7.12.2021

Path integrals in quantum mechanics

(Lecture notes - a.a. 2021/22)

Fiorenzo Bastianelli

Quantum mechanics can be formulated in two equivalent ways: (i) canonical quantization, also known as operatorial quantization, which is based on linear operators acting on a Hilbert space of physical states, (ii) path integrals, based on integration over a space of functions. The former was the first one to be developed, through the work of Heisenberg, Schrödinger, Dirac and others. The latter was introduced later on by Feynman, who extended previous suggestions by Dirac. Nowadays it is useful to know both formulations, as depending on the problem at hand, one may find technical advantages in using one over the other. In worldline approaches one often uses the operatorial formulation to define the problem, and path integrals to calculate the answer.

The operatorial formulation of quantum mechanics is the one usually presented in introductory courses on quantum mechanics. Path integrals are introduced later on, when approaching the problem of quantizing gauge fields. Indeed path integrals have become quite popular since the advent of gauge theories, because the quantization of the latter is much more intuitive and transparent in that context.

In these notes we introduce path integrals for the quantization of point particles, as opposed to the quantization of field theories. The former contains a finite number of degrees of freedom, the latter deals with an infinite number of degrees of freedom. We assume only elementary notions of quantum mechanics in its operatorial form, and start developing path integrals from the beginning. In particular, we discuss path integrals for a non relativistic point particle, which contains already the essence of path integrals, and then extend them to fermionic systems.

1 Canonical quantization

Canonical quantization is constructed starting from the hamiltonian formulation of a classical system. It is obtained by lifting its phase space coordinates, the generalized coordinates x^i and their conjugate momenta p_i, to linear operators $\hat{x}^i$ and $\hat{p}_i$ that act on a linear space endowed with a positive definite norm, the Hilbert space of physical states $\mathcal{H}$. The basic operators must satisfy commutation relations required to be equal $i\hbar$ times the value of the corresponding classical Poisson brackets

$$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j \ , \quad [\hat{x}^i, \hat{x}^j] = 0 \ , \quad [\hat{p}_i, \hat{p}_j] = 0 \ . \tag{1}$$

All classical observables $A(x,p)$, which are functions on phase space, become linear operators $\hat{A}(\hat{x},\hat{p})$ acting on the Hilbert space $\mathcal{H}$. The most important example is given by the hamiltonian function $H(x,p)$, which upon quantization becomes the hamiltonian operator $\hat{H}(\hat{x},\hat{p})$. The latter generates the time evolution of any state $|\psi\rangle \in \mathcal{H}$ through the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \ . \tag{2}$$

The corresponding solution is a time dependent state $|\psi(t)\rangle$ that describes the evolution of the quantum system. This set up is known as the Schrödinger picture of quantum mechanics. It is a formal quantization procedure that becomes operative once one finds an irreducible unitary representation of the operator algebra in eq. (1). A mathematical result, known as the Stone–von Neumann theorem, states that in quantum mechanics all irreducible representations of (1) are unitarily equivalent, so that there is a unique procedure of quantizing a classical system[1]. Historically, this theorem made it clear that the Schrödinger formulation of quantum mechanics was equivalent to the one proposed by Heisenberg with its matrix mechanics (known as the Heisenberg picture).

Let us consider, more specifically, the motion of a non relativistic particle in one dimension in the presence of an external potential $V(x)$. The classical dynamics is fixed by the action

$$S[x] = \int dt \left(\frac{m}{2}\dot{x}^2 - V(x)\right) \ . \tag{3}$$

The quantum theory is recognized by first developing the hamiltonian formulation, defined on phase space with the Poisson bracket structure and phase space action

$$S[x,p] = \int dt \left(p\dot{x} - H(x,p)\right) \ , \qquad H(x,p) = \frac{p^2}{2m} + V(x) \ . \tag{4}$$

Then, one recognizes that the quantum theory has fundamental operators $\hat{x}$ and $\hat{p}$, linear operators on the Hilbert space $\mathcal{H}$ of quantum states, with commutation relations

$$[\hat{x},\hat{p}] = i\hbar \ , \quad [\hat{x},\hat{x}] = 0 \ , \quad [\hat{p},\hat{p}] = 0 \ . \tag{5}$$

The quantum hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ is an operator on the Hilbert space and generates the time evolution through the Schrödinger equation (2).

