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Heisenberg-Euler effective lagrangians

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We derive here effective lagrangians for the electromagnetic field induced by a loop of charged virtual particles of spin 0 and 1/2. Historically, Heisenberg and Euler derived in 1936 an effective lagrangian that includes the virtual effects of electrons. Soon after Weisskopf extended that calculation to a charged scalar particle.

For a particle of charge q and mass m the results, that include the classical lagrangian plus the leading effective quartic interaction in the electromagnetic field, are the following

$$\begin{aligned} \mathcal{L}^{(0)} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{q^4}{16\pi^2 m^4}\left[\frac{1}{288}(F^{\mu\nu}F_{\mu\nu})^2 + \frac{1}{360}F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}\right] \\ \mathcal{L}^{(1/2)} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{q^4}{8\pi^2 m^4}\left[\frac{1}{72}(F^{\mu\nu}F_{\mu\nu})^2 - \frac{7}{180}F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}\right] \end{aligned} \tag{1}$$

or, equivalently[1],

$$\begin{aligned} \mathcal{L}^{(0)} &= \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \frac{\alpha^2}{360\, m^4}\left[7(\vec{E}^2 - \vec{B}^2)^2 + 4(\vec{E}\cdot\vec{B})^2\right] \\ \mathcal{L}^{(1/2)} &= \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \frac{2\alpha^2}{45\, m^4}\left[(\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E}\cdot\vec{B})^2\right] \end{aligned} \tag{2}$$

where $\alpha = \frac{q^2}{4\pi}$ is the fine-structure constant when q is taken as the electron charge.

We use here a first quantized approach to derive these results. We consider first the case of a spin 0 particle, it exemplifies the main points of the calculation. Then we extend it to the spin 1/2 particle, of more direct physical relevance. All charged particles induce similar nonlinearities, but since the outcome is proportional to m^{-4}, it is the lightest charged particle that gives the dominant contribution, i.e. the electron. This nonlinear effect has not been detected experimentally yet, because of the smallness of the coupling.

[1]One may express the result in terms of the Lorentz invariants $\vec{E}^2 - \vec{B}^2$ and $\vec{E}\cdot\vec{B}$. Using the components

$$F^{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix} , \qquad F_{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix}$$

one immediately finds

$$F^{\mu\nu}F_{\mu\nu} = 2(\vec{B}^2 - \vec{E}^2) \ .$$

As for the term $F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}$ one may first consider the case with only $E^1 \neq 0$, that annihilates the second invariant, and then the case $E^1 = B^1 \neq 0$, that annihilates the first invariant, to find

$$F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu} = 2(\vec{E}^2 - \vec{B}^2)^2 + 4(\vec{E}\cdot\vec{B})^2 \ .$$

1 Spin 0 case

We outline the calculation for a scalar particle of charge q and mass m, quoting the relevant formulas. The action to be used for the scalar particle is best described in hamiltonian form

$$S[x,p,e;A] = \int_0^1 d\tau \Big(p_\mu \dot{x}^\mu - e \underbrace{\frac{1}{2}(\pi^\mu \pi_\mu + m^2)}_{H} \Big) \tag{3}$$

where the minimal coupling to the electromagnetic field A_μ has been introduced in the hamiltonian constraint H by using the covariant momentum π_μ instead of p_μ

$$\pi_\mu = p_\mu - qA_\mu(x) \; . \tag{4}$$

Eliminating p_μ by its own algebraic equation of motion, one finds the configuration space action

$$S_c[x,e;A] = \int_0^1 d\tau \Big(\frac{1}{2} e^{-1} \dot{x}^2 - \frac{1}{2} e m^2 + qA_\mu \dot{x}^\mu \Big) \; . \tag{5}$$

The calculation to be described is better performed using euclidean conventions, so that one performs a Wick rotation ($iS_c \to -S_E$) to find the euclidean action

$$S_E[x,e;A] = \int_0^1 d\tau \Big(\frac{1}{2} e^{-1} \dot{x}^2 + \frac{1}{2} e m^2 - iqA_\mu \dot{x}^\mu \Big) \; . \tag{6}$$

