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Path integrals in quantum mechanics

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Quantum mechanics can be formulated in two equivalent ways: (i) canonical quantization, also known as operatorial quantization, which is based on linear operators acting on a Hilbert space of physical states, (ii) path integrals, based on integration over a space of functions. The former was the first one to be developed, through the work of Heisenberg, Schrödinger, Dirac and others. The latter was introduced later on by Feynman, who extended previous suggestions by Dirac. Nowadays it is useful to know both formulations, as depending on the problem one is studying, one may find technical advantages in using one over the other. In worldline approaches one often uses the operatorial formulation to define the problem, and path integrals to calculate the answer.

The operatorial formulation of quantum mechanics is the one usually presented in introductory courses on quantum mechanics. Path integrals are introduced later on, when approaching the problem of quantizing gauge fields. Indeed path integrals have become quite popular since the advent of gauge theories, because the quantization of the latter is much more intuitive and transparent in that context.

In part I of this book we introduce path integrals for the quantization of point particles, as opposed to the quantization of field theories. The former contains a finite number of degrees of freedom, while the latter deals with an infinite number of degrees of freedom. We assume only elementary notions of quantum mechanics in its operatorial form, and start developing path integrals from the beginning.

We begin this chapter by introducing path integrals for a non relativistic point particle. This case contains already the essence of path integrals. Then in the following chapter we provide a derivation of path integrals for fermionic systems, i.e. those which at the canonical level are quantized using anticommutators. Fermionic path integrals make use of Grassmann variables, anticommuting variables that allow the description of spin at the "classical level". Path integrals with bosonic and fermionic variables can be used to discuss supersymmetric systems, that often arise in the description of point particles with spin. In later chapters we consider path integrals in the presence of various background fields. For the case of bosonic path integrals, a scalar background potential $V(x)$ is treated without any particular effort already in the present chapter. In chapter 4 we discuss the regularization issues needed for treating the coupling of the particle to a vector potential $A_i(x)$. In chapter 5 similar issues are dealt with for the coupling to a tensor potential $g_{ij}(x)$, the metric of the space on which the particle propagates. Extensions to fermionic path integrals with background fields are presented as well.

1 Canonical quantization

Canonical quantization is constructed starting from the hamiltonian formulation of a classical system. It is obtained by lifting its phase space coordinates, the generalized coordinates x^i and their conjugate momenta p_i, to linear operators $\hat{x}^i$ and $\hat{p}_i$ that act on a linear space endowed

with a positive definite norm, the Hilbert space of physical states $\mathcal{H}$. The basic operators must satisfy commutation relations required to be equal $i\hbar$ times the value of the corresponding classical Poisson brackets

$$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j \ , \quad [\hat{x}^i, \hat{x}^j] = 0 \ , \quad [\hat{p}_i, \hat{p}_j] = 0 \ . \tag{1}$$

All classical observables $A(x,p)$, which are just functions on phase space, become linear operators $\hat{A}(\hat{x},\hat{p})$ acting on the Hilbert space $\mathcal{H}$. The most important example is given by the hamiltonian $H(x,p)$, which upon quantization turns into the hamiltonian operator $\hat{H}(\hat{x},\hat{p})$. The latter generates the time evolution of any state $|\psi\rangle \in \mathcal{H}$ through the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \ . \tag{2}$$

The corresponding solution is a time dependent state $|\psi(t)\rangle$ that describes the evolution of the system. This set up is known as the Schrödinger picture of quantum mechanics. It is a formal quantization procedure that becomes operative once one finds an irreducible unitary representation of the operator algebra in eq. (1). A mathematical result, known as the Stone–von Neumann theorem, states that in quantum mechanics all irreducible representations of (1) are unitarily equivalent, so that there is a unique procedure of quantizing a classical system[1]. Historically, this theorem made it clear that the Schrödinger formulation of quantum mechanics was equivalent to the one proposed by Heisenberg with its matrix mechanics (known as the Heisenberg picture).

Let us consider, more specifically, the motion of a non relativistic particle in one dimension. This system has just one degree of freedom, but extension to more degrees of freedom is straightforward. Using the coordinate representation, obtained by considering the eigenstates $|x\rangle$ of the position operator $\hat{x}$, that satisfy $\hat{x}|x\rangle = x|x\rangle$ with x a real number, and projecting the various states of the Hilbert space onto them to identify the wave functions, one finds the familiar way of realizing quantum mechanics as wave mechanics

$$\begin{array}{lcll}
|\psi\rangle & \to & \psi(x) & \Big(\psi(x) = \langle x|\psi\rangle\Big) \\
\hat{x} & \to & x & \Big(\langle x|\hat{x}|x'\rangle = x\langle x|x'\rangle = x\delta(x-x')\Big) \\
\hat{p} & \to & -i\hbar\dfrac{\partial}{\partial x} & \Big(\langle x|\hat{p}|x'\rangle = -i\hbar\dfrac{\partial}{\partial x}\langle x|x'\rangle = -i\hbar\dfrac{\partial}{\partial x}\delta(x-x')\Big) \\
\hat{H} & \to & -\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2} + V(x) & \Big(\langle x|\hat{H}|x'\rangle = \Big(-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2} + V(x)\Big)\delta(x-x')\Big)
\end{array} \tag{3}$$

together with the standard form of the Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \ . \tag{4}$$

Given a ket $|\psi_i\rangle$ that describes the system at an initial time t_i, the solution of the Schrödinger equation for time independent hamiltonians can be written as

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t_i)}|\psi_i\rangle \tag{5}$$

[1]Up to the problem of resolving ordering ambiguities, often present when one tries to relate the classical hamiltonian $H(x,p)$ to its quantum counterpart $\hat{H}(\hat{x},\hat{p})$.

which indeed satisfies the equation and the boundary condition $|\psi(t_i)\rangle = |\psi_i\rangle$. The amplitude to find the system at time t_f in state $|\psi_f\rangle$ is obtained by projecting the solution evaluated at time t_f onto the state $|\psi_f\rangle$

$$\langle\psi_f|\psi(t_f)\rangle = \langle\psi_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|\psi_i\rangle \ . \tag{6}$$

This amplitude is called "transition amplitude". Thus, we see that amplitudes correspond to matrix element of $e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}$, the unitary operator that evolves the states of the system. In the following sections we shall find a path integral representation for such amplitudes.

2 Path integrals in phase space

To start with, it is useful to insert twice the identity operator $\mathbb{1}$, expressed using the eigenstates of the position operator

$$\mathbb{1} = \int dx\, |x\rangle\langle x| \qquad\qquad \text{with } \langle x|x'\rangle = \delta(x-x')\,, \tag{7}$$

and rewrite (6) as

$$\begin{aligned}\langle\psi_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|\psi_i\rangle &= \langle\psi_f|\,\mathbb{1}\,e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}\,\mathbb{1}\,|\psi_i\rangle \\ &= \int dx_f \int dx_i\, \psi_f^*(x_f)\,\langle x_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|x_i\rangle\,\psi_i(x_i)\end{aligned} \tag{8}$$

where $\psi_i(x_i) = \langle x|\psi_i\rangle$ and $\psi_f(x_f) = \langle x|\psi_f\rangle$ are the wave functions for the initial and final states. This rewriting shows that it is enough to consider the matrix element of the evolution operator between position eigenstates

$$A(x_i, x_f; T) = \langle x_f|e^{-\frac{i}{\hbar}\hat{H}T}|x_i\rangle \tag{9}$$

where $T = (t_f - t_i)$ is the total propagation time.

We consider quantum hamiltonians of the form

$$\hat{H}(\hat{x},\hat{p}) = \frac{1}{2m}\hat{p}^2 + \hat{V}(\hat{x}) \tag{10}$$

where V is a scalar potential. The derivation of the path integral proceeds now as follows. One splits the transition amplitude $A(x_i, x_f; T)$ as the product of N factors, and inserts the completeness relation (7) $N-1$ times in between the factors

$$\begin{aligned} A &= \langle x_f|e^{-\frac{i}{\hbar}\hat{H}T}|x_i\rangle = \langle x_f|\Big(e^{-\frac{i}{\hbar}\hat{H}\frac{T}{N}}\Big)^N|x_i\rangle = \langle x_f|\underbrace{e^{-\frac{i\epsilon}{\hbar}\hat{H}}e^{-\frac{i\epsilon}{\hbar}\hat{H}}\cdots e^{-\frac{i\epsilon}{\hbar}\hat{H}}}_{N\text{ times}}|x_i\rangle \\ &= \langle x_f|e^{-\frac{i\epsilon}{\hbar}\hat{H}}\,\mathbb{1}\,e^{-\frac{i\epsilon}{\hbar}\hat{H}}\,\mathbb{1}\cdots\mathbb{1}\,e^{-\frac{i\epsilon}{\hbar}\hat{H}}|x_i\rangle = \int\Big(\prod_{k=1}^{N-1}dx_k\Big)\prod_{k=1}^{N}\langle x_k|e^{-\frac{i\epsilon}{\hbar}\hat{H}}|x_{k-1}\rangle\ . \end{aligned} \tag{11}$$

For convenience we have denoted $x_0 \equiv x_i$, $x_N \equiv x_f$, and $\epsilon \equiv \frac{T}{N}$. Then, one can use the resolution of the identity N more times, now expressed in terms of the momentum eigenstates

$$\mathbb{1} = \int \frac{dp}{2\pi\hbar}\,|p\rangle\langle p| \qquad\qquad \text{with } \langle p|p'\rangle = 2\pi\hbar\,\delta(p-p')\,, \tag{12}$$

to obtain

$$
\begin{aligned}
A &= \int \Big(\prod_{k=1}^{N-1} dx_k\Big) \prod_{k=1}^{N} \langle x_k | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | x_{k-1}\rangle = \int \Big(\prod_{k=1}^{N-1} dx_k\Big) \prod_{k=1}^{N} \langle x_k | \, \mathbb{1} \, e^{-\frac{i\epsilon}{\hbar}\hat{H}} | x_{k-1}\rangle \\
&= \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}\Big) \prod_{k=1}^{N} \langle x_k | p_k\rangle\langle p_k | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | x_{k-1}\rangle \ . \qquad (13)
\end{aligned}
$$

This is an exact expression. Note that there is one more integration over momenta than integrations over coordinates, consequence of choosing coordinate eigenstates as initial and final states in the transition amplitude. Now one can manipulate this expression further by making approximations that are valid in the limit $N \to \infty$ (i.e. $\epsilon \to 0$). The crucial point is the evaluation of the following matrix element

$$
\begin{aligned}
\langle p | e^{-\frac{i\epsilon}{\hbar}\hat{H}(\hat{x},\hat{p})} | x\rangle &= \langle p | \Big(\mathbb{1} - \frac{i\epsilon}{\hbar}\hat{H}(\hat{x},\hat{p}) + \cdots\Big) | x\rangle \\
&= \langle p | x\rangle - \frac{i\epsilon}{\hbar}\langle p | \hat{H}(\hat{x},\hat{p}) | x\rangle + \cdots \\
&= \langle p | x\rangle \left(1 - \frac{i\epsilon}{\hbar} H(x,p) + \cdots\right) \\
&= \langle p | x\rangle \, e^{-\frac{i\epsilon}{\hbar}H(x,p)+\cdots} \ . \qquad (14)
\end{aligned}
$$

These approximations are all valid in the limit of small ϵ. The substitution $\langle p | \hat{H}(\hat{x},\hat{p}) | x\rangle = \langle p | x\rangle \, H(x,p)$ follows from the simple structure of the hamiltonian (10), that allows one to act with the momentum operator on the left, and with the position operator on the right, to have the operators replaced by the corresponding eigenvalues. Notice that there is no need of commuting operators inside the hamiltonian, because of the simplicity of the hamiltonian we considered. The final result is that all operators are simply replaced by eigenvalues. This way the quantum hamiltonian $\hat{H}(\hat{x},\hat{p})$ gets replaced by the classical function $H(x,p) = \frac{p^2}{2m} + V(x)$. There exists a rigorous proof that these manipulations are correct for a wide class of physically interesting potentials $V(x)$ (the "Trotter formula"). We shall not review this mathematical point, as the physically intuitive derivation given above is enough for our purposes.

Using now eq. (14), and remembering that the wave functions of the momentum eigenstates (the plane waves) are normalized as

$$
\langle x | p\rangle = e^{\frac{i}{\hbar}px} \ , \quad \langle p | x\rangle = \langle x | p\rangle^* = e^{-\frac{i}{\hbar}px} \ , \qquad (15)
$$

that follows from the normalization chosen in (7) and (12), one obtains

$$
\langle x_k | p_k\rangle\langle p_k | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | x_{k-1}\rangle = e^{\frac{i}{\hbar}p_k(x_k - x_{k-1}) - \frac{i\epsilon}{\hbar}H(x_{k-1},p_k)} \qquad (16)
$$

up to terms that vanish for $\epsilon \to 0$. This expression can now be inserted in (13). At this stage the transition amplitude does not contain any more operators, bra or kets. It contains just integrations, though a big number of them, of ordinary functions

$$
\begin{aligned}
A &= \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}\Big) e^{\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\left[p_k \frac{(x_k - x_{k-1})}{\epsilon} - H(x_{k-1},p_k)\right]} \\
&= \int Dx Dp \ e^{\frac{i}{\hbar}S[x,p]} \ . \qquad (17)
\end{aligned}
$$

This is the path integral in phase space. One recognizes in the exponent a discretization of the classical phase space action

$$S[x,p] = \int_{t_i}^{t_f} dt \left(p\dot{x} - H(x,p) \right) \quad \rightarrow \quad \sum_{k=1}^{N} \epsilon \Big(p_k \frac{(x_k - x_{k-1})}{\epsilon} - H(x_{k-1}, p_k) \Big) \tag{18}$$

where $t_f - t_i = T = N\epsilon$ is the total propagation time, with the paths in phase space discretized as

$$x(t)\ ,\ p(t) \quad \rightarrow \quad x_k = x(t_i + k\epsilon)\ ,\ p_k = p(t_i + k\epsilon)\ . \tag{19}$$

The last way of writing the amplitude in (17) is symbolic but suggestive: it indicates the sum over all paths in phase space weighted by the exponential of $\frac{i}{\hbar}$ times the classical action. It depends implicitely on the boundary conditions assigned to the paths $x(t)$

3 Path integrals in configuration space

The path integral in configurations space is easily derived by integrating over the momenta. Indeed the dependence on momenta in the exponent of (17) is at most quadratic and can be eliminated by gaussian integration

$$\int_{-\infty}^{\infty} dp\, e^{-\frac{K}{2}p^2} = \sqrt{\frac{2\pi}{K}} \tag{20}$$

extended analytically to include complex values of K (for details see section 4.1). Considering the form of the hamiltonian $H(x,p) = \frac{p^2}{2m} + V(x)$, and completing the squares[2], one may perform the gaussian integrations over the momenta to obtain

$$\begin{aligned} A &= \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k \Big) \Big(\frac{m}{2\pi i\hbar\epsilon} \Big)^{\frac{N}{2}} e^{\frac{i\epsilon}{\hbar}\sum_{k=1}^{N} \left[\frac{m}{2}\frac{(x_k - x_{k-1})^2}{\epsilon^2} - V(x_{k-1}) \right]} \\ &= \int Dx\, e^{\frac{i}{\hbar}S[x]}\ . \end{aligned} \tag{21}$$

This is the path integral in configurations space. It contains in the exponent the configuration space action suitably discretized

$$S[x] = \int_{t_i}^{t_f} dt \left(\frac{m}{2}\dot{x}^2 - V(x) \right) \rightarrow \sum_{k=1}^{N} \epsilon \Big[\frac{m}{2} \Big(\frac{x_k - x_{k-1}}{\epsilon} \Big)^2 - V(x_{k-1}) \Big]\ . \tag{22}$$

Again, the last way of writing the path integral in (21) is symbolic, and indicates the formal sum over paths in configurations space, weighted by the exponential of $\frac{i}{\hbar}$ times the classical action. The space of paths is given by the space of functions $x(t)$ with boundary values $x(t_i) = x_i$ and $x(t_f) = x_f$. It is an infinite dimensional space. How to perform concretely the path integral over this functional space is defined precisely by the discretization, that approximates the function $x(t)$ by its $N+1$ values $x_k = x(t_i + k\epsilon)$ obtained for $k = 0, 1, 2, ...N$.

[2]First rewrite $p_k \frac{(x_k - x_{k-1})}{\epsilon} - \frac{1}{2m}p_k^2 = -\frac{1}{2m}(p_k - m\frac{(x_k - x_{k-1})}{\epsilon})^2 + \frac{m}{2}\frac{(x_k - x_{k-1})^2}{\epsilon^2}$. Then change integration variables $p_k \rightarrow \tilde{p}_k = p_k - m\frac{(x_k - x_{k-1})}{\epsilon}$. It leaves the measure invariant and produces $-\frac{\tilde{p}_k^2}{2m} + \frac{m}{2}\frac{(x_k - x_{k-1})^2}{\epsilon^2}$ in the exponent.

3.1 Free particle

For a free particle ($V(x) = 0$) one may use repeatedly gaussian integrations and calculate from eq. (21) the exact transition amplitude

$$A(x_i, x_f; T) = \sqrt{\frac{m}{2\pi i\hbar T}}\, e^{\frac{i}{\hbar}\frac{m(x_f-x_i)^2}{2T}} \ . \tag{23}$$

It satisfies the free Schrödinger equation

$$i\hbar\frac{\partial}{\partial T}A(x_i, x_f; T) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_f^2}A(x_i, x_f; T) \tag{24}$$

with initial conditions

$$A(x_i, x_f; 0) = \delta(x_f - x_i) \ . \tag{25}$$

The result is very suggestive: up to a prefactor it is given by the exponential of $\frac{i}{\hbar}$ times the classical action evaluated on the classical path, i.e. the path that satisfies the classical equations of motion. This is typical for the cases in which the semiclassical approximation is exact. One may interpret the prefactor as due to quantum "one-loop" corrections to the classical ("tree-level") result. The free particle case is also quite special: the exact result is obtained for any N, so that there is no need to take the limit $N \to \infty$. The case $N = 1$, which carries no integration at all in x, is already exact.

A formal but useful way of calculating gaussian path integrals is obtained by working directly in the continuum limit. One does not consider the precise definition of the path integral measure, but uses only its formal properties like translational invariance. The calculation is formal in the sense that one assumes properties of the path integral measure, that eventually should be proven by an explicit regularization and construction. The calculation goes as follows. The action is $S[x] = \int_0^T dt\, \frac{m}{2}\dot{x}^2$, and the classical equations of motion with the above boundary conditions are solved by

$$x_{cl}(t) = x_i + (x_f - x_i)\frac{t}{T} \ . \tag{26}$$

Now one can represent a generic path $x(t)$ as a classical path $x_{cl}(t)$ plus quantum fluctuations $q(t)$

$$x(t) = x_{cl}(t) + q(t) \tag{27}$$

where the quantum fluctuations $q(t)$ must vanish at $t = 0$ and $t = T$ to preserve the boundary conditions. One may interpret $x_{cl}(t)$ as the origin in the space of functions. Then one computes the path integral as follows

$$\begin{aligned} A(x_i, x_f; T) &= \int Dx\, e^{\frac{i}{\hbar}S[x]} = \int D(x_{cl} + q)\, e^{\frac{i}{\hbar}S[x_{cl}+q]} \\ &= \int Dq\, e^{\frac{i}{\hbar}(S[x_{cl}]+S[q])} = e^{\frac{i}{\hbar}S[x_{cl}]}\int Dq\, e^{\frac{i}{\hbar}S[q]} \\ &= Ne^{\frac{i}{\hbar}S[x_{cl}]} = Ne^{\frac{i}{\hbar}\frac{m(x_f-x_i)^2}{2T}} \end{aligned} \tag{28}$$

where translational invariance of the path integral measure has been used in the form $Dx = D(x_{cl} + q) = Dq$. There is no linear term in q in the action because x_{cl} solves the classical equations of motion: for quadratic actions one has $S[x_{cl} + q] = S[x_{cl}] + S[q]$. The normalization factor $N = \int Dq\, e^{\frac{i}{\hbar}S[q]}$ is undetermined by this method, but one can fix it by requiring that the final result satisfies the Schrödinger equation, thus finding $N = \sqrt{\frac{m}{2\pi i\hbar T}}$.

3.2 Euclidean time and statistical mechanics

Quantum mechanics can be related to statistical mechanics by an analytic continuation. We introduce this relation by considering the free particle just described.

Continuing analytically the time parameter to purely imaginary values by $T \to -i\beta$ with real β, and setting $\hbar = 1$, the free Schrödinger equation (24) turns into the heat equation

$$\frac{\partial}{\partial \beta} A = \frac{1}{2m} \frac{\partial^2}{\partial x_f^2} A \ . \tag{29}$$

Its fundamental solution, i.e. the solution with boundary condition $A \stackrel{\beta \to 0}{\longrightarrow} \delta(x_f - x_i)$, is given by

$$A = \sqrt{\frac{m}{2\pi\beta}}\, e^{-\frac{m(x_f - x_i)^2}{2\beta}} \ , \tag{30}$$

and can be obtained from (23) by the same analytic continuation. This continuation is called "Wick rotation".

A Wick rotation can be performed directly on the path integral to obtain euclidean path integrals. Analytically continuing the time variable as $t \to -i\tau$, one finds that the action with "minkowskian" time (i.e. with a real time t) turns into an "euclidean" action S_E defined by

$$iS[x] = i \int_0^T dt\, \frac{m}{2} \dot{x}^2 \quad \to \quad -S_E[x] = - \int_0^\beta d\tau\, \frac{m}{2} \dot{x}^2 \tag{31}$$

where in the euclidean action one defines $\dot{x} = \frac{dx}{d\tau}$, with τ usually called "euclidean time". The euclidean action thus defined is positive definite. It appears in the path integral that, after the Wick rotation, takes the form

$$\int Dx\, e^{-S_E[x]} \ . \tag{32}$$

For a free theory it is truly gaussian, with exponential damping rather than with increasingly rapid phase oscillations. In this form it coincides with the functional integral introduced by Wiener in the 1920's to study brownian motion and the heat equation.

Such euclidean path integrals are quite useful in statistical mechanics, where β is related to the inverse temperature Θ by $\beta = \frac{1}{k\Theta}$, where k is the Boltzmann's constant. To understand this, let us consider the trace of the evolution operator $e^{-\frac{i}{\hbar}\hat{H}T}$. It can be written using energy eigenstates (labeled by n if the spectrum is discrete), or equivalently using position eigenstates (labeled by x), as

$$Z \equiv \mathrm{Tr}\, e^{-\frac{i}{\hbar}\hat{H}T} = \sum_n e^{-\frac{i}{\hbar}E_n T} = \int dx\, \langle x | e^{-\frac{i}{\hbar}\hat{H}T} | x \rangle \ . \tag{33}$$

It can be Wick rotated $Z \to Z_E$ (with $T \to -i\beta$) to obtain the statistical partition function Z_E of the quantum system with hamiltonian $\hat{H}$. Setting $\hbar = 1$, it reads

$$Z_E \equiv \mathrm{Tr}\, e^{-\beta\hat{H}} = \sum_n e^{-\beta E_n} = \int dx\, \langle x | e^{-\beta\hat{H}} | x \rangle \ . \tag{34}$$

At this stage it is immediate to find a path integral representation of the statistical partition function: one performs a Wick rotation of the path integral action, sets the initial state (at

euclidean time $\tau = 0$) equal to the final state (at euclidean time $\tau = \beta$), and sums over all possible states, as indicated in (34). The paths become closed, $x(0) = x(\beta)$, and the partition function becomes

$$Z_E = \mathrm{Tr}\, e^{-\beta \hat{H}} = \int_{PBC} Dx\, e^{-S_E[x]} \tag{35}$$

where PBC stands for "periodic boundary conditions", indicating the sum over all paths that close onto themselves in an euclidean time β.

Introduced here for the free theory, the Wick rotation is supposed to be of more general value, relating quantum mechanics to statistical mechanics in the interacting case as well. Even if one is interested in the theory with a real time, nowadays one often works in the euclidean version of the theory, where factors of the imaginary unit i are absent, and path integral convergence is more easily kept under control. Only at the very end one performs the inverse Wick rotation to read off the result for the minkowskian theory.

The Wick rotation procedure is better appreciated by considering the usual time as the real line of a complex plane: denoting the complex time by $t_\theta = te^{-i\theta}$, the usual real time corresponds to $\theta = 0$, while the euclidean time τ appears at $\theta = \frac{\pi}{2}$ as $t_{\frac{\pi}{2}} = -i\tau$. The analytical continuation of all physical quantities is achieved by continually increasing θ form 0 to $\frac{\pi}{2}$, a clockwise rotation of the real axis into the imaginary one. The generalized partition function $Z_\theta \equiv \mathrm{Tr}\, e^{-\frac{i}{\hbar}\hat{H}t_\theta}$ with a complex time $t_\theta = te^{-i\theta}$ with positive t has a damping factor for all $0 < \theta \leq \frac{\pi}{2}$ and for all hamiltonians that are bounded from below (up to an overall factor due to the value of the ground state energy, if that happens to be negative).

Similar considerations can be made for path integrals in minkowskian and euclidean times with other boundary conditions. Path integrals in euclidean times are mathematically better defined (one may develop a mathematically well-defined measure theory on the space of functions), at least for quadratic actions and perturbations thereof. Path integral with a minkowskian time are more delicate, and physicists usually use the argument of rapid phase oscillations to deduce that unwanted terms vanish. The Wick rotation suggests a way of defining the path integral in real time starting from the euclidean time one. These points of mathematical rigor are not needed for the applications described in this book, and the derivation of path integrals described previously is enough for our purposes.

3.3 Miscellaneous comments

We have seen that the quantization of a classical system with action $S[x]$ is achieved by the path integral $\int Dx\, e^{\frac{i}{\hbar}S[x]}$ that computes the transition amplitude.

At the classical level, the solutions of the equations of motion are those that extremize the action and make it a minimum ("principle of minimal action"): $x_{cl}(t)$ solves the equations of motion if the extremality condition $\delta S[x_{cl}] = 0$ is satisfied. This is generically all that one needs in classical physics. In the quantum theory one needs, instead, the value of the action for all configurations $x(t)$, as each configuration contributes to the total amplitude with the phase $e^{\frac{i}{\hbar}S[x(t)]}$.

In the path integral formulation the classical limit is intuitive: macroscopic systems have large values of action in $\hbar$ units. Macroscopically small variations of paths can still make the phase variations $\frac{\delta S[x]}{\hbar}$ much bigger than π, so that amplitudes of nearby paths get canceled by destructive interference. This is true except for variations that make $\delta S[x] = 0$, which is the condition that identifies the classical path. Nearby paths have amplitudes that sum coherently with the classical one, and the path integral is dominated by the classical trajectory.

The notation $\int Dx$ is symbolic and indicates the formal integration over the space of functions $x(t)$. To make it precise one has to regulate the functional space by making it finite dimensional ("regularization"). Then one integrates over the regulated finite-dimensional space, and eventually takes the continuum limit by removing the regularization parameters. If this procedure is done with enough care, the limit exists and gives the correct transition amplitude. In the previous derivation we have seen that the space of paths is regulated by approximating the functions $x(t)$ by their $N-1$ values computed at intermediate points, the x_k's with $k = 1, ..., N-1$. This makes the space of functions finite dimensional. The action is discretized and evaluated using the approximated functions. At this stage the integration over the regulated functional space is well defined. Eventually one takes the continuum limit ($N \to \infty$): if the integration measure is chosen appropriately, as in eq. (21), this limit exists and gives a viable definition of the path integral.

We have started from canonical quantization and derived the above discretized form of the space of functions. This regularization is often called Time Slicing (TS). Viceversa, one can start directly with the path integral, regulate it suitably, and use it to construct the quantum theory. The path integral evaluates a transition amplitude that is eventually seen to satisfy a Schördinger wave equation. This can be viewed as an alternative approach to quantization. In the regularization procedure one must make several choices, and they may produce different transition amplitudes. For example, in a TS regularization one may discretize the potential term $V(x(t))$ in the action to $V(x_k)$ or $V(x_{k-1})$ or $V(\frac{1}{2}(x_k + x_{k-1})$. In the present case this makes no difference, and one obtains the same continuum limit. For more complicated interactions, such as those arising from the coupling to gauge fields or in the presence of a nontrivial background metric, different discretizations may produce different final answers. These ambiguities are the path integral counterparts of the ordering ambiguities of canonical quantization. Given a regulated path integral, one may compute the transition amplitude and find the corresponding quantum hamiltonian by checking which Schrödinger equation it satisfies.

In general, any regularization can be used to solve a physical problem, as different regularizations are related by "counterterms", that is extra potential terms that added to the action in one regularization scheme produce the result of another regularization scheme. Of course, a given regularization may be easier to deal with than others. It is often the case that one is interested in studying a system characterized by a particular quantum hamiltonian. To treat the problem with path integrals, once one has chosen a consistent regularization, he must add counterterms to make sure that the correct hamiltonian is reproduced by the regulated path integral. This is typical when dealing with motions on a curved space with metric $g_{ij}(x)$ and, in a milder form, when considering the coupling to a gauge field $A_i(x)$. Counterterms are always local function. In the language of quantum field theory, quantum mechanics is a super-renormalizable QFT in one dimension, with coupling constants parametrized by the scalar $V(x)$, vector $A_i(x)$, and tensor $g_{ij}(x)$ potentials.

A regularization scheme, alternative to TS, can be obtained by considering the Fourier expansion of the functions belonging to the space of paths (or alternatively an expansion in a complete set of orthonormal functions). The regularization is achieved by truncating the expansion at a given high mode M. Path integration is defined as the integration over the finite number of Fourier coefficients. The continuum limit is obtained by removing the cut-off M, i.e. sending $M \to \infty$. Such a regularization is called Mode Regularization (MR).

Yet another regularization is based on extending the time dimension to higher dimensional spacetimes of dimension D, analytically continued to be a complex number. It is called Dimensional Regularization (DR), and though more abstract than the previous ones, it has its own

virtues. These regularizations will be exemplified when studying the motion in curved space in chapter **??**.

We have introduced path integrals by considering a single degree of freedom. Extension to a finite number of degrees of freedom is immediate, so that quantizing the motion of one or more particles in a finite dimensional space does not pose any new conceptual problem. For example, the motion of a nonrelativistic particle in $\mathbb{R}^3$ with cartesian coordinates $\vec{x}$, in the presence of a scalar potential $V(\vec{x})$, is quantized by the following discretized path integral

$$\int Dx\, e^{\frac{i}{\hbar}S[x]} \;=\; \lim_{N\to\infty}\int \Big(\prod_{k=1}^{N-1} d^3x_k\Big)\Big(\frac{m}{2\pi i\hbar\epsilon}\Big)^{\frac{3N}{2}} e^{\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\left[\frac{m}{2}\frac{(\vec{x}_k-\vec{x}_{k-1})^2}{\epsilon^2}-V(\vec{x}_{k-1})\right]}\,. \tag{36}$$

Formally, one can also consider the case of an infinite number of degrees of freedom, as appropriate for a field theory. In this case convergence is not guaranteed, and the removal of the regularization may lead to infinite results. In the class of theories called renormalizable, the infinites can be removed consistently by a renormalization procedure that redefines the field variables and the coupling constants, and allows to obtain, at least at the level of perturbation theory, finite results.

4 Correlation functions

Correlation functions are quantities used to describe many physical observables in the quantum theory. They are also useful to develop the perturbative expansion around the solvable gaussian path integral that corresponds to a free theory.

Correlation functions are normalized averages of the product of n dynamical variables, evaluated at different times and weighted by $e^{\frac{i}{\hbar}S}$. In our one dimensional example, the normalized "n-point correlation function" is defined by

$$\langle x(t_1)x(t_2)\dots x(t_n)\rangle = \frac{1}{Z}\int Dx\, x(t_1)x(t_2)\dots x(t_n)\, e^{\frac{i}{\hbar}S[x]} \tag{37}$$

where $Z=\int Dx\, e^{\frac{i}{\hbar}S[x]}$ provides the normalization to guarantee that $\langle 1\rangle = 1$. Of particular importance is the 2-point function $\langle x(t_1)x(t_2)\rangle$, often called the propagator. It is understood that correlation functions depend implicitly on the boundary conditions that specify the initial and final states. Very often, especially in quantum field theory, one chooses the initial and final states to be the vacuum state (the state with lowest energy) and, in additions, considers an infinite propagation time. We mostly consider amplitudes between positions eigenstates, but one can insert any desired state as boundary state by using eq. (8).

In this book we are going to use mostly path integrals. However, it is useful to compare with the corresponding definition of correlation functions given in canonical quantization. We have employed the Schrödinger picture to evaluate the transition amplitude. In this picture operators are time independent and states acquire the time dependence by the Schrödinger equation. To state the equivalent definition of the n-point correlation function, given the times $t_1, t_2, ..., t_n$, one has to reorder them from the earliest to the latest one, i.e. use the permutation $T(1), T(2), ..., T(n)$ of the numbers $1, 2, ..., n$ such that $t_{T(1)} < t_{T(2)} < ... < t_{T(n)}$. Then one

defines

$$\begin{aligned}\langle x(t_1)x(t_2)\cdots x(t_n)\rangle &= \frac{1}{Z}\langle x_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_{T(n)})}\hat{x}\, e^{-\frac{i}{\hbar}\hat{H}(t_{T(n)}-t_{T(n-1)})}\cdots \\ &\cdots\ e^{-\frac{i}{\hbar}\hat{H}(t_{T(3)}-t_{T(2)})}\hat{x}\, e^{-\frac{i}{\hbar}\hat{H}(t_{T(2)}-t_{T(1)})}\hat{x}\, e^{-\frac{i}{\hbar}\hat{H}(t_{T(1)}-t_i)}|x_i\rangle\end{aligned} \tag{38}$$

where $Z = \langle x_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|x_i\rangle$ is the transition amplitude. The time ordering guarantees that in the path integral derivation, due to the time slicing procedure, each position operator is substituted by the eigenvalue of the eigenstate carried by the resolution of the identity inserted next to the operator under consideration. This is always the case as, for very large N, the time discretization is sufficiently fine to have a resolution of the identity next to the position of one of the operators $\hat{x}$, which is then substituted by an eigenvalue.

Equivalently, in the Heisenberg picture, one assigns the time evolution to the operators while states are time independent. Heisenberg's equations of motion (Heisenberg's "matrix mechanics") read as

$$i\hbar\frac{d\hat{x}_H}{dt} = [\hat{x}_H, \hat{H}] \tag{39}$$

where the subscript H refers to operators in the Heisenberg picture. They correspond to the quantum version of Hamilton's equations, with the Poisson bracket substituted by a commutator that takes the value of $i\hbar$ times the classical Poisson bracket. For a time independent hamiltonian the solution can formally be written as

$$\hat{x}_H(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{x}_H(0)e^{-\frac{i}{\hbar}\hat{H}t} \tag{40}$$

where the value of $\hat{x}_H(0)$ can be identified with the time independent Schrödinger operator $\hat{x}$. Eigenstates of $\hat{x}_H(t)$ can be written as $|x,t\rangle_H$

$$\hat{x}_H(t)|x,t\rangle_H = x|x,t\rangle_H\ . \tag{41}$$

The relation to the Schrödinger picture is simply given in terms of the unitary operator $e^{\frac{i}{\hbar}\hat{H}t}$. This operator relates the two pictures which are then unitarily equivalent, as guaranteed by the Stone–von Neumann theorem. Correlation functions in the Heisenberg picture are defined by

$$\langle x(t_1)x(t_2)...x(t_n)\rangle = \frac{1}{Z}{}_H\langle x_f,t_f|T\,\hat{x}(t_1)\hat{x}(t_2)\cdots\hat{x}(t_n)\,|x_i,t_i\rangle_H \tag{42}$$

where the symbol T indicates time ordering, i.e. the prescription of ordering the operators in such a way that they have an increasing value of time when going form right to left. The value of the transition amplitude that normalizes the expression is written, in such a picture, as $Z = {}_H\langle x_f,t_f|x_i,t_i\rangle_H$. As already mentioned, in extensions to QFT one normally chooses $t_i = -\infty$, $t_f = \infty$, and picks as final states the vacuum state, rather than position eigenstates.

It is useful to collect all correlation functions into a single object $Z[J]$, called the generating functional of correlation functions. One uses an arbitrary function $J(t)$, called "source", and defines

$$\begin{aligned}Z[J] &= \int Dx\, e^{\frac{i}{\hbar}(S[x]+\int dt\, Jx)} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n\int dt_1dt_2\ldots dt_n\ \langle x(t_1)x(t_2)\cdots x(t_n)\rangle_U\ J(t_1)J(t_2)\cdots J(t_n)\end{aligned} \tag{43}$$

where the subscript "U" indicates un-normalized correlation functions, i.e. correlation functions obtained without dividing by Z. The last expression is proven by expanding the exponential $e^{\frac{i}{\hbar}\int dt\, Jx}$.

Correlation functions are obtained by computing the functional derivatives of the generating functional $Z[J]$, normalised as follows

$$\langle x(t_1)x(t_2)\cdots x(t_n)\rangle = \frac{1}{Z}\left(\frac{\hbar}{i}\right)^n \frac{\delta^n Z[J]}{\delta J(t_1)\delta J(t_2)\cdots\delta J(t_n)}\bigg|_{J=0} \tag{44}$$

where $Z = Z[0]$.

4.1 Digression over gaussian integrals

Gaussian integrals in one or more variables are easily computed. For a real variable $\phi \in \mathbb{R}$ they are given by

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}}\ \mathrm{e}^{-\frac{1}{2}K\phi^2} &= \frac{1}{\sqrt{K}}\\ \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}}\ \mathrm{e}^{-\frac{1}{2}K\phi^2+J\phi} &= \frac{1}{\sqrt{K}}\,\mathrm{e}^{\frac{1}{2}\frac{1}{K}J^2}\end{aligned} \tag{45}$$

with K a real positive number. The first one is the standard gaussian integral, whose square is easily computed in polar coordinates. The second one is obtained by square completion, i.e. writing $-\frac{1}{2}K\phi^2 + J\phi = -\frac{1}{2}K(\phi - \frac{J}{K})^2 + \frac{1}{2}\frac{1}{K}J^2$, and then shifting the measure from ϕ to $\phi - \frac{J}{K}$ to obtain the desired result.

They are straightforwardly extended to n real variables

$$\begin{aligned}\int \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}}\ \mathrm{e}^{-\frac{1}{2}\phi^i K_{ij}\phi^j} &= (\det K_{ij})^{-\frac{1}{2}}\\ \int \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}}\ \mathrm{e}^{-\frac{1}{2}\phi^i K_{ij}\phi^j + J_i\phi^i} &= (\det K_{ij})^{-\frac{1}{2}}\,\mathrm{e}^{\frac{1}{2}J_iG^{ij}J_j}\end{aligned} \tag{46}$$

where K_{ij} is a real, symmetric, positive definite matrix (all eigenvalues must be strictly positive), and G^{ij} its inverse (so that $K_{ij}G^{jk} = \delta_i^k$). The first integral is immediate if K_{ij} is diagonal, and valid in full generality by noting that K_{ij} is diagonalizable by an orthogonal transformation which leaves the measure invariant. The last integral is obtained again by square completion.

These gaussian integrals are suitable for euclidean path integrals. Moreover, in a hypercondensed notation to be explained shortly, path integrals look very much like ordinary integrals. Of course the definition of determinants for infinite dimensional matrices is delicate, and requires a regularization procedure.

By analytical extension one obtains gaussian integrals suitable for quantum mechanics

$$\int \frac{d^n\phi}{(-2\pi i)^{\frac{n}{2}}}\ \mathrm{e}^{-\frac{i}{2}\phi^i K_{ij}\phi^j + iJ_i\phi^i} = (\det K_{ij})^{-\frac{1}{2}}\ \mathrm{e}^{\frac{i}{2}J_iG^{ij}J_j} \tag{47}$$

where again G^{ij} is the inverse of K_{ij}. Convergence to the given values is guaranteed if K_{ij} has a small negative imaginary part (e.g. $K_{ij} = \tilde{K}_{ij} - i\epsilon\delta_{ij}$ with $\tilde{K}_{ij}$ real and $\epsilon > 0$) that assure a gaussian damping for $|\phi| \to \infty$ and which is eventually set to vanish (in quantum field

theory this gives rise to the causal $i\epsilon$ Feynman prescription). In a hypercondensed notation these formulae give the formal solution of path integrals (without gauge invariances) in either quantum mechanics or quantum field theory. Gauge invariances correspond to the vanishing of $\det K_{ij}$, and one must apply a gauge fixing procedure to obtain a finite answer.

4.2 Hypercondensed notation and generating functionals

To proceed swiftly, it is useful to introduce a hypercondensed notation. It allows to treat path integrals, including those for field theories, formally as ordinary integrals. The hypercondensed notation is defined by lumping together discrete and continuous indices into a single index, so that a variable ϕ^i can be used as a shorthand notation for the position $x(t)$ of the particle, identifying

$$x(t) \;\to\; \phi^i \tag{48}$$

that is

$$\begin{aligned} x &\to \phi \\ t &\to i\,. \end{aligned} \tag{49}$$

Similarly for fields, as for example the vector quadripotential $A_\mu(x^\nu)$, the hypercondensed notation is obtained by denoting

$$A_\mu(x^\nu) \;\to\; \phi^i \tag{50}$$

that is

$$\begin{aligned} A &\to \phi \\ \mu, x^\nu &\to i \end{aligned} \tag{51}$$

where now the index i contains a discrete part (the discrete index $\mu = 0, 1, 2, 3$) and a continuous part (the space-time coordinates $x^\nu = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$). Indices may be lowered with a metric, simply given by the identity matrix in many cases, though one may consider more general situations. Repeated indices are understood to be summed over (the Einstein summation convention). Thus, in the above cases the notation $\phi^i\phi_i$ stands for $\int dt\, x(t)x(t)$ and $\int d^4x\, A_\mu(x)A^\mu(x)$, respectively. Of course, one must pay attention to simple looking expressions, as they include integrations or infinite sums, and might not converge.

With such a notation we are ready to review the definition of correlation functions, introduce generating functionals, and present gaussian integration formulae. We are also going to describe the Wick theorem, that gives a simple way of computing correlation functions in a free theory in terms of the 2-point functions.

The path integral in (22) can be written as

$$\int D\phi\, e^{\frac{i}{\hbar}S[\phi]} \tag{52}$$

and correlation functions by

$$\langle \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}\rangle = \frac{1}{Z}\int D\phi\ \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}\mathrm{e}^{\frac{i}{\hbar}S[\phi]}\ . \tag{53}$$

where $Z = \int D\phi\, e^{\frac{i}{\hbar}S[\phi]}$. The generating functional takes the form

$$Z[J] = \int D\phi\ \mathrm{e}^{\frac{i}{\hbar}(S[\phi]+J_i\phi^i)} \tag{54}$$

and generates all correlation functions by differentiation (in hypercondensed notation functional derivatives read as usual derivatives, though we keep using the symbol δ of functional derivative)

$$\langle \phi^{i_1} \phi^{i_2} \cdots \phi^{i_n} \rangle = \frac{1}{Z[0]} \left(\frac{\hbar}{i} \right)^n \frac{\delta}{\delta J_{i_1}} \frac{\delta}{\delta J_{i_2}} \cdots \frac{\delta}{\delta J_{i_n}} Z[J] \Big|_{J=0} . \tag{55}$$

We can now define the generating functional of connected correlation functions $W[J]$ by the relation

$$Z[J] = \mathrm{e}^{\frac{i}{\hbar} W[J]} \qquad \Rightarrow \qquad W[J] = \frac{\hbar}{i} \ln Z[J] . \tag{56}$$

One can prove that it generates connected correlation functions by

$$\langle \phi^{i_1} \phi^{i_2} \cdots \phi^{i_n} \rangle_c = \left(\frac{\hbar}{i} \right)^{n-1} \frac{\delta}{\delta J_{i_1}} \frac{\delta}{\delta J_{i_2}} \cdots \frac{\delta}{\delta J_{i_n}} W[J] \Big|_{J=0} . \tag{57}$$

We will check this in the free theory.

It is also useful to define the effective action $\Gamma[\varphi]$ as the Legendre transform of the $W[J]$ functional

$$\Gamma[\varphi] = \min_J \Big\{ W[J] - J_i \varphi^i \Big\} \tag{58}$$

which can be treated as a classical action that includes all quantum corrections. It generates the so called one-particle irreducible (1PI) correlation functions. The minimum in J is obtained at $\varphi^i = \frac{\delta W[J]}{\delta J_i}$, a relation that should be inverted to obtain $J_i = J_i(\varphi)$ and inserted into the right hand side of (58) to obtain the effective action as a functional of the variable φ^i.

The last two functionals find their main applications in quantum field theory, but we will have the chance of employing them in later applications of the worldline approach. Equivalent definitions for euclidean path integrals are collected at the end of the chapter.

4.3 Free theory

It is useful to study the free theory as it provides a simple application of the previous formulae, giving at the same time additional intuition. A free theory is described by a quadratic action

$$S[\phi] = -\frac{1}{2} \phi^i K_{ij} \phi^j \tag{59}$$

which produces the linear equations of motion $K_{ij}\phi^j = 0$. We assume K_{ij} invertible, which means that there are no gauge symmetries. Denoting $\mathcal{D}\phi \equiv \frac{d^n \phi}{(-2\pi i)^{\frac{n}{2}}}$, setting $\hbar = 1$, and using eq. (47), one can calculate the path integral with sources

$$Z[J] = \int \mathcal{D}\phi \ \mathrm{e}^{i(S[\phi] + J_i \phi^i)} = (\det K_{ij})^{-\frac{1}{2}} \ \mathrm{e}^{\frac{i}{2} J_i G^{ij} J_j} . \tag{60}$$

Then, recalling eq. (55), one immediately obtains the following correlation functions

$$\begin{aligned} &\langle 1 \rangle = 1 \\ &\langle \phi^i \rangle = 0 \\ &\langle \phi^i \phi^j \rangle = -i G^{ij} . \end{aligned} \tag{61}$$

The first one is a consequence of the normalization, the second one reflects the symmetry $\phi^i \to -\phi^i$, and the third one is the propagator, which we recognize as proportional to the inverse of the kinetic matrix K_{ij}.

Continuing with the calculation of higher point functions, we see that all correlation functions with an odd number of points vanish, again a consequence of the symmetry $\phi^i \to -\phi^i$. Those with an even number n factorize instead into a sum of $(n-1)!!$ terms, the latter given by the product of the 2-point functions which connect any two points in all possible ways. This fact is known as the "Wick theorem". For example, the 4-point correlation function is given by

$$\langle\phi^1\phi^2\phi^3\phi^4\rangle = \langle\phi^1\phi^2\rangle\langle\phi^3\phi^4\rangle + \langle\phi^1\phi^3\rangle\langle\phi^2\phi^4\rangle + \langle\phi^1\phi^4\rangle\langle\phi^2\phi^3\rangle \tag{62}$$

that indeed contains the sum of 3!! terms. This correlation function is not connected, as it disconnects into the sum of products of correlation functions of lower order. This is true for all higher point correlation functions of the free theory.

The generating functional of connected correlation functions $W[J]$ is obtained from eq. (60) using (56)

$$W[J] = \frac{1}{2} J_i G^{ij} J_j - \Lambda \tag{63}$$

where $\Lambda = -\frac{i}{2}\ln\det(K_{ij}) = -\frac{i}{2}\,\mathrm{tr}\ln(K_{ij})$ is a constant, whose precise value is often not needed. One verifies that all of its correlation functions (0- and 2-points) are connected.

Let us also calculate the effective action. The minimum in J of eq. (58) is achieved for

$$\frac{\delta W}{\delta J_i} = \varphi^i \quad \Longrightarrow \quad \varphi^i = G^{ij} J_j \quad \Longrightarrow \quad J_i = K_{ij}\varphi^j \tag{64}$$

so that

$$\Gamma[\varphi] = -\frac{1}{2}\varphi^i K_{ij}\varphi^j - \Lambda\ . \tag{65}$$

We see that for a free theory the effective action $\Gamma[\varphi]$ coincides with the original action $S[\varphi]$ up to the additive constant Λ, that can be interpreted as a vacuum energy of quantum origin. The latter can be disregarded if the gravitational interactions are neglected. In general, the effective action can be considered as a classical action that contains the effects of quantization in its couplings (and thus should not be quantized again).

4.4 Harmonic oscillator

Let us work out explicitely the case of a harmonic oscillator with unit mass

$$S[x] = \int_{-\infty}^{\infty} dt \Big(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2\Big)\ , \qquad Z[J] = \int Dx\, e^{\frac{i}{\hbar}(S[x]+\int dt\, Jx)} \tag{66}$$

solved formally in the previous section. We can repeat briefly the deduction without using the hypercondensed notation. We consider an infinite propagation time and a transition amplitude between the ground state, which classically is achieved for $x = 0$. The action in the exponent can be manipulated with an integration by parts without producing boundary terms (imposing that $x(t)$ be in its classical vacuum at initial and final times gives a vanishing boundary term,

another justification will be given later on when treating the euclidean version of the problem). One may re-write the action with an integration by parts as

$$\begin{aligned} S[x] &= -\int dt\, \frac{1}{2}x(t)\Big(\frac{d^2}{dt^2}+\omega^2\Big)x(t) \\ &= -\int dtdt'\, \frac{1}{2}x(t)\Big(\frac{d^2}{dt^2}+\omega^2\Big)\delta(t-t')x(t') \\ &\equiv -\frac{1}{2}\int dtdt'\, x(t)K(t,t')x(t') \qquad \Big(-\frac{1}{2}\phi^i K_{ij}\phi^j \text{ in hyp. notation}\Big) \end{aligned} \tag{67}$$

where a Dirac delta function $\delta(t-t')$ has been introduced to expose the "kinetic matrix" $K(t,t') = (\frac{d^2}{dt^2}+\omega^2)\delta(t-t')$, i.e. the differential operator of the harmonic oscillator. The inverse of this matrix (i.e. the Green function of the differential operator) is conveniently written in terms of its Fourier transform

$$G(t,t') = \int \frac{dp}{2\pi}\, \frac{e^{-ip(t-t')}}{-p^2+\omega^2}\ . \tag{68}$$

Indeed it satisfies the defining equation

$$\int dt''\, K(t,t'')G(t'',t') = \Big(\frac{d^2}{dt^2}+\omega^2\Big)G(t,t') = \delta(t-t')\ . \tag{69}$$

Adding the Feynman $i\epsilon$ prescription for specifying how to integrate around the poles $p=\pm\omega$ ($\omega^2 \to \omega^2 - i\epsilon$ with $\epsilon \to 0^+$) one computes

$$G(t,t') = \int \frac{dp}{2\pi}\, \frac{e^{-ip(t-t')}}{-p^2+\omega^2-i\epsilon} = \frac{i}{2\omega}e^{-i\omega|t-t'|}\ . \tag{70}$$

One uses it to complete the square in (66) and find

$$\begin{aligned} Z[J] &= \int Dx\, e^{\frac{i}{\hbar}(S[x]+\int dt\, Jx)} \\ &= \int Dx\, \exp\frac{i}{\hbar}\Big[-\int dtdt'\, \Big(\frac{1}{2}x(t)K(t,t')x(t') - J(t)\delta(t-t')x(t') \\ &\qquad\qquad +\frac{1}{2}J(t)G(t,t')J(t') - \frac{1}{2}J(t)G(t,t')J(t')\Big)\Big] \\ &= \exp\Big(\frac{i}{2\hbar}\int dtdt'\, J(t)G(t,t')J(t')\Big)\underbrace{\int D\tilde{x}\, \exp\Big(-\frac{i}{\hbar}\int dtdt'\, \frac{1}{2}\tilde{x}(t)K(t,t')\tilde{x}(t')\Big)}_{N\sim \det{}^{-1/2}[K(t,t')]} \\ &= N\exp\Big(\frac{i}{2\hbar}\int dtdt'\, J(t)G(t,t')J(t')\Big) \end{aligned} \tag{71}$$

where $\tilde{x}(t) = x(t) - \int dt' G(t,t')J(t')$ is the shifted variable that completes the square.

The 2-point function, i.e. the Feynman propagator, is then given by

$$\begin{aligned} \langle x(t)x(t')\rangle &= \frac{\int Dx\, x(t)x(t')e^{\frac{i}{\hbar}S[x]}}{\int Dx\, e^{\frac{i}{\hbar}S[x]}} = \frac{1}{Z[0]}\left(\frac{\hbar}{i}\right)^2 \frac{\delta^2 Z[J]}{\delta J(t)\delta J(t')}\bigg|_{J=0} = -i\hbar G(t,t') \\ &= \frac{\hbar}{2\omega}e^{-i\omega|t-t'|}\ . \end{aligned} \tag{72}$$

4.4.1 Harmonic oscillator in euclidean time

The statistical partition function in the limit of vanishing temperature ($\Theta \to 0$), or equivalently for an infinite euclidean propagation time ($\beta \to \infty$), takes a simple form

$$Z_E = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_n e^{-\beta E_n} \quad \stackrel{\beta\to\infty}{\longrightarrow} \quad e^{-\beta E_0} + \text{subleading terms.} \tag{73}$$

This is true even in the presence of a source J if one assumes that the source is nonvanishing for a finite interval of time only: the remaining infinite time is sufficient to project the operator $e^{-\beta \hat{H}}$ onto the ground state. This allows to rewrite the generating functional $Z[J]$ in the euclidean case in a simpler way, justifying again the dropping of boundary terms in the integration by parts. The statistical partition function is obtained by using periodic boundary conditions, and for large β one gets the projection onto the ground state

$$\begin{aligned} Z_E[J] &= \int_{PBC} Dx\, e^{-S_E[x]+\int d\tau\, Jx} \;\to\; \lim_{\beta\to\infty} e^{-\beta E_0(J)} \\ S_E[x] &= \int_{-\infty}^{\infty} d\tau \Big(\frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2\Big) \end{aligned} \tag{74}$$

where $E_0(J)$ is the ground state energy in the presence of the source J. We can now repeat the previous calculation in the present context. One may integrate by parts without encountering boundary terms, as the paths are closed, and the path integral is strictly gaussian

$$\begin{aligned} Z_E[J] &= \int_{PBC} Dx\, \exp\Big[-\int d\tau \Big(\frac{1}{2}x(\tau)\Big(-\frac{d^2}{d\tau^2}+\omega^2\Big)x(\tau) - J(\tau)x(\tau)\Big] \\ &= N \exp\Big[\frac{1}{2}\int d\tau d\tau'\, J(\tau)G_E(\tau,\tau')J(\tau')\Big] \end{aligned} \tag{75}$$

where the euclidean Green function G_E is given by

$$G_E(\tau,\tau') = \Big[-\frac{d^2}{d\tau^2}+\omega^2\Big]^{-1} = \int \frac{dp_E}{2\pi}\,\frac{e^{-ip_E(\tau-\tau')}}{p_E^2+\omega^2} = \frac{1}{2\omega}e^{-\omega|\tau-\tau'|}\,. \tag{76}$$

This euclidean Green function is unique: there are no poles and related prescriptions to specify how to perform the integration. The inverse Wick rotation implies $\tau \equiv t_E \to it_M \equiv it$ and $p_E \to -ip_M \equiv -ip$, with the latter arising form the requirement that the correct Fourier transform is kept during the analytic deformation. Indeed for the two-point function one finds

$$\begin{aligned} \langle x(\tau)x(\tau')\rangle &= G_E(\tau,\tau') = \int \frac{dp_E}{2\pi}\,\frac{e^{-ip_E(\tau-\tau')}}{p_E^2+\omega^2} \quad \to \\ &\to \quad -i\int \frac{dp_M}{2\pi}\,\frac{e^{-ip_M(t-t')}}{-p_M^2+\omega^2} = -iG_M(t,t') = \langle x(t)x(t')\rangle \end{aligned} \tag{77}$$

which is the Feynman propagator in (72) (with $\hbar = 1$).

We recognize that the Feynman propagator is the unique analytical extension of the euclidean two point function. All other Green functions, such as the retarded or advanced ones, correspond to different boundary conditions that can be implemented with different prescriptions for performing the integration around the poles. They cannot be Wick rotated as one would encounter poles in the analytic continuation.

5 Perturbative expansion

The free theory corresponds to a gaussian path integral which is exactly solvable. With interactions one is often unable to compute exactly the path integral, and one must resort to some sort of approximation. The simplest one is the perturbative expansion around a free theory, which consists in expanding the solution in power series in the coupling constants that parametrize the interactions. If the couplings are small enough, the perturbative expansion might give a good approximation to the searched for solution.

We describe the perturbative expansion taking as guiding example the case of an anharmonic oscillator

$$S[x] \;=\; \int dt\Big(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2 - \frac{g}{3!}x^3 - \frac{\lambda}{4!}x^4\Big)\;. \tag{78}$$

When the coupling constants g and λ vanish, the theory is exactly solvable. Thus one may try to include perturbatively the corrections that arise when g and λ are small enough. It is convenient to split the action as the sum of two terms, a free part S_0 and an interacting one S_{int}

$$\begin{aligned} S[x] &= S_0[x] + S_{int}[x] \\ S_0[x] &= \int dt\,\Big(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2\Big) \\ S_{int}[x] &= \int dt\,\Big(-\frac{g}{3!}x^3 - \frac{\lambda}{4!}x^4\Big)\;. \end{aligned} \tag{79}$$

Including a source term, one considers the path integral and expands in a Taylor series the exponential of the interaction term

$$\begin{aligned} Z[J] &= \int \mathcal{D}x\, e^{\frac{i}{\hbar}(S[x]+\int dt\, Jx)} \qquad (80)\\ &= \int \mathcal{D}x\, e^{\frac{i}{\hbar}(S_0[x]+S_{int}[x]+\int dt\, Jx)} \\ &= \int \mathcal{D}x\, e^{\frac{i}{\hbar}S_{int}[x]}\, e^{\frac{i}{\hbar}(S_0[x]+\int dt\, Jx)} \\ &= \int \mathcal{D}x\, \Big[1 + \frac{i}{\hbar}S_{int}[x] + \frac{1}{2}\Big(\frac{i}{\hbar}S_{int}[x]\Big)^2 + \cdots \\ &\qquad \cdots + \frac{1}{n!}\Big(\frac{i}{\hbar}S_{int}[x]\Big)^n + \cdots\Big]\, e^{\frac{i}{\hbar}(S_0[x]+\int dt\, Jx)}\;. \end{aligned}$$

Written in the last form, one may proceed in computing it term by term. Using an obvious notation, the path integral can be written equivalently as

$$Z[J] = \Big\langle e^{\frac{i}{\hbar}S_{int}[x]}\Big\rangle_{U,0,J} \tag{81}$$

where the subscripts $U, 0, J$ denote un-normalized averaging (U) with the free theory (0) and in the presence of a source (J). This last expression is sometimes called the "Dyson formula". It generates the perturbative expansion in terms of Feynman diagrams, as we shall see.

An alternative way of writing the perturbative series is the following one

$$\begin{aligned} Z[J] &= \int \mathcal{D}x\ e^{\frac{i}{\hbar}(S[x]+\int dt\, Jx)} = \int \mathcal{D}x\ e^{\frac{i}{\hbar}S_{int}[x]}\, e^{\frac{i}{\hbar}(S_0[x]+\int dt\, Jx)} \\ &= e^{\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}]} \int \mathcal{D}x\ e^{\frac{i}{\hbar}(S_0[x]+\int dt\, Jx)} \\ &= e^{\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}]}\, Z_0[J] \end{aligned} \tag{82}$$

which presents the solution as a (quite complicated) differential operator acting on the solution of the free theory $Z_0[J]$. In particular, all vacuum diagrams are generated by

$$Z[0] = \int \mathcal{D}x\ e^{\frac{i}{\hbar}S[x]} = e^{\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}]}\, Z_0[J]\Big|_{J=0} . \tag{83}$$

The expansion in terms of Feynman diagrams is obtained expanding the interactions potential and using the Wick theorem to compute the correlation functions of the free theory: the vertices generated by the interactions potential contain a coupling constant plus quantum variables that are tied together two by two in all possible ways with the free propagators (graphically each vertex is denoted by a dot and propagators are denoted by lines). This will be exemplified next in the case of vacuum diagrams for the anharmonic oscillator.

5.1 Vacuum diagrams

As a first example we compute perturbatively the corrections to the ground state energy of the harmonic oscillator due to the anharmonic potential terms. As already mentioned, it is often the case that one computes using the euclidean version of the theory and only at the very end performs the inverse Wick rotation to obtain the results in minkowskian time. Thus we will proceed with the euclidean version of the theory, which is the one used later on in presenting worldline applications.

We wish to compute

$$\begin{aligned} Z_E[J] &= \int \mathcal{D}x\ e^{-S_E[x]+\int d\tau Jx} \\ S_E[x] &= \lim_{\beta\to\infty} \int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2 + \frac{g}{3!}x^3 + \frac{\lambda}{4!}x^4\right) \end{aligned} \tag{84}$$

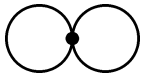
with $\beta \to \infty$. The corrections to the ground state energy can be recognized from calculating

$$\begin{aligned} Z_E[0] &= \langle 1\rangle_U = \lim_{\beta\to\infty} \langle 0|e^{-\beta\hat{H}}|0\rangle = \lim_{\beta\to\infty} e^{-\beta E_0} \\ &= \left\langle e^{-S_{E,int}[x]} \right\rangle_{U,0} = \lim_{\beta\to\infty} e^{-\beta(E_0^{(0)}+\Delta E_0)} \end{aligned} \tag{85}$$

where the exact energy E_0 of the ground state $|0\rangle$ differs from the ground state energy of the harmonic oscillator $E_0^{(0)}$ by the term ΔE_0 due to the anharmonic potential. The latter can be computed perturbatively. We consider the first non vanishing corrections only.

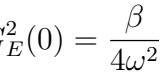
Let us look first at the case with $g = 0$ and focus on the first correction in λ

$$\begin{aligned} Z_E[0] &= \langle 1\rangle_U = \Big\langle e^{-S_{E,int}[x]}\Big\rangle_{U,0} = \Big\langle (1 - S_{E,int}[x] + \cdots)\Big\rangle_{U,0} \\ &= \langle 1\rangle_{U,0} - \frac{\lambda}{4!}\int_{-\beta/2}^{\beta/2} d\tau\, \langle x^4(\tau)\rangle_{U,0} + \cdots \\ &= \langle 1\rangle_{U,0}\Big[1 - \frac{\lambda}{4!}\int_{-\beta/2}^{\beta/2} d\tau\, \langle x^4(\tau)\rangle_0 + \cdots\Big] \\ &= \langle 1\rangle_{U,0}\Big[1 - \frac{\lambda}{4!}\Big[3 \times \quad\Big] + \cdots\Big] . \end{aligned} \tag{86}$$



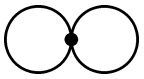
In the last line we have used Wick contractions to calculate normalized correlations functions in the free theory, and then introduced a graphical representation in terms of Feynman diagrams. In this graphical representation a line denotes a propagator that joins two points in time, while vertices arising form the interactions are denoted by dots. Recalling the euclidean propagator, computed in eq. (76),

$$\langle x(\tau)x(\tau')\rangle_0 = G_E(\tau - \tau') = \frac{1}{2\omega}e^{-\omega|\tau-\tau'|} = \quad \tau \quad \tau' \tag{87}$$



one immediately finds

$$\quad = \int_{-\beta/2}^{\beta/2} d\tau\, G_E^2(0) = \frac{\beta}{4\omega^2} . \tag{88}$$



Thus, to this perturbative order one gets

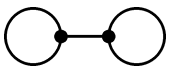
$$Z_E[0] = \langle 1\rangle_{U,0}\Big[1 - \frac{\lambda}{4!}\Big[3\frac{\beta}{4\omega^2}\Big] + \cdots\Big] = \langle 1\rangle_{U,0}\; e^{-\frac{\beta\lambda}{32\omega^2}+\cdots} \tag{89}$$

so that, comparing with eq. (85), one finds

$$\Delta E_0 = \frac{1}{32}\frac{\lambda}{\omega^2} . \tag{90}$$

Similarly one may consider the case with $g \neq 0$ and $\lambda = 0$. The first non vanishing correction is obtained from

$$\begin{aligned} Z_E[0] &= \langle 1\rangle_U = \Big\langle \Big(1 - S_{E,int} + \frac{1}{2}S_{E,int}^2 + \cdots\Big)\Big\rangle_{U,0} \qquad (91) \\ &= \langle 1\rangle_{U,0} - \frac{g}{3!}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau\, \langle x^3(\tau)\rangle_{U,0} \\ &\quad + \frac{1}{2}\Big(\frac{g}{3!}\Big)^2 \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau'\, \langle x^3(\tau)x^3(\tau')\rangle_{U,0} + \cdots \\ &= \langle 1\rangle_{U,0}\Big[1 + 0 + \frac{1}{2}\Big(\frac{g}{3!}\Big)^2\Big[3! \times \quad + 3^2 \times \quad\Big] + \cdots\Big] . \end{aligned}$$

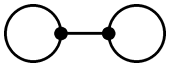


Now

$$= \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau'\, G_E^3(\tau-\tau') = \frac{1}{8\omega^3}\int_{-\beta/2}^{\beta/2} d\tau \int_{-\infty}^{\infty} d\sigma\, e^{-3\omega|\sigma|}$$

$$= \frac{\beta}{8\omega^3}\frac{2}{3\omega} \tag{92}$$

and



$$= \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau'\, G_E(0)G_E(\tau-\tau')G_E(0)$$

$$= \frac{1}{8\omega^3}\int_{-\beta/2}^{\beta/2} d\tau \int_{-\infty}^{\infty} d\sigma\, e^{-\omega|\sigma|} = \frac{\beta}{8\omega^3}\frac{2}{\omega} \tag{93}$$

where the limit $\beta \to \infty$ has been used suitably to calculate the integrals. Therefore

$$Z_E[0] = \langle 1\rangle_{U,0}\left[1 + \frac{1}{2}\Big(\frac{g}{3!}\Big)^2\Big(3!\frac{\beta}{12\omega^4} + 3^2\frac{\beta}{4\omega^4}\Big) + \cdots\right] = \langle 1\rangle_{U,0}\; e^{\beta\frac{11}{8(3!)^2}\frac{g^2}{\omega^4}+\cdots} \tag{94}$$

and one finds

$$\Delta E_0 = -\frac{11}{288}\frac{g^2}{\omega^4}\ . \tag{95}$$

5.2 Heat kernel

A second example to illustrate perturbation theory is the calculation of the heat kernel using path integrals. The heat kernel is the fundamental solution of the heat equation

$$-\frac{\partial}{\partial\beta}\psi = \hat{H}\psi \tag{96}$$

where $\hat{H}$ is a second order elliptic differential operator. It can be obtained by Wick rotating the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = \hat{H}\psi \tag{97}$$

with $t \to -i\beta$, as discussed previously. We consider the simple case of a hamiltonian for a particle with unit mass in D dimensions in the presence of a smooth scalar potential V

$$\hat{H} = -\frac{1}{2}\nabla^2 + V(x) \tag{98}$$

where ∇^2 is the laplacian on $\mathbb{R}^D$ in cartesian coordinates x^i.

The heat kernel is the fundamental solution that can be represented in operatorial form by

$$\psi(x,y;\beta) = \langle y|e^{-\beta\hat{H}}|x\rangle\ . \tag{99}$$

It satisfies eq. (96) plus the boundary condition $\psi(x,y;0) = \delta^D(x-y)$. It is well-known (see eq. (30)) that the solution in the free case (i.e. for $V=0$) is given by

$$\psi(x,y;\beta) = \frac{1}{(2\pi\beta)^{\frac{D}{2}}}\, e^{-\frac{(x-y)^2}{2\beta}}\ . \tag{100}$$

The path integral which computes the transition amplitude in euclidean time (99) can be written as

$$\psi(x,y;\beta) = \int_{x(0)=x}^{x(\beta)=y} \mathcal{D}x \; e^{-S[x]} \tag{101}$$

where $\int_{x(0)=x}^{x(\beta)=y} \mathcal{D}x$ indicates the sum over all functions $x^i(t)$ which satisfy the boundary conditions $x^i(0) = x^i$ and $x^i(\beta) = y^i$, whereas the euclidean action $S[x]$ is given by

$$S[x] = \int_0^\beta dt \left(\frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + V(x) \right) . \tag{102}$$

No confusion should arise in denoting the paths by $x^i(t)$ and their boundary value at $t = 0$ by $x^i(0) = x^i$. The path integral cannot be computed exactly for an arbitrary potential V, but one can compute it perturbatively for small propagation times β.

To start with let us rescale the euclidean time variable $t = \beta\tau$, so that $\tau \in [0, 1]$. The action (102) can now be written as follows

$$S[x] = \frac{1}{\beta} \int_0^1 d\tau \left(\frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \beta^2 V(x) \right) \tag{103}$$

where of course dots (as in $\dot{x}^i$) now represent derivatives with respect to τ. This rescaling is useful since we are going to compute the path integral in a perturbative expansion valid for small β.

One can decompose all paths by

$$x^i(\tau) = x^i_{bg}(\tau) + q^i(\tau) \tag{104}$$

where $x^i_{bg}(\tau)$ is a fixed path (sometimes called the background path, or the classical path) which can be taken to satisfy the boundary conditions and the classical equations of motion for $V = 0$. The classical path is given by

$$x^i_{bg}(\tau) = x^i + (y^i - x^i)\tau \tag{105}$$

as it satisfies the boundary conditions $x^i_{bg}(0) = x^i$ and $x^i_{bg}(1) = y^i$, as well as the equations of motion $\ddot{x}^i_{bg}(\tau) = 0$ valid when $V = 0$. Indeed for small β the potential can be neglected, as evident from (103). The remaining arbitrary "quantum fluctuations" $q^i(\tau)$ must then have vanishing boundary conditions, $q^i(0) = q^i(1) = 0$.

The inclusion of an arbitrary potential V makes the problem quite difficult to solve in full generality. However it can be treated in perturbation theory. The emerging solution will be of the form

$$\psi(x,y;\beta) = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} \, e^{-\frac{(x-y)^2}{2\beta}} \left(a_0(x,y) + a_1(x,y)\beta + a_2(x,y)\beta^2 + \dots \right) \tag{106}$$

where the so-called Seeley-DeWitt coefficients a_n depend on the points x^i and y^i and on the potential V. Instead of y^i it is useful to use the displacement variable

$$\xi^i = (y^i - x^i) \tag{107}$$

whose length (i.e. the distance between the two points) may be considered of order $\sqrt{\beta}$ for the brownian motion. Thus β controls the perturbative expansion. Of course $a_0(x,y) = 1$, as dictated by the free theory.

5.2.1 Perturbative expansion

The perturbative expansion is based on the gaussian averages of the path integral with the free quadratic action S_0 (i.e. the one with $V = 0$) and in terms of the quantum fluctuations $q^i(\tau)$, namely

$$\begin{aligned} A &= \int \mathcal{D}q\, e^{-S_0[q]} = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} \\ \langle q^i(\tau)\rangle &= \frac{1}{A}\int \mathcal{D}q\, q^i(\tau)\, e^{-S_0[q]} = 0 \\ \langle q^i(\tau)q^j(\sigma)\rangle &= \frac{1}{A}\int \mathcal{D}q\, q^i(\tau)q^j(\sigma)\, e^{-S_0[q]} = -\beta\delta^{ij}\Delta(\tau,\sigma) \end{aligned} \tag{108}$$

and so on. Here $\Delta(\tau,\sigma)$ is the Green function of the operator $\frac{\partial^2}{\partial\tau^2}$ on the space of functions $f(\tau)$ with vanishing boundary conditions at $\tau = 0$ and $\tau = 1$

$$\frac{\partial^2}{\partial\tau^2}\Delta(\tau,\sigma) = \delta(\tau-\sigma)\ . \tag{109}$$

It reads (for τ and σ in $[0,1]$)

$$\begin{aligned} \Delta(\tau,\sigma) &= (\tau-1)\sigma\,\theta(\tau-\sigma) + (\sigma-1)\tau\,\theta(\sigma-\tau) \\ &= \frac{1}{2}|\tau-\sigma| - \frac{1}{2}(\tau+\sigma) + \tau\sigma \end{aligned} \tag{110}$$

where in the first form $\theta(x)$ is the standard step function ($\theta(x) = 1$ for $x > 0$, $\theta(0) = 1/2$ for $x = 0$, and $\theta(x) = 0$ for $x < 0$), while in the second form we used the absolute value. One may check that it satisfies the differential equation with the correct boundary conditions. The two point function $\langle q^i(\tau)q^j(\sigma)\rangle$ is the propagator in the free theory.

Recalling that we indicate the normalized average of an arbitrary functional $F[q]$ by

$$\langle F[q]\,\rangle = \frac{1}{A}\int \mathcal{D}q\, F[q]\, e^{-S_0[q]} \tag{111}$$

we compute the perturbative expansion as follows. The full action is written as

$$S[x] = S_0[x] + S_{int}[x] \tag{112}$$

where the free part is given by

$$S_0[x] = \frac{1}{\beta}\int_0^1 d\tau\, \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j \tag{113}$$

and the interaction part by

$$S_{int}[x] = \beta\int_0^1 d\tau\, V(x)\ . \tag{114}$$

The path integral for the paths $x^i(\tau)$ can now be manipulated as follows

$$\begin{aligned}
&\int_{x(0)=x}^{x(1)=y} \mathcal{D}x\ e^{-S[x]} = \int_{x(0)=x}^{x(1)=y} \mathcal{D}x\ e^{-S_0[x]-S_{int}[x]} \\
&= e^{-S_0[x_{bg}]} \int_{q(0)=0}^{q(1)=0} \mathcal{D}q\ \Big(e^{-S_{int}[x_{bg}+q]}\Big) e^{-S_0[q]} \\
&= A\, e^{-S_0[x_{bg}]} \Big\langle e^{-S_{int}[x_{bg}+q]}) \Big\rangle \\
&= \frac{1}{(2\pi\beta)^{\frac{D}{2}}}\, e^{-\frac{(x-y)^2}{2\beta}} \Big\langle \Big(1 - S_{int}[x_{bg}+q] + \frac{1}{2} S^2_{int}[x_{bg}+q] + \cdots \Big) \Big\rangle \ . \qquad (115)
\end{aligned}$$

The transition from the first to the second line is due to the translation invariance of the path integral measure, $\mathcal{D}x = \mathcal{D}(x_{bg}+q) = \mathcal{D}q$, since $x_{bg}(\tau) = x^i + \xi^i\tau$ is a fixed function. Then we used the notation in (111) to indicate normalized averages with the free path integral. Finally, the perturbative expansion is generated by expanding the interaction part, as shown in the last line.

Let us compute systematically the various terms appearing in the last line of eq. (115). The first one is trivial, $\langle 1\rangle = 1$, since correlations functions are normalized. Next we have to consider $\langle S_{int}[x_{bg}+q]\rangle$. We can Taylor expand the potential around the initial point x^i

$$\begin{aligned}
S_{int}[x_{bg}+q] &= \beta \int_0^1 d\tau\, V(x_{bg}+q) \qquad (116) \\
&= \beta \int_0^1 d\tau \Big(V(x) + [\xi^i\tau + q^i(\tau)]\partial_i V(x) \\
&+ \frac{1}{2}[\xi^i\tau + q^i(\tau)][\xi^j\tau + q^j(\tau)]\, \partial_i\partial_j V(x) + \cdots \Big)
\end{aligned}$$

so that one obtains

$$\begin{aligned}
\langle -S_{int}[x_{bg}+q]\rangle &= -\beta V(x) - \frac{\beta}{2}\xi^i\partial_i V(x) - \frac{\beta}{6}\xi^i\xi^j\partial_i\partial_j V(x) \\
&- \frac{\beta}{2}\partial_i\partial_j V(x) \int_0^1 d\tau\, \langle q^i(\tau) q^j(\tau)\rangle + \cdots \qquad (117)
\end{aligned}$$

where the last term is computed using the free propagator in (108) and (110)

$$\begin{aligned}
\int_0^1 d\tau\, \langle q^i(\tau) q^j(\tau)\rangle &= \int_0^1 d\tau\, (-\beta\delta^{ij}\Delta(\tau,\tau)) \\
&= -\beta\delta^{ij} \int_0^1 d\tau\, \tau(\tau-1) = \frac{\beta}{6}\delta^{ij} \qquad (118)
\end{aligned}$$

to give

$$\begin{aligned}
\langle -S_{int}[x_{bg}+q]\rangle &= -\beta V(x) - \frac{\beta}{2}\xi^i\partial_i V(x) - \frac{\beta}{6}\xi^i\xi^j\partial_i\partial_j V(x) \\
&- \frac{\beta^2}{12}\nabla^2 V(x) + \cdots \ . \qquad (119)
\end{aligned}$$

Similarly, at lowest order in β one gets for the next term in (115)

$$\Big\langle \frac{1}{2} S^2_{int}[x_{bg}+q] \Big\rangle = \frac{\beta^2}{2} V^2(x) + \cdots \ . \qquad (120)$$

Collecting all the terms, one finds that at this order the heat kernel is given by

$$\begin{aligned}\psi(x,y;\beta) &= \frac{1}{(2\pi\beta)^{\frac{D}{2}}}\, e^{-\frac{(x-y)^2}{2\beta}}\Big[1-\beta V(x)-\frac{\beta}{2}\xi^i\partial_i V(x)-\frac{\beta}{6}\xi^i\xi^j\partial_i\partial_j V(x)\\ &- \frac{\beta^2}{12}\nabla^2 V(x)+\frac{\beta^2}{2}V^2(x)+\cdots\Big] \end{aligned} \tag{121}$$

from which one reads off the expansion around the point x of the Seeley-DeWitt coefficients a_0, a_1 and a_2

$$\begin{aligned} a_0(x,y) &= 1\\ a_1(x,y) &= -V(x)-\frac{1}{2}\xi^i\partial_i V(x)-\frac{1}{6}\xi^i\xi^i\partial_j\partial_j V(x)+\cdots\\ a_2(x,y) &= \frac{1}{2}V^2(x)-\frac{1}{12}\nabla^2 V(x)+\cdots\ . \end{aligned} \tag{122}$$

In particular, their values at coinciding points $y^\mu = x^\mu$ (i.e. for $\xi^\mu = 0$) are given exactly by

$$\begin{aligned} a_0(x,x) &= 1\\ a_1(x,x) &= -V(x)\\ a_2(x,x) &= \frac{1}{2}V^2(x)-\frac{1}{12}\nabla^2 V(x)\ . \end{aligned} \tag{123}$$

With this calculation we have exemplified the use of the path integrals to compute heat kernels as well as perturbation theory.

6 Gaussian formulae

We collect here some gaussian formulae used in the previous examples plus some additional ones to be used later in the book. We use the hypercondensed notation for quantum mechanics and quantum field theory in euclidean time. Setting $\mathcal{D}\phi = \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}}$ one also recovers the finite dimensional case.

The required formulae arise from the following gaussian integrals

$$\begin{aligned} Z &\equiv \int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi^i K_{ij}\phi^j} = (\det K_{ij})^{-\frac{1}{2}}\\ Z[J] &\equiv \int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi^i K_{ij}\phi^j + J_i\phi^i} = (\det K_{ij})^{-\frac{1}{2}}\, e^{\frac{1}{2}J_i G^{ij} J_j} \end{aligned} \tag{124}$$

where G^{ij} is the inverse of the kinetic matrix K_{ij} (i.e. $K_{ij}G^{jk} = \delta_i^k$) and gives the propagator. It corresponds to a Green function when K_{ij} is a differential operator. The first one is the standard gaussian integral. The second one is obtained by "completing the square" and shifting integration variables.

The normalized correlation functions follow from differentiating $Z[J]$

$$\begin{aligned} \langle\phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}\rangle &= \frac{1}{Z}\int D\phi\ \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n} e^{-\frac{1}{2}\phi^i K_{ij}\phi^j}\\ &= \frac{1}{Z}\frac{\delta^n Z[J]}{\delta J_{i_1}\delta J_{i_2}\cdots\delta J_{i_n}}\Big|_{J=0}\\ &= \frac{\delta^n\, e^{\frac{1}{2}J_i G^{ij}J_j}}{\delta J_{i_1}\delta J_{i_2}\cdots\delta J_{i_n}}\Big|_{J=0} \end{aligned} \tag{125}$$

obtaining in particular

$$\begin{aligned}
&\langle 1\rangle = 1\\
&\langle \phi^i\rangle = 0\\
&\langle \phi^i\phi^j\rangle = G^{ij}\\
&\langle \phi^i\phi^j\phi^k\rangle = 0\\
&\langle \phi^i\phi^j\phi^k\phi^l\rangle = G^{ij}G^{kl} + G^{ik}G^{jl} + G^{il}G^{jk}
\end{aligned} \tag{126}$$

and so on. In particular, correlation functions of an odd number of variables ϕ^i vanish. Those with an even number of variables are given by sums of products of two-point functions. The two-point function is also known as the Feynman propagator.

In the second part of the book we will need more general correlation functions containing the insertion of an exponential $e^{ip_i\phi^i}$. Denoting by $Z[J,p]$ the following functional

$$Z[J,p] = \int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi^i K_{ij}\phi^j + (J_i+ip_i)\phi^i} = (\det K_{ij})^{-\frac{1}{2}}\, e^{\frac{1}{2}(J_i+ip_i)G^{ij}(J_j+ip_j)} \tag{127}$$

with $Z[0,0] = Z$, we find again that the desired correlation functions are obtained by differentiating with respect to the source J and then setting it to zero

$$\begin{aligned}
\langle \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}e^{ip_k\phi^k}\rangle &= \frac{1}{Z}\int D\phi\ \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}e^{ip_k\phi^k}\, e^{-\frac{1}{2}\phi^i K_{ij}\phi^j}\\
&= \frac{1}{Z}\frac{\delta^n Z[J,p]}{\delta J_{i_1}\delta J_{i_2}\cdots\delta J_{i_n}}\Big|_{J=0}\\
&= \frac{\delta^n\, e^{\frac{1}{2}(J_m+ip_m)G^{mn}(J_n+ip_n)}}{\delta J_{i_1}\delta J_{i_2}\cdots\delta J_{i_n}}\Big|_{J=0}
\end{aligned} \tag{128}$$

obtaining in particular

$$\begin{aligned}
&\langle e^{ip_k\phi^k}\rangle = e^{-\frac{1}{2}p_m G^{mn}p_n}\\
&\langle \phi^i e^{ip_k\phi^k}\rangle = iG^{ij}p_j\, e^{-\frac{1}{2}p_m G^{mn}p_n}\\
&\langle \phi^i\phi^j e^{ip_k\phi^k}\rangle = (G^{ij} - p_k G^{ki}p_l G^{lj})\, e^{-\frac{1}{2}p_m G^{mn}p_n}
\end{aligned} \tag{129}$$

and so on. The exponential $e^{ip_k\phi^k}$ corresponds to "vertex operators" appearing in worldline applications.

These identities were originally derived by Gian-Carlo Wick for QFT's using the operatorial formalism. The mnemonics to write them down, without computing all the times the derivatives with respect to the source is as follows:

(i) In the absence of the exponential (i.e. with $p_i = 0$), one connects any two quantum variables in the correlation function in all possible ways. Each times a pair of variables is connected (this is called a Wick-contraction, or simply contraction) the pair is substituted by the propagator G^{ij}. As an example, in the 4-point function in (126) one contracts the first variable with any other (it can be done in three ways) and substitutes this pair with the propagator, which is a function that can be brought outside the correlation function. The remaining last two variables necessarily gets paired together and substituted again by the propagator. Thus we get a sum of 3!! terms containing the products of 2 free propagators. Similarly, the correlation function of $2m$ variables is given by the sum of $(2m-1)!!$ terms containing each the products of m free propagators.

(ii) In the presence of the exponential, one again connects any quantum variable not in the exponent with any other variable in all possible ways, remembering that in addition it can be Wick-contracted with the exponential as well. In the latter case it acts as a derivation: it regenerates the exponential (that remains in the correlation function and participates to further contractions) multiplied by the contraction of the field with the exponent. The latter produces a propagator times the factor that multiplies the quantum variable in the exponent. One continues this procedure iteratively with all the variables left which are not in the exponent. Eventually, there is only the exponential left, that gives rise to the first line in (129). Again, an example can clarify this recipe. Let us consider the 2-point function in (129). The first term in the bracket is due to the Wick-contraction of the two variables not in the exponent together, the second one corresponds to the Wick-contraction of each variable with the exponential.

Finally, let us collect the formulae defining the various generating functionals in euclidean conventions, in the hypercondensed notation. The generating functional for connected correlation functions $W[J]$ is obtained from the path integral with sources by

$$Z[J] = e^{W[J]} = \int D\phi \, e^{-S[\phi]+J_i\phi^i} \tag{130}$$

so that

$$W[J] = \ln Z[J] \ . \tag{131}$$

The effective action is defined by a Legendre transform as

$$\Gamma[\phi] = \min_J \Big\{ J_i\phi^i - W[J] \Big\} \ . \tag{132}$$

The functionals $S[\phi]$, $Z[J]$, $W[J]$, $\Gamma[\phi]$ are very useful, especially in quantum field theory. For a free theory (without gauge invariances) one has

$$\begin{aligned} S[\phi] &= \frac{1}{2}\phi^i K_{ij}\phi^j \\ Z[J] &= (\det K)^{-\frac{1}{2}} e^{\frac{1}{2}J_i G^{ij} J_j} \\ W[J] &= \frac{1}{2}J_i G^{ij} J_j - \frac{1}{2}\mathrm{tr}\, \ln K \\ \Gamma[\phi] &= \frac{1}{2}\phi^i K_{ij}\phi^j + \frac{1}{2}\mathrm{tr}\, \ln K \ . \end{aligned} \tag{133}$$

The constant $\mathrm{tr}\, \ln K$ generically depends on the coupling constants of the theory and on possible background fields. When computing one loop effective actions for QFT in the presence of external fields, without a source for the quantum variable itself, this constant identifies directly the (one-loop) effective action. Indeed, setting $\phi^i = 0$ in there last formula, one gets

$$\Gamma \equiv \Gamma[0] = \frac{1}{2}\mathrm{tr}\, \ln K \ . \tag{134}$$

Using the identity

$$\ln \frac{a}{b} = -\int_0^\infty \frac{dT}{T} \left(e^{-aT} - e^{-bT} \right) \ , \tag{135}$$

proven by verifying that both sides identify the same function (both sides have the same derivative in a, which means that at most they can differ by a constant; the latter vanishes as both

sides have the same value for $a = b$), and extending it formally to operators, one finds the representation of the effective action in terms of the Fock-Schwinger proper time T

$$\Gamma = \frac{1}{2}\mathrm{tr}\,\ln K = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\,\mathrm{tr}\,\left(e^{-KT} - e^{-T}\right)\ . \tag{136}$$

Dropping the last (infinite) constant, that is taken care of by the QFT renormalization procedure, one finds the famous Schwinger formula

$$\Gamma = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\,\mathrm{tr}\,e^{-KT}\ . \tag{137}$$

In subsequent parts of the book, this one-loop QFT effective action is reinterpreted and analyzed in terms of worldline path integrals.