Using the coordinate representation, obtained by considering the eigenstates $|x\rangle$ of the position operator $\hat{x}$, that satisfy $\hat{x}|x\rangle = x|x\rangle$ with x a real number, and projecting the various states of the Hilbert space onto them to identify the wave functions, one finds the familiar way of realizing quantum mechanics as wave mechanics

$$\begin{array}{lcll} |\psi\rangle & \to & \psi(x) & \Big(\psi(x) = \langle x|\psi\rangle\Big) \\ \hat{x} & \to & x & \Big(\langle x|\hat{x}|x'\rangle = x\langle x|x'\rangle = x\delta(x-x')\Big) \\ \hat{p} & \to & -i\hbar\dfrac{\partial}{\partial x} & \Big(\langle x|\hat{p}|x'\rangle = -i\hbar\dfrac{\partial}{\partial x}\langle x|x'\rangle = -i\hbar\dfrac{\partial}{\partial x}\delta(x-x')\Big) \\ \hat{H} & \to & -\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2} + V(x) & \Big(\langle x|\hat{H}|x'\rangle = \Big(-\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2} + V(x)\Big)\delta(x-x')\Big) \end{array} \tag{6}$$

with the Schrödinger equation taking the familiar form

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \ . \tag{7}$$

Returning to the Dirac bra and ket notation, let us consider the solution of the Schrödinger equation. Given a ket $|\psi_i\rangle$ that describes the system at initial time t_i, the solution of the Schrödinger equation for time independent hamiltonians can be written as

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t_i)}|\psi_i\rangle \tag{8}$$

[1]Up to the problem of resolving ordering ambiguities, often present when one tries to relate the classical hamiltonian $H(x,p)$ to its quantum counterpart $\hat{H}(\hat{x},\hat{p})$.

which indeed satisfies the equation and the boundary condition $|\psi(t_i)\rangle = |\psi_i\rangle$. The amplitude to find the system at time t_f in state $|\psi_f\rangle$ is obtained by projecting the solution evaluated at time t_f onto the state $|\psi_f\rangle$

$$\langle\psi_f|\psi(t_f)\rangle = \langle\psi_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|\psi_i\rangle \ . \tag{9}$$

This amplitude is called "transition amplitude". Thus, we see that amplitudes correspond to matrix element of $e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}$, the unitary operator that evolves in time the states of the system. In the following sections we shall find a path integral representation for such amplitudes.

2 Path integrals in phase space

To derive a path integral expression for the transition amplitudes, we start by inserting twice the identity operator $\mathbb{1}$, expressed using the eigenstates of the position operator

$$\mathbb{1} = \int dx\, |x\rangle\langle x| \qquad\qquad \text{with } \langle x|x'\rangle = \delta(x-x')\,, \tag{10}$$

and rewrite (9) as

$$\begin{aligned}\langle\psi_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|\psi_i\rangle &= \langle\psi_f|\,\mathbb{1}\,e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}\,\mathbb{1}\,|\psi_i\rangle \\ &= \int dx_f \int dx_i\, \psi_f^*(x_f)\,\langle x_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|x_i\rangle\,\psi_i(x_i)\end{aligned} \tag{11}$$

where $\psi_i(x_i) = \langle x|\psi_i\rangle$ and $\psi_f(x_f) = \langle x|\psi_f\rangle$ are the wave functions for the initial and final states. This rewriting shows that it is enough to consider the matrix element of the evolution operator between position eigenstates

$$A(x_i, x_f; T) = \langle x_f|e^{-\frac{i}{\hbar}\hat{H}T}|x_i\rangle \tag{12}$$

where $T = (t_f - t_i)$ is the total propagation time. It satisfies the Schrödinger equation

$$i\hbar\frac{\partial}{\partial T}A(x_i, x_f; T) = \hat{H}(x_f, p_f = -i\hbar\partial_{x_f})A(x_i, x_f; T) \tag{13}$$

with initial conditions $A(x_i, x_f; 0) = \delta(x_f - x_i)$.

We are going to consider quantum hamiltonians of the form

$$\hat{H}(\hat{x}, \hat{p}) = \frac{1}{2m}\hat{p}^2 + \hat{V}(\hat{x}) \tag{14}$$

where V is a generic scalar potential. The derivation of the path integral proceeds now as follows. One splits the transition amplitude $A(x_i, x_f; T)$ as the product of N factors, and inserts the completeness relation (10) $N-1$ times in between the factors

$$\begin{aligned} A &= \langle x_f|e^{-\frac{i}{\hbar}\hat{H}T}|x_i\rangle = \langle x_f|\Big(e^{-\frac{i}{\hbar}\hat{H}\frac{T}{N}}\Big)^N|x_i\rangle = \langle x_f|\underbrace{e^{-\frac{i\epsilon}{\hbar}\hat{H}}e^{-\frac{i\epsilon}{\hbar}\hat{H}}\cdots e^{-\frac{i\epsilon}{\hbar}\hat{H}}}_{N\text{ times}}|x_i\rangle \\ &= \langle x_f|e^{-\frac{i\epsilon}{\hbar}\hat{H}}\,\mathbb{1}\,e^{-\frac{i\epsilon}{\hbar}\hat{H}}\,\mathbb{1}\,\cdots\,\mathbb{1}\,e^{-\frac{i\epsilon}{\hbar}\hat{H}}|x_i\rangle = \int\Big(\prod_{k=1}^{N-1}dx_k\Big)\prod_{k=1}^{N}\langle x_k|e^{-\frac{i\epsilon}{\hbar}\hat{H}}|x_{k-1}\rangle \end{aligned} \tag{15}$$

where for convenience we have denoted $x_0 \equiv x_i$, $x_N \equiv x_f$, and $\epsilon \equiv \frac{T}{N}$. To evaluate this expression better, it is convenient to use the resolution of the identity N more times, now expressed in terms of the momentum eigenstates

$$\mathbb{1} = \int \frac{dp}{2\pi\hbar} |p\rangle\langle p| \qquad \text{with } \langle p|p'\rangle = 2\pi\hbar\,\delta(p-p') \tag{16}$$

to obtain

$$\begin{aligned} A &= \int \Big(\prod_{k=1}^{N-1} dx_k\Big) \prod_{k=1}^{N} \langle x_k| e^{-\frac{i\epsilon}{\hbar}\hat{H}} |x_{k-1}\rangle = \int \Big(\prod_{k=1}^{N-1} dx_k\Big) \prod_{k=1}^{N} \langle x_k|\, \mathbb{1}\, e^{-\frac{i\epsilon}{\hbar}\hat{H}} |x_{k-1}\rangle \\ &= \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}\Big) \prod_{k=1}^{N} \langle x_k|p_k\rangle\langle p_k| e^{-\frac{i\epsilon}{\hbar}\hat{H}} |x_{k-1}\rangle \ . \end{aligned} \tag{17}$$

This is an exact expression. Note that there is one more integration over momenta than integrations over coordinates, consequence of choosing coordinate eigenstates as initial and final states in the transition amplitude. Now one can manipulate this expression further by making approximations that are valid in the limit $N \to \infty$ (i.e. $\epsilon \to 0$). The crucial point is the evaluation of the following matrix element

$$\begin{aligned} \langle p| e^{-\frac{i\epsilon}{\hbar}\hat{H}(\hat{x},\hat{p})} |x\rangle &= \langle p|\Big(\mathbb{1} - \frac{i\epsilon}{\hbar}\hat{H}(\hat{x},\hat{p}) + \cdots\Big)|x\rangle \\ &= \langle p|x\rangle - \frac{i\epsilon}{\hbar}\langle p|\hat{H}(\hat{x},\hat{p})|x\rangle + \cdots \\ &= \langle p|x\rangle \left(1 - \frac{i\epsilon}{\hbar} H(x,p) + \cdots\right) \\ &= \langle p|x\rangle\, e^{-\frac{i\epsilon}{\hbar}H(x,p)+\cdots} \ . \end{aligned} \tag{18}$$

These approximations are all valid in the limit of small ϵ. The substitution $\langle p|\hat{H}(\hat{x},\hat{p})|x\rangle = \langle p|x\rangle\, H(x,p)$ follows from the simple structure of the hamiltonian (14), that allows one to act with the momentum operator on the left, and with the position operator on the right, to have the operators replaced by the corresponding eigenvalues. Notice that there is no need of commuting operators inside the hamiltonian, because of the simplicity of the hamiltonian we considered. The final result is that all operators are simply replaced by eigenvalues. This way the quantum hamiltonian $\hat{H}(\hat{x},\hat{p})$ gets replaced by the classical function $H(x,p) = \frac{p^2}{2m} + V(x)$. There exists a mathematically rigorous proof that these manipulations are correct for a wide class of physically interesting potentials $V(x)$ (the "Trotter formula"). We shall not review these subtleties, as the physically intuitive derivation given above is enough for our purposes.

Using now eq. (18), and remembering that the wave functions of the momentum eigenstates (the plane waves) are normalized as

$$\langle x|p\rangle = e^{\frac{i}{\hbar}px} \ , \quad \langle p|x\rangle = \langle x|p\rangle^* = e^{-\frac{i}{\hbar}px} \ , \tag{19}$$

that follows from the normalization chosen in (10) and (16), one obtains

$$\langle x_k|p_k\rangle\langle p_k| e^{-\frac{i\epsilon}{\hbar}\hat{H}} |x_{k-1}\rangle = e^{\frac{i}{\hbar}p_k(x_k - x_{k-1}) - \frac{i\epsilon}{\hbar}H(x_{k-1},p_k)} \tag{20}$$

up to terms that vanish for $\epsilon \to 0$. This expression can now be inserted in (17). At this stage the transition amplitude does not contain any more operators, bra and kets. It contains just

integrations, though a big number of them, of ordinary functions

$$\begin{aligned} A &= \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}\Big) e^{\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\left[p_k\frac{(x_k-x_{k-1})}{\epsilon}-H(x_{k-1},p_k)\right]} \\ &= \int DxDp\; e^{\frac{i}{\hbar}S[x,p]} \ . \end{aligned} \tag{21}$$

This is the path integral in phase space. One recognizes in the exponent a discretization of the classical phase space action

$$S[x,p] = \int_{t_i}^{t_f} dt\,\Big(p\dot{x} - H(x,p)\Big) \quad \to \quad \sum_{k=1}^{N}\epsilon\Big(p_k\frac{(x_k-x_{k-1})}{\epsilon} - H(x_{k-1},p_k)\Big) \tag{22}$$

where $t_f - t_i = T = N\epsilon$ is the total propagation time, with the paths in phase space discretized as

$$x(t)\ ,\ p(t) \quad \to \quad x_k = x(t_i + k\epsilon)\ ,\ p_k = p(t_i + k\epsilon)\ . \tag{23}$$

The last way of writing the amplitude in (21) is symbolic but suggestive: it indicates the sum over all paths in phase space weighted by the exponential of $\frac{i}{\hbar}$ times the classical action. It depends implicitely on the boundary conditions assigned to the paths $x(t)$.

3 Path integrals in configuration space

The path integral in configurations space is easily derived by integrating over the momenta. The dependence on momenta in the exponent of (21) is at most quadratic and can be eliminated by gaussian integration

$$\int_{-\infty}^{\infty} dp\, e^{-\frac{\alpha}{2}p^2} = \sqrt{\frac{2\pi}{\alpha}} \tag{24}$$

which is valid for $\alpha > 0$, but then extended analytically to include complex values of α (see section 4.1 for details). In particular, we consider gaussian integrals of the type

$$\int_{-\infty}^{\infty} dp\, e^{-\frac{\alpha}{2}p^2+\beta p} = \sqrt{\frac{2\pi}{\alpha}}\, e^{\frac{\beta^2}{2\alpha}} \tag{25}$$

obtained by square completion. Note that the final exponential is the original exponential inside the integral with argument evaluated at the minimum in p.

Returning to the path integral, and considering the hamiltonian $H(x,p) = \frac{p^2}{2m} + V(x)$, one completes the squares[2] and performs the gaussian integrations over the momenta

$$\begin{aligned} A &= \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\frac{m}{2\pi i\hbar\epsilon}\Big)^{\frac{N}{2}} e^{\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\left[\frac{m}{2}\frac{(x_k-x_{k-1})^2}{\epsilon^2}-V(x_{k-1})\right]} \\ &= \int Dx\, e^{\frac{i}{\hbar}S[x]} \ . \end{aligned} \tag{26}$$

[2]First, rewrite $p_k\frac{(x_k-x_{k-1})}{\epsilon} - \frac{1}{2m}p_k^2 = -\frac{1}{2m}(p_k - m\frac{(x_k-x_{k-1})}{\epsilon})^2 + \frac{m}{2}\frac{(x_k-x_{k-1})^2}{\epsilon^2}$. Then, change integration variables $p_k \to \tilde{p}_k = p_k - m\frac{(x_k-x_{k-1})}{\epsilon}$. The measure is invariant under a translation, and it produces the term $-\frac{\tilde{p}_k^2}{2m} + \frac{m}{2}\frac{(x_k-x_{k-1})^2}{\epsilon^2}$ in the exponent.

This is the path integral in configuration space. It contains in the exponent the configuration space action suitably discretized

$$S[x] \;=\; \int_{t_i}^{t_f} dt\,\Big(\frac{m}{2}\dot{x}^2 - V(x)\Big) \;\to\; \sum_{k=1}^{N} \epsilon\Big[\frac{m}{2}\Big(\frac{x_k - x_{k-1}}{\epsilon}\Big)^2 - V(x_{k-1})\Big]\,. \tag{27}$$

Again, the last way of writing the path integral in (26) is symbolic, and indicates the formal sum over paths in configuration space, weighted by the exponential of $\frac{i}{\hbar}$ times the classical action. The space of paths is given by the space of functions $x(t)$ with boundary values $x(t_i) = x_i$ and $x(t_f) = x_f$. It is an infinite dimensional space. How to perform concretely the path integral over this functional space is defined precisely by the discretization, that approximates a function $x(t)$ by its $N+1$ values $x_k = x(t_i + k\epsilon)$ at $k = 0, 1, 2, ..., N$, as shown in fig. 1.

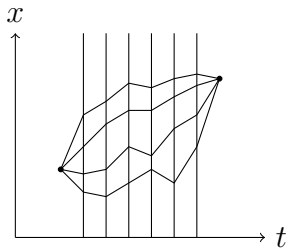


Figure 1: The discretized path integral in configuration space.

Thus, we have found a path integral that computes quantum mