integral in this gauge fixed form, an integration over the proper time T arises.

A similar procedure can be employed for the spin 1/2 case. In this case one performs the minimal substitution in the susy charge constraint Q. Then, the correct hamiltonian constraint H follows by computing the susy algebra. This gives rise to non minimal couplings in H, necessary for consistency. One finds (for the massless case)

$$Q = \psi^\mu (p_\mu - qA_\mu(x)) \equiv \psi^\mu \pi_\mu \ , \qquad H = \frac{1}{2}\pi^\mu \pi_\mu + \frac{iq}{2} F_{\mu\nu}\psi^\mu \psi^\nu \tag{122}$$

with an analogous formula for the massive case. Evidently, the susy constraint gives rise to the minimally coupled Dirac equation. Inserting the constraints in the phase space action, going to configuration space, Wick rotating, and considering the gauge fixing conditions $e(\tau) = 2T$ and

$\chi(\tau) = 0$, appropriate when considering a loop as worldline, produces the following euclidean action for a massive spin 1/2 particle

$$S[x^\mu, \psi^\mu; A_\mu] = \int d\tau \ \left(\frac{1}{4T} \dot{x}^\mu \dot{x}_\mu + Tm^2 - iqA_\mu(x)\dot{x}^\mu + \frac{1}{2}\psi_\mu \dot{\psi}^\mu + iqTF_{\mu\nu}(x)\psi^\mu\psi^\nu \right) \ . \tag{123}$$

The last two terms are responsible for the spinning degrees of freedom. This form of the action will be used in later chapters.

Trying to employ the same procedure for the massless spin 1 particle finds an obstruction, as the minimal coupling of the susy charge produce constraints that do not satisfy a first class algebra. This signals the fact that it is problematic to introduce abelian gauge coupling for particles with spin higher then 1/2. Non abelian couplings can nevertheless be introduced, indeed we know that Yang Mills theory exists, and this case will be briefly described in later chapters.

Finally, let us mention how to introduce a gravitational coupling. This is achieved by using a background metric $g_{\mu\nu}$. For the spin zero particle it is enough to covariantize the hamiltonian constraint H by

$$p^\mu p_\mu \equiv \eta^{\mu\nu} p_\mu p_\nu \ \to \ g^{\mu\nu}(x) p_\mu p_\nu + \xi R(x) \tag{124}$$

where in the right hand side we have inserted also a non minimal coupling with the same mass dimension (ξ is a dimensionless coupling constant). Non minimal coupling of higher dimensions can also be introduced, as long as they are scalars under an arbitrary change of coordinates, but their effect is negligeable at low energies and are typically disregarded. The corresponding euclidean action in the gauge $e = 2T$ reads

$$S[x^\mu; g_{\mu\nu}] = \int d\tau \left(\frac{1}{4T} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + T(m^2 + \xi R(x)) \right) \ . \tag{125}$$

For spin 1/2 and 1 one can proceed minimally covariantizing the susy charges, working out the modifications to the other constraint necessary to keep the algebra first class. This can be achieved successfully to find the correctly coupled action to be used in the path integral applications of later chapter. We refer to those chapters for the explicit formulae.

4 Path integral quantization

We conclude this chapter with a brief discussion of the path integral quantization of relativistic particles, taking into account a covariant gauge fixing of the local symmetries.

The main points are already present in the quantization of the scalar particle, so that we start considering its free euclidean action in configuration space, eq. (66), that we report here for convenience

$$S[x, e] = \int_0^1 d\tau \ \frac{1}{2}(e^{-1}\dot{x}^\mu \dot{x}_\mu + em^2) \ . \tag{126}$$

We have chosen to parametrize the worldline with the parameter $\tau \in [0, 1]$. The most important topologies of the worldline are the interval I, suitable for describing the propagation of the particle, and the circle S^1 (or one dimensional torus T^1), which enters in the first quantized representation of the one-loop effective action induced by relativistic particles (FIGURES).

The path integral quantization is formally given by

$$Z \sim \int \frac{\mathcal{D}x\mathcal{D}e}{\text{Vol(Gauge)}} \ e^{-S[x,e]} \tag{127}$$

where the overcounting from summing over gauge equivalent configurations is formally taken into account by dividing by the volume of the gauge group. Concretely, the factorization of the volume of the gauge group can be achieved by using the Feddeev-Popov method or using the modern BRST approach. We briefly describe the latter to justify the gauge-fixed path integral to be used extensively in later chapters.

The BRST quantization method builds on the existence of a rigid symmetry, the BRST symmetry. The latter arises by elevating the gauge parameter of the local symmetry to a ghost variable with opposite Grassmann character of the former. It is used to show that physical observables are independent of the gauge chosen, so that one may fix the gauge symmetries in the most convenient way to perform the path integral calculations. In spirit, it is based on the same structure described in the hamiltonian analysis of section 1.2. We adapt it here to the lagrangian situation, and take the relativistic scalar particle as guiding example.

The gauge symmetry of action (126), reported below eq. (65), can be presented in the abelian form by redefining $\zeta = e\xi$ as local parameter

$$\delta x^\mu = \zeta e^{-1} \dot{x}^\mu \ , \quad \delta e = \dot{\zeta} \ . \tag{128}$$

From the local symmetry, one finds a rigid BRST symmetry by setting $\zeta(\tau) = \Lambda c(\tau)$ in the gauge transformations. Here $c(\tau)$ is the ghost, a real anticommuting variable, and Λ is the BRST constant parameter, a purely imaginary Grassmann number needed to keep the combination $\Lambda c(\tau)$ real and bosonic. One then requires nilpotency, i.e. $[\delta_B(\Lambda_1), \delta_B(\Lambda_2)] = 0$ on all variables including the ghosts. This requirement fixes the BRST transformation rule on the ghost $c(\tau)$. Generically it results proportional to the structure constants of the gauge algebra, and in our case it vanishes as the gauge algebra is abelian. The nilpotent BRST symmetry is then given by

$$\delta_B x^\mu = \Lambda c e^{-1} \dot{x}^\mu \ , \quad \delta_B e = \Lambda \dot{c} \ , \quad \delta_B c = 0 \ . \tag{129}$$

It is obviously a symmetry of (126), as at this stage it just reproduces the original gauge symmetry.

One could now rewrite the path integral as

$$Z \sim \int \mathcal{D}x \mathcal{D}e \mathcal{D}c \; e^{-S[x,e]} \tag{130}$$

identifying $\int \mathcal{D}c \sim \text{Vol(Gauge)}^{-1}$, but this is still of no practical use, as there is no freedom to perform the gauge fixing yet. For that purpose, one introduces extra variables, the non-minimal fields b and π, where b is called the antighost and is anticommuting, while π is called the auxiliary field and is commuting. They are defined to have the BRST transformation

$$\delta_B b = i\Lambda\pi \ , \quad \delta_B \pi = 0 \tag{131}$$

which is nilpotent. One can interpret b as a variable that can be shifted away completely by the gauge symmetry $i\Lambda\pi$, while π is interpreted as the bosonic ghost of this gauge symmetry. Thus these non-minimal variables are not physical and their introduction does not modify the cohomology, i.e. the physical observables of the model. However, they allow to select quite general gauge fixing terms. At this stage it is useful to assign ghost numbers to the variables introduced thus far. One assigns zero ghost number to the original variables, x^μ and e, as well as to the auxiliary field π. Then one assigns ghost number 1 to the ghost c, and ghost number -1 to the antighost b. It will be a conserved number if the gauge fixing term is chosen accordingly.

Now let's come to the choice of the gauge fixing term. It must be BRST invariant, as the BRST symmetry is crucial to show independence of physical results from the chosen gauge. This is achieved by adding to the classical action (126) a term with vanishing ghost number of the form

$$S_{fix}[x,e,c,b,\pi] = \frac{\delta}{\delta\Lambda}\Psi\ , \qquad \Psi \equiv \int_0^1 d\tau\, b f(x,e,\pi) \tag{132}$$

where Ψ is called the gauge fixing fermion, that depends on the antighost b and on the arbitrary function $f(x,e,\pi)$. The symbol $\frac{\delta}{\delta\Lambda}$ indicates the BRST transformation with the constant parameter Λ removed from the left. It is automatically BRST invariant, as the BRST transformation is nilpotent. Also, it conserves the ghost number. The function $f(x,e,\pi)$ parametrizes the arbitrariness of selecting a gauge fixing condition. One could use an even more general form of the gauge fermion Ψ, but the above one is general enough for our applications.

The path integral

$$Z \sim \int \mathcal{D}x\mathcal{D}e\mathcal{D}c\mathcal{D}b\mathcal{D}\pi\ e^{-S_q}\ , \quad S_q \equiv S[x,e] + S_{fix}[x,e,c,b,\pi] \tag{133}$$

is now in a well-defined form, explicitly calculable for suitable choices of the gauge fixing function f. The quantum action S_q is gauge fixed and BRST invariant. Using the BRST symmetry one can show that it does not depend on the gauge fixing fermion, so that one may choose the most convenient one. Choosing the gauge is an art. Generic choices, such as $f = 0$, are formally correct, but produce the final result in the singular form $\infty \cdot 0$, that is rather useless.

We choose now a suitable gauge fixing condition. We require the einbein $e(\tau)$ to be constant, by convention $e(\tau) = 2T$ with T a constant. This can be achieved by the gauge fermion

$$\Psi = \int_0^1 d\tau\, b(2T - e) \quad \to \quad S_{fix} = \int_0^1 d\tau \left(i\pi(2T-e) + b\dot{c}\right) \tag{134}$$

Path integrating over π produces the functional Dirac delta $\delta(e - 2T)$, and path integrating over e fixes $e = 2T$. One is left with

$$Z \sim \int \mathcal{D}x\mathcal{D}c\mathcal{D}b\ e^{-\int_0^1 d\tau\,(\frac{1}{4T}\dot{x}^2 + m^2 T + b\dot{c})} \tag{135}$$

and one recognizes the Faddeev-Popov ghosts b, c that can be path integrated to produce a Faddeev-Popov determinant

$$Z \sim \int \mathcal{D}x\, \mathrm{Det}(\partial_\tau)\ e^{-\int_0^1 d\tau\,(\frac{1}{4T}\dot{x}^2 + m^2 T)}\ . \tag{136}$$

This is almost the end of the story, except for one important detail related to global properties of the worldline. The einbein e cannot be completely gauged away, and the constant T is not arbitrary. Indeed the length of the worldline $\int_0^1 d\tau\, e = 2T$ is gauge invariant, so that one should still integrate over all possible values of T, taken to be positive. The final answer is obtained by computing the determinant, but to fix appropriately the overall normalization it is expedient to compare with the Schwinger proper-time representation for the propagator and one-loop effective action, for the latter see the end of section **??**.

For the worldline topology of the interval I, one obtains the QFT propagator of the scalar particle. The Faddeev-Popov determinant is just a constant and can be absorbed in the overall normalization, and the path integral formula takes the form

$$Z_I = \int_0^\infty dT\ e^{-m^2 T} \int_I \mathcal{D}x\, e^{-\int_0^1 d\tau\, \frac{1}{4T}\dot{x}^2}\ . \tag{137}$$

The last path integral on the interval I is a free one, similar to that computed in chapter **??** for non relativistic particles. The known answer (see eqs. (**??**) and (**??**), adapting notations and using boundary conditions $x_f = x$ and $x_i = x'$) reads

$$\int_I \mathcal{D}x\, e^{-\int_0^1 d\tau\, \frac{1}{4T}\dot{x}^2} = \frac{1}{(4\pi T)^{\frac{D}{2}}}\, e^{-\frac{(x'-x)^2}{4T}} = \int \frac{d^D p}{(2\pi)^D}\, e^{ip(x-x')} e^{-p^2 T}\,. \tag{138}$$

Here we have used the Fourier representation of the result, obtainable directly also from the phase space path integral. It can be inserted in (137) to produce the standard euclidean Feynman QFT propagator for the free complex Klein Gordon field $\phi(x)$

$$\begin{aligned} Z_I &= \int \frac{d^D p}{(2\pi)^D}\, e^{ip(x-x')} \int_0^\infty dT\, e^{-(p^2+m^2)T} = \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip(x-x')}}{p^2+m^2} \\ &= \langle \phi(x)\phi^*(x')\rangle_{QFT}\,. \end{aligned} \tag{139}$$

In this example, we see explicitly how second quantized objects can be reproduced in a first quantized picture.

For the topology of the circle S^1, one obtains the QFT one-loop effective action

$$Z_{S^1} = -\int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int_P \mathcal{D}x\, e^{-\int_0^1 d\tau\, \frac{1}{4T}\dot{x}^2}\,. \tag{140}$$

It contains the extra factor T^{-1} due to the fact that there is a zero mode in the ghost determinant that signals a translational symmetry of the circle (T is basically the volume of the circle and one should divide by this overcounting). The subscript P stands for periodic boundary conditions for the coordinates $x(\tau)$, appropriate when defined on the circle S^1. The overall normalization (-1) is chosen to agree with the QFT definition of the one-loop effective action. To compute the remaining free path integral it is convenient to switch back to the operatorial picture

$$\int_P \mathcal{D}x\, e^{-\int_0^1 d\tau\, \frac{1}{4T}\dot{x}^2} = \mathrm{Tr}\, e^{-\hat{p}^2 T} = \int \frac{d^D p}{(2\pi)^D} e^{-p^2 T} \tag{141}$$

so that

$$Z_{S^1} = \int \frac{d^D p}{(2\pi)^D} \left(-\int_0^\infty \frac{dT}{T}\, e^{-(p^2+m^2)T} \right) = \int \frac{d^D p}{(2\pi)^D} \ln(p^2+m^2) = \Gamma_{eff}^{QFT} \tag{142}$$

which gives the expected one-loop QFT effective action. It contains the (diverging) contribution to the vacuum energy of complex scalar field. The ultraviolet divergence can be seen as arising from the $T \to 0$ limit of the proper-time integration. One may regulate it and apply the QFT renormalization procedure.

A similar program can be carried out for the spin 1/2 fermionic particle. Let us start from the action in eq. (80). After Wick rotation, and passing to the configuration space, one finds the euclidean action

$$S[x,\psi,\psi_5,e,\chi] = \int_0^1 d\tau\, \frac{1}{2} \left(e^{-1}(\dot{x}-\chi\psi)^2 + \psi\dot{\psi} + \psi_5\dot{\psi}_5 + em^2 + 2i\chi m\psi_5 \right) \tag{143}$$

where we have suppressed obvious indices. It is again quantized by a path integral

$$Z \sim \int \frac{\mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi_5\mathcal{D}e\mathcal{D}\chi}{\mathrm{Vol(Gauge)}}\, e^{-S[x,\psi,\psi_5,e,\chi]}\,. \tag{144}$$

There are two local symmetries to take care of, local supersymmetry and reparametrizations, with gauge fields χ and e, respectively. On the interval I, one can choose a gauge with constant $e(\tau) = 2T$ and $\chi(\tau) = \theta$, with T and θ gauge invariant constants that must be integrated over in the path integral. The variable T is the usual Schwinger proper-time, while θ is a corresponding supersymmetric partner, the super proper-time of Grassmann character. This produces a formula of the type

$$Z_I = \int d\theta \int_0^\infty dT\; e^{-m^2T} \int_I \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi_5 e^{-\int_0^1 d\tau\,(\frac{1}{4T}(\dot{x}-\theta\psi)^2+\frac{1}{2}\psi\dot{\psi}+\frac{1}{2}\psi_5\dot{\psi}_5+i\theta m\psi_5)}\;. \qquad (145)$$

We will not analyze this formula much further. However, to explain its meaning, let us just notice that in the equivalent phase space path integral the integration over the proper time T produces a term $(p^2+m^2)^{-1}$, the denominator of the Feynman propagator of the second quantized Dirac field, while the integration over the super proper time θ produces the numerator, proportional to $\gamma^\mu p_\mu + m\gamma_5$, arising naturally for a Dirac equation written as in eq. (81). Redefining the gamma matrices as explained there would then give a numerator proportional to the standard term $-i\gamma^\mu p_\mu + m$, the proportionality factor being just the matrix γ_5.

Applications of the worldline approach for fermions have been considered mostly for the loop, i.e. for the worldline with the topology of the circle S^1. In such a case, choosing anti periodic boundary conditions for the ψ's and χ, as appropriate for fermions, shows that a constant configuration for χ is not allowed, and it is consistent to select the gauge $\chi(\tau) = 0$. The gravitino can be gauged away completely. In this gauge the variable ψ_5 decouples completely and can be ignored. The integration over the proper time remains, and one is left with the formula

$$Z_{S^1} = \frac{1}{2}\int_0^\infty \frac{dT}{T}\; e^{-m^2T}\int_P \mathcal{D}x \int_A \mathcal{D}\psi\; e^{-\int_0^1 d\tau\,(\frac{1}{4T}\dot{x}^2+\frac{1}{2}\psi\dot{\psi})} \qquad (146)$$

where A stands for anti periodic boundary conditions. The decoupling of ψ^5 remains valid also when considering gauge and gravitational couplings, and the formula above gives a useful representation of the one loop effective action of the quantum field theory of a Dirac spinor in terms of worldlines. The overall normalization 1/2 is inserted to agree with the QFT normalization. To check this last statement one notices that the fermionic path integral factorizes and computes the trace in the finite dimensional Hilbert yspace corresponding to the worldline fermions (i.e. the trace of the identity in the space of gamma matrices). It produces the factor $2^{\frac{D}{2}}$ in even dimensions D. For $D = 4$, proceeding as for the scalar particle, one finds

$$Z_{S^1} = -2\int \frac{d^4p}{(2\pi)^4}\ln(p^2+m^2) \qquad (147)$$

which takes into account the correct number of degrees of freedom of the fermion, together with the correct sign for fermionic loops.

Having fixed the correct overall normalization, one can proceed to calculate amplitudes arising from coupling to background fields, which is the main task of the following parts of the book.

5 Appendix: the case of the electromagnetic field

The electromagnetic field $F_{\mu\nu}$ in Heaviside–Lorentz units has components

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} . \tag{148}$$

It can be described in terms of the gauge potential $A_\mu = (A_0, \vec{A}) = (-\Phi, \vec{A})$, where Φ is the electric potential, by setting $F_{\mu\nu} = \partial_\mu A_\mu - \partial_\mu A_\mu$. Indeed the splitting of time and space components reproduces the usual definition of electric and magnetic fields

$$\begin{aligned} F_{i0} &= \partial_i A_0 - \partial_0 A_i = -\partial_i \Phi - \partial_0 A_i = E_i \\ F_{ij} &= \partial_i A_j - \partial_j A_i = \epsilon_{ijk} B_k . \end{aligned} \tag{149}$$

In terms of the potential A_μ the dynamics is fixed by the lagrangian

$$L = \int d^3x \, \mathcal{L} , \qquad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} F_{0i}^2 - \frac{1}{4} F_{ij}^2 = \frac{1}{2} (\vec{E}^2 - \vec{B}^2) \tag{150}$$

which gives rise to the Maxwell's equations in empty space. It is a singular lagrangian because of the gauge invariance $\delta A_\mu(x) = \partial_\mu \alpha(x)$.

From the general theory of constrained hamiltonian systems, one expects that the hamiltonian treatment will contain first class constraints. Defining the momenta (at fixed time t) by

$$\pi^\mu(\vec{x}) = \frac{\delta L}{\delta(\dot{A}_\mu(\vec{x}))} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0}(\vec{x}) \tag{151}$$

one finds the primary constraint

$$\pi^0(\vec{x}) \approx 0 \tag{152}$$

while

$$\pi_i(\vec{x}) = \partial_0 A_i(\vec{x}) + \partial_i \Phi(\vec{x}) = -E_i(\vec{x}) \tag{153}$$

just identifies the remaining momenta as (minus) the components of electric field. These relations are used to express the velocities $\partial_0 A_i$ in terms of the the momenta π_i in the construction of the hamiltonian. The Poisson brackets (at fixed time t) are given by

$$\{A_i(\vec{x}), \pi_j(\vec{y})\} = \delta_{ij} \delta^{(3)}(\vec{x} - \vec{y}) , \qquad \{A_0(\vec{x}), \pi^0(\vec{y})\} = \delta^{(3)}(\vec{x} - \vec{y}) . \tag{154}$$

The canonical hamiltonian H_0 is found by the Legendre transform and must be suitably simplified by using the constraints

$$H_0 = \int d^3x \, (\pi_i \partial_0 A_i - \mathcal{L}) = \int d^3x \left(\frac{1}{2} \vec{\pi}^2 + \frac{1}{2} \vec{B}^2 - A_0 \vec{\nabla} \cdot \vec{\pi} \right) \tag{155}$$

where an integration by part has been performed, and where $\vec{B}$ has to be interpreted as function of $\vec{A}$ by $\vec{B} = \vec{\nabla} \times \vec{A}$. Then, the extended hamiltonian is

$$H_E = H_0 + \int d^3x \, \lambda_1(\vec{x}) \pi^0(\vec{x}) \tag{156}$$

with $\lambda_1(\vec{x})$ a Lagrange multiplier. Now computing

$$\dot{\pi}^0 = \{\pi^0, H_E\} = \vec{\nabla} \cdot \vec{\pi} \tag{157}$$

and requiring that $\dot{\pi}^0 \approx 0$, one finds the secondary constraints

$$\vec{\nabla} \cdot \vec{\pi} \approx 0 \tag{158}$$

which is recognized as the Gauss's law ($\vec{\nabla} \cdot \vec{E} = 0$). No other constraint emerges, and the set

$$C_1 \equiv \pi^0 \approx 0 \ , \qquad C_2 \equiv \vec{\nabla} \cdot \vec{\pi} \approx 0 \tag{159}$$

form a set of first class constraint, as expected because of the gauge invariance of the configuration space lagrangian

$$\{C_\alpha(\vec{x}), C_\beta(\vec{y})\} = 0 \qquad \alpha, \beta = 1, 2 \ . \tag{160}$$

These constraints can be used to further simplify the canonical hamiltonian H_0 and they appear in the extended hamiltonian H_E multiplied by the corresponding Lagrange multipliers $\lambda_1(\vec{x})$ and $\lambda_2(\vec{x})$

$$H_E = \underbrace{\int d^3x \left(\frac{1}{2}\vec{\pi}^2(\vec{x}) + \frac{1}{2}\vec{B}^2(\vec{x})\right)}_{H_0} + \int d^3x \left(\lambda_1(\vec{x})\pi^0(\vec{x}) + \lambda_2(\vec{x})\vec{\nabla} \cdot \vec{\pi}(\vec{x})\right) \ . \tag{161}$$

Note that the last term in (155) has been eliminated by using the newly found constraint. The latter reappears in H_E with A_0 replaced by the Lagrange multiplier λ_2.

The treatment in terms of the *reduced phase space method* proceeds by setting the gauge fixing conditions

$$F_1 \equiv A_0 \approx 0 \ , \qquad F_2 \equiv \vec{\nabla} \cdot \vec{A} \approx 0 \tag{162}$$

which added to first class constraints turn them into a set of second class constraints. The gauge is well-fixed by F_1 and F_2 as

$$\begin{aligned} \{C_1(\vec{x}), F_1(\vec{y})\} &= \{\pi^0(\vec{x}), A_0(\vec{y})\} = -\delta^{(3)}(\vec{x} - \vec{y}) \neq 0 \\ \{C_2(\vec{x}), F_2(\vec{y})\} &= \{\vec{\nabla} \cdot \vec{\pi}(\vec{x}), \vec{\nabla} \cdot \vec{A}(\vec{y})\} = \partial_i^{(x)} \partial_j^{(y)} \{\pi^i(\vec{x}), A^j(\vec{y})\} \\ &= \partial_i^{(x)} \partial_j^{(y)} (-\delta^{ij} \delta^{(3)}(\vec{x} - \vec{y})) = \nabla^2_{(x)} \delta^{(3)}(\vec{x} - \vec{y}) \neq 0 \ . \end{aligned} \tag{163}$$

This is enough to prove that the set of constraints (C_α, F_α) is second class.

Evidently, the phase space coordinates (A_0, π^0) are unphysical, as they are set to vanish. The Dirac brackets eliminates them straight away. The remaining vector fields $\vec{A}$ and $\vec{\pi}$ are purely transversal[3], and their Dirac bracket takes the form

$$\{A_i(\vec{x}), \pi_j(\vec{y})\}_{DB} = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right) \delta^{(3)}(\vec{x} - \vec{y}) \tag{165}$$

[3]We recall that a vector field $\vec{A}$ can be decomposed into a transversal part $\vec{A}_T$ and a longitudinal part $\vec{A}_L$

$$\vec{A} = \vec{A}_T + \vec{A}_L \tag{164}$$

where $\vec{\nabla} \cdot \vec{A}_T = 0$ and $\vec{\nabla} \times \vec{A}_L = 0$ (Helmholtz decomposition).

where the derivatives act on the x variable. As a check one may notice that these Dirac brackets are consistent with the transversality of the dynamical variables A_i and π_i. We recall that the symbol $\frac{1}{\nabla^2}\delta^{(3)}(\vec{x}-\vec{y})$ stands for the Green function $G(\vec{x}-\vec{y})$ that satisfies $\nabla^2 G(\vec{x}-\vec{y}) = \delta^{(3)}(\vec{x}-\vec{y})$. It emerges when inverting the Poisson brackets of the second class constraint in (163), as needed in the definition of the Dirac brackets.

At last, we rewrite our final result in terms of the vector potential A_i and electric field E_i constrained to be transverse ($\vec{\nabla}\cdot\vec{A}=0$ and $\vec{\nabla}\cdot\vec{E}=0$). These constrained fields are the coordinates of the reduced phase space, and have the Dirac bracket

$$\{A_i(\vec{x}), E_j(\vec{y})\}_{DB} = -\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta^{(3)}(\vec{x}-\vec{y}) \ . \tag{166}$$

The hamiltonian is given by

$$H = \int d^3x\, \frac{1}{2}(\vec{E}^2 + \vec{B}^2) = \int d^3x\, \frac{1}{2}\left(\vec{E}^2 + (\vec{\nabla}\times\vec{A})^2\right) = \int d^3x\, \frac{1}{2}\left(\vec{E}^2 + (\partial_i A_j)^2\right) \tag{167}$$

where the last form follows for the fact that $\vec{A}$ is tranverse. The hamiltonian is a function of the coordinates of the reduced phase space, and generates the evolution of the system by the Hamilton's equations of motion (written in terms of the Dirac brackets). The latter are calculated as follows

$$\begin{aligned} \dot{A}_i(\vec{x}) &= \{A_i(\vec{x}), H\}_{DB} = -E_i(\vec{x}) \\ \dot{E}_i(\vec{x}) &= \{E_i(\vec{x}), H\}_{DB} = -\nabla^2 A_i(\vec{x}) \ . \end{aligned} \tag{168}$$

Indeed these equations, first order in time, are equivalent to the wave equation

$$\ddot{A}_i - \nabla^2 A_i = 0 \tag{169}$$

for the two independent functions contained in the transverse vector potential A_i.