Taking the worldline to be a loop, i.e. a manifold with the topology of the circle S^1, one gets the worldline representation of the effective action $\Gamma[A]$ for the electromagnetic field as induced by the particle

$$\begin{aligned} \Gamma[A] &= Z_{S^1}[A] = \int \frac{\mathcal{D}x \mathcal{D}e}{\text{Vol(Gauge)}} \, e^{-S_E[x,e;A]} \\ &= -\int_0^\infty \frac{dT}{T} \, e^{-m^2 T} \int_P \mathcal{D}x \, e^{-S_q[x;A]} \end{aligned} \tag{7}$$

where in the last line the path integral has been gauge fixed by setting $e(\tau) = 2T$, and the gauge fixed action (without the mass term that is factored out) is given by

$$S_q[x^\mu; A_\mu] = \int_0^1 d\tau \, \left(\frac{1}{4T} \dot{x}^\mu \dot{x}_\mu - iqA_\mu(x) \dot{x}^\mu \right) \; . \tag{8}$$

The overall normalization of (7), as well as a check on the integration measure over the proper time T, can be fixed by comparing with similar formulas obtained directly from the quantum field theory of a charged Klein-Gordon field coupled to electromagnetism[2].

[2] The details are as follows. In QFT the effective action $\Gamma[A]$ induced by a charged Klein-Gordon field ϕ with euclidean action $S[\phi,\phi^*;A] = \int d^4x \, (D^\mu \phi^* D_\mu \phi + m^2 \phi^* \phi) = \int d^4x \, \phi^* (-\Box_A + m^2)\phi)$ is given by

$$e^{-\Gamma[A]} = \int \mathcal{D}\phi \mathcal{D}\phi^* e^{-S[\phi,\phi^*;A]} = \text{Det}^{-1}(-\Box_A + m^2)$$

so that

$$\Gamma[A] = -\ln \text{Det}^{-1}(-\Box_A + m^2) = \text{Tr} \ln(-\Box_A + m^2) = -\int_0^\infty \frac{dT}{T} \text{Tr} \, e^{-(-\Box_A + m^2)T}$$

up to an irrelevant constant term. The proper time representation of the logarithm follows from the equality

$$\ln \frac{a}{b} = -\int_0^\infty \frac{dT}{T} (e^{-aT} - e^{-bT}) \; .$$

The formula in (7) is a general representation of the effective action, but its evaluation in closed form is a difficult task. However, for external low-energy photons, one can consider the electromagnetic field $F_{\mu\nu}$ to be constant, and find an exact result. For such a constant configuration one may use the potential

$$A_\mu(x) = \frac{1}{2} x^\nu F_{\nu\mu} \tag{9}$$

and the action takes a quadratic form

$$\begin{aligned} S_q[x^\mu; A_\mu] &= \int_0^1 d\tau \left(\frac{1}{4T} \dot{x}^\mu \dot{x}_\mu - \frac{iq}{2} x^\mu F_{\mu\nu} \dot{x}^\nu \right) \\ &= \int_0^1 d\tau \, \frac{1}{2} x^\mu \left(-\frac{1}{2T} \delta_{\mu\nu} \partial_\tau^2 - iq F_{\mu\nu} \partial_\tau \right) x^\nu \, . \end{aligned} \tag{10}$$

In the integration by part used to obtain the second expression no boundary term is produced, as one must use periodic boundary conditions (i.e. there is no real boundary on the circle S^1). The path integral is gaussian and is computed exactly. The result is expressed in terms of functional determinants. In four spacetime dimensions it reads

$$\int_P \mathcal{D}x \, e^{-S_q[x;A]} = \int \frac{d^4x}{(4\pi T)^2} \left[\frac{\mathrm{Det}'_P\Big(-\frac{1}{2T}\partial_\tau^2 - iqF\partial_\tau \Big)}{\mathrm{Det}'_P\Big(-\frac{1}{2T}\partial_\tau^2 \Big)} \right]^{-\frac{1}{2}} \, . \tag{11}$$

To prove this formula, and its precise normalization, one may perform the path integral by defining it as an integration over the Fourier coefficients of the Fourier expansion of a generic periodic path

$$x^\mu(\tau) = \sum_{n \in Z} x_n^\mu \, e^{2\pi i n \tau} \, . \tag{12}$$

The zero modes x_0^μ drop out of the action, and they can be identified with the space-time coordinates. One must integrate over them, as well as over the other Fourier coefficients. The latter produce a functional determinant over periodic functions without the zero modes. This is indicated by the prime in Det'. For vanishing $F_{\mu\nu}$ it evaluates the path integral for a free particle, which has been computed earlier. Adapting notations one finds

$$\int_P \mathcal{D}x \, e^{-S_q[x]} = \int d^4x \left[\mathrm{Det}'_P \Big(-\frac{1}{2T}\partial_\tau^2 \Big) \right]^{-\frac{1}{2}} = \int \frac{d^4x}{(4\pi T)^2} \, . \tag{13}$$

Then, for non vanishing and constant $F_{\mu\nu}$ one gets the expression in (11).

The evaluation of the determinant is carried out as product of all eigenvalues. The determinants in (11) can be simplified to

$$\frac{\mathrm{Det}'_P\Big(-\frac{1}{2T}\partial_\tau^2 - iqF\partial_\tau \Big)}{\mathrm{Det}'_P\Big(-\frac{1}{2T}\partial_\tau^2 \Big)} = \frac{\mathrm{Det}'_P\big(\partial_\tau + 2iTqF\big)}{\mathrm{Det}'_P\big(\partial_\tau\big)} \, . \tag{14}$$

Let us consider the case of a single dimension with the matrix F replaced by a number. A set of eigenfunctions is given by the periodic functions

$$f_n(\tau) = e^{2\pi i n \tau} \qquad n \in Z \tag{15}$$

so that removing the zero mode one computes

$$\begin{aligned}
\frac{\mathrm{Det}'_P\big(\partial_\tau + iw\big)}{\mathrm{Det}'_P\big(\partial_\tau\big)} &= \prod_{n\in Z, n\neq 0} \frac{2\pi i n + iw}{2\pi i n} = \prod_{n\in Z, n\neq 0} \Big(1+\frac{w}{2\pi n}\Big) \\
&= \prod_{n>0} \Big(1-\frac{w^2}{4\pi^2 n^2}\Big) = \frac{\sin\left(\frac{w}{2}\right)}{\frac{w}{2}} \ . \qquad (16)
\end{aligned}$$

Now, in four dimensions we have a matrix F, that one could diagonalize so to proceed by applying the result above. One finds the final result as a determinant of a 4×4 matrix, expressed in terms of its eigenvalues. Thus one may just rewrite it as

$$\frac{\mathrm{Det}'_P\big(\partial_\tau + 2iTqF\big)}{\mathrm{Det}'_P\big(\partial_\tau\big)} = \det\left(\frac{\sin(qTF)}{qTF}\right) \qquad (17)$$

where "Det" denotes a functional determinant and "det" the determinant of a finite dimensional matrix. Inserting this into (11), and then into (7), one finds

$$\Gamma[A] = \int d^4x \left[-\int_0^\infty \frac{dT}{T}\, \frac{e^{-m^2T}}{(4\pi T)^2} \det{}^{-\frac{1}{2}} \left(\frac{\sin(qTF)}{qTF}\right)\right] \ . \qquad (18)$$

The effective lagrangian is then given by

$$\mathcal{L} = -\int_0^\infty \frac{dT}{T}\, \frac{e^{-m^2T}}{(4\pi T)^2} \det{}^{-\frac{1}{2}} \left(\frac{\sin(qTF)}{qTF}\right) \ . \qquad (19)$$

As it stands it contains ultraviolet divergences that must be renormalized away. These divergences are seen as coming from the $T \to 0$ integration limit.

To remove them let us consider the expansion of the determinant piece

$$\begin{aligned}
\det{}^{-\frac{1}{2}}\Big(\frac{\sin(x)}{x}\Big) &= \exp \ln \det{}^{-\frac{1}{2}}\Big(\frac{\sin(x)}{x}\Big) = \exp\Big[-\frac{1}{2}\ln\det\Big(\frac{\sin(x)}{x}\Big)\Big] \\
&= \exp\Big[-\frac{1}{2}\mathrm{tr}\ \ln\Big(\frac{\sin(x)}{x}\Big)\Big] \\
&= \exp\Big[-\frac{1}{2}\mathrm{tr}\Big(-\frac{x^2}{6}-\frac{x^4}{180}+\cdots\Big)\Big] \\
&= 1+\frac{1}{12}\mathrm{tr}(x^2)+\frac{1}{288}(\mathrm{tr}(x^2))^2+\frac{1}{360}\mathrm{tr}(x^4)+\cdots \ . \qquad (20)
\end{aligned}$$

Considering $x = qTF$, one checks that only the first two terms give divergent contributions to the proper time integral in (19). One may subtract them with counterterms, and express the renormalized lagrangian, with the classical Maxwell lagrangian added, as

$$\mathcal{L}^{(0)}_{\rm ren} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T}\,\frac{e^{-m^2T}}{T^2}\left[\det{}^{-\frac{1}{2}}\left(\frac{\sin(qTF)}{qTF}\right) - 1 + \frac{q^2T^2}{12}F^{\mu\nu}F_{\mu\nu}\right] \ . \qquad (21)$$

This is the full, non perturbative result, for a constant electromagnetic field (in a euclidean spacetime), known as the Heisenberg-Euler effective lagrangian induced by a scalar particle.

Expanding the term in square brackets, and keeping only the first contribution, i.e. the term quartic in $F_{\mu\nu}$, one finds

$$\mathcal{L}^{(0)}_{\rm ren} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{q^4}{16\pi^2 m^4}\left[\frac{1}{288}(F^{\mu\nu}F_{\mu\nu})^2 + \frac{1}{360}F^{\mu\nu}F_{\nu\rho}F^{\rho\sigma}F_{\sigma\mu}\right] + \cdots \tag{22}$$

The calculation has been done in euclidean time, but at the end one must perform the inverse Wick rotation, which amounts to a global change of sign (plus the use of the Minkowski metric in raising and lowering indices). This sign can be understood as due to $L_M = T - V$ versus $L_E = T + V$ in a Wick rotation, and considering that (21) and (22) give just a constant contribution to V. Thus, back in Minkowski space, one finds the result anticipated in (1) for the scalar particle.

2 Spin 1/2 case

For a spin 1/2 particle one starts from the action in phase space

$$\begin{aligned} S[x,p,\psi,\psi_5,e,\chi;A] &= \int_0^1 d\tau \Big(p_\mu \dot{x}^\mu + \frac{i}{2}\psi_\mu\dot{\psi}^\mu + \frac{i}{2}\psi^5\dot{\psi}^5 - i\chi \underbrace{(\pi_\mu\psi^\mu + m\psi^5)}_{Q} \\ &\qquad - e\underbrace{\frac{1}{2}(\pi^\mu\pi_\mu + iqF_{\mu\nu}\psi^\mu\psi^\nu + m^2)}_{H}\Big) \end{aligned} \tag{23}$$

where the minimal coupling to the electromagnetic field A_μ has been introduced in the susy constraint Q by using the covariant momentum $\pi_\mu = p_\mu - qA_\mu(x)$, while the hamiltonian constraint H is found by imposing the Poisson bracket susy algebra $\{Q,Q\} = -2iH$.

Eliminating p_μ by its own algebraic equation of motion, one obtains the configuration space action

$$\begin{aligned} S_c[x,\psi,\psi^5,e,\chi;A] &= \int_0^1 d\tau \Big(\frac{1}{2}e^{-1}(\dot{x}^\mu - i\chi\psi^\mu)^2 + \frac{i}{2}\psi_\mu\dot{\psi}^\mu + \frac{i}{2}\psi^5\dot{\psi}^5 - \frac{1}{2}em^2 - im\chi\psi^5 \\ &\qquad + qA_\mu\dot{x}^\mu - \frac{ieq}{2}F_{\mu\nu}\psi^\mu\psi^\nu\Big) . \end{aligned} \tag{24}$$

Again, the calculation is better performed using euclidean conventions, so that we perform a Wick rotation ($iS_c \to -S_E$) to obtain the euclidean action

$$\begin{aligned} S_E[x,\psi,\psi^5,e,\chi;A] &= \int_0^1 d\tau \Big(\frac{1}{2}e^{-1}(\dot{x}^\mu - \chi\psi^\mu)^2 + \frac{1}{2}\psi_\mu\dot{\psi}^\mu + \frac{1}{2}\psi^5\dot{\psi}^5 + \frac{1}{2}em^2 + im\chi\psi^5 \\ &\qquad - iqA_\mu\dot{x}^\mu + \frac{ieq}{2}F_{\mu\nu}\psi^\mu\psi^\nu\Big) . \end{aligned} \tag{25}$$

Taking the worldline to be a loop, i.e. a circle S^1, one gets the worldline representation of the effective action $\Gamma[A]$ for the electromagnetic field as induced by the spin 1/2 particle

$$\begin{aligned} \Gamma[A] &= Z_{S^1}[A] = \int \frac{\mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^5\mathcal{D}e\mathcal{D}\chi}{\text{Vol(Gauge)}}\, e^{-S_E} \\ &= \frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}\int_P \mathcal{D}x \int_A \mathcal{D}\psi\, e^{-S_q[x,\psi;A]} \end{aligned} \tag{26}$$

where in the last line the path integral has been gauge fixed. This is achieved by considering periodic boundary conditions (P) for bosons and antiperiodic boundary conditions (A) for fermions; by setting $e(\tau) = 2T$ and $\chi(\tau) = 0$ as gauge fixing conditions for the local symmetries; recognizing that ψ^5 becomes free and can be dropped (at most it contributes to the overall normalization that is anyhow fixed by comparison with QFT). At the end one has a gauge fixed action (without the mass term that is factored out)

$$S_q[x, \psi; A] = \int_0^1 d\tau \,\Big(\frac{1}{4T}\dot{x}^\mu \dot{x}_\mu + \frac{1}{2}\psi_\mu \dot{\psi}^\mu - iqA_\mu \dot{x}^\mu + iqTF_{\mu\nu}\psi_\mu\psi^\nu\Big) \,. \tag{27}$$

The overall normalization of (26), as well as a check on the integration measure over the proper time T, can be fixed by comparing with similar formulas obtained from the quantum field theory of a charged Dirac field coupled to electromagnetism[3].

Again, the formula in (26) is a useful representation of the effective action, but its evaluation in closed form is not known. For external low-energy photons, one takes the external electromagnetic field $F_{\mu\nu}$ constant, and finds an exact result. For such a constant configuration one may use the potential

$$A_\mu(x) = \frac{1}{2} x^\nu F_{\nu\mu} \tag{28}$$

and the action takes a quadratic form. It splits into two parts

$$\begin{aligned} S_q[x, \psi; A] &= \int_0^1 d\tau \,\Big[\Big(\frac{1}{4T}\dot{x}^\mu \dot{x}_\mu - \frac{iq}{2} x^\mu F_{\mu\nu}\dot{x}^\nu\Big) + \Big(\frac{1}{2}\psi_\mu \dot{\psi}^\mu + iqTF_{\mu\nu}\psi_\mu\psi^\nu\Big)\Big] \\ &= \int_0^1 d\tau \,\Big[\frac{1}{2} x^\mu \Big(-\frac{1}{2T}\delta_{\mu\nu}\partial_\tau^2 - iqF_{\mu\nu}\partial_\tau\Big) x^\nu + \frac{1}{2}\psi^\mu \Big(\delta_{\mu\nu}\partial_\tau + 2iqTF_{\mu\nu}\Big)\psi^\nu\Big] \\ &= S_q^{(b)}[x; A] + S_q^{(f)}[\psi; A] \,. \end{aligned} \tag{29}$$

The path integral is gaussian. The bosonic part has already been treated previously. The fermionic part reads

$$\int_A \mathcal{D}\psi \, e^{-S_q^{(f)}[\psi;A]} = \left[\frac{\mathrm{Det}_A\Big(\partial_\tau + 2iqTF\Big)}{\mathrm{Det}_A\Big(\partial_\tau\Big)}\right]^{\frac{1}{2}} \mathrm{Det}_A^{\frac{1}{2}}\Big(\partial_\tau\Big) \tag{30}$$

where the last determinant corresponds to the free path integral that computes the trace of the identity (as the hamiltonian vanishes) in the Hilbert space of the worldline fermions. This Hilbert space is four dimensional, as it gives the number of spinorial components of a Dirac spinor in four dimensions

$$\mathrm{Det}_A^{\frac{1}{2}}\Big(\partial_\tau\Big) = \mathrm{Tr}\,\mathbb{1} = 2^2 = 4 \,. \tag{31}$$

[3] The details are as follows. In QFT the effective action induced by a charged Dirac field Ψ with action $S[\Psi, \bar{\Psi}; A] = \int d^4x \, \bar{\Psi}(\not{D} + m)\Psi$ is given

$$e^{-\Gamma[A]} = \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} \, e^{-S[\Psi,\bar{\Psi};A]} = \mathrm{Det}(\not{D} + m)$$

so that

$$\Gamma[A] = -\mathrm{Tr}\ln(\not{D} + m) = -\mathrm{Tr}\ln(-\not{D} + m) = -\frac{1}{2}\ln(-\not{D}^2 + m^2) = \frac{1}{2}\int_0^\infty \frac{dT}{T}\mathrm{Tr}\, e^{-(-\not{D}^2+m^2)T}$$

up to constant irrelevant terms.

The remaining determinants are evaluated as product of all eigenvalues. A set of eigenfunctions is given by the antiperiodic functions

$$f_n(\tau) = e^{2\pi i(n+\frac{1}{2})\tau} \qquad n \in Z \tag{32}$$

so that if one would have just a number

$$\begin{aligned}\frac{\mathrm{Det}_A\Big(\partial_\tau + i\omega\Big)}{\mathrm{Det}_A\Big(\partial_\tau\Big)} &= \prod_{n\in Z} \frac{2\pi i(n+\frac{1}{2}) + iw}{2\pi i(n+\frac{1}{2})} = \prod_{n\in Z}\Big(1 + \frac{w}{2\pi(n+\frac{1}{2})}\Big) \\ &= \prod_{n=0}^{\infty}\Big(1 - \frac{w^2}{\pi^2(2n+1)^2}\Big) = \cos\Big(\frac{w}{2}\Big) \ . \end{aligned}\tag{33}$$

Now, in four dimensions we have a matrix F, that one could diagonalize so to use the result above. One finds the final result written as the determinant of a 4×4 matrix expressed in terms of its eigenvalues, which one rewrites it back as

$$\frac{\mathrm{Det}_A\Big(\partial_\tau + 2iqTF\Big)}{\mathrm{Det}_A\Big(\partial_\tau\Big)} = \det\big(\cos(qTF)\big) \ . \tag{34}$$

Inserting everything into (26) one finds

$$\Gamma[A] = \int d^4x \left[2\int_0^\infty \frac{dT}{T}\,\frac{e^{-m^2T}}{(4\pi T)^2}\det{}^{-\frac{1}{2}}\left(\frac{\tan(qTF)}{qTF}\right)\right] \ . \tag{35}$$

The effective lagrangian is thus given by

$$\mathcal{L} = 2\int_0^\infty \frac{dT}{T}\,\frac{e^{-m^2T}}{(4\pi T)^2}\det{}^{-\frac{1}{2}}\left(\frac{\tan(qTF)}{qTF}\right) \ . \tag{36}$$

As it stands it contains ultraviolet divergences that must be renormalized away. These divergences are seen as emerging from the $T\to 0$ integration limit.

To remove them let us consider the expansion of the determinant piece

$$\begin{aligned}\det{}^{-\frac{1}{2}}\Big(\frac{\tan(x)}{x}\Big) &= \exp\ln\det{}^{-\frac{1}{2}}\Big(\frac{\tan(x)}{x}\Big) = \exp\Big[-\frac{1}{2}\ln\det\Big(\frac{\tan(x)}{x}\Big)\Big] \\ &= \exp\Big[-\frac{1}{2}\mathrm{tr}\,\ln\Big(\frac{\tan(x)}{x}\Big)\Big] \\ &= \exp\Big[-\frac{1}{2}\mathrm{tr}\Big(\frac{x^2}{3} + \frac{7}{90}x^4 + \cdots\Big)\Big] \\ &= 1 - \frac{1}{6}\mathrm{tr}(x^2) + \frac{1}{72}(\mathrm{tr}(x^2))^2 - \frac{7}{180}\mathrm{tr}(x^4) + \cdots \end{aligned}\tag{37}$$

Considering $x = qTF$, one checks that only the first two terms give divergent contributions to the proper time integral in (36), so that one subtracts them with counterterms, and expresses the renormalized lagrangian, with the classical Maxwell lagrangian added for comparison, as

$$\mathcal{L}^{(1/2)}_{\mathrm{ren}} = \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T}\,\frac{e^{-m^2T}}{T^2}\left[\det{}^{-\frac{1}{2}}\left(\frac{\tan(qTF)}{qTF}\right) - 1 - \frac{q^2T^2}{6}F^{\mu\nu}F_{\mu\nu}\right] \ . \tag{38}$$

This is the full, non perturbative result, for a constant electromagnetic field (in euclidean time), which is known as the Heisenberg-Euler effective lagrangian. Expanding the term in square brackets, and keeping only the first contribution, which is quartic in $F_{\mu\nu}$, one finds

$$\mathcal{L}_{\rm ren}^{(1/2)} = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{q^4}{8\pi^2 m^4} \left[\frac{1}{72} (F^{\mu\nu} F_{\mu\nu})^2 - \frac{7}{180} F^{\mu\nu} F_{\nu\rho} F^{\rho\sigma} F_{\sigma\mu} \right] + \cdots \tag{39}$$

The calculation has been done in euclidean time. One may perform the inverse Wick rotation, which amounts to a global change of sign (and the use of the Minkowski metric in raising and lowering indices). Back in Minkowski space, one finds the result anticipated in (1) for the spin 1/2 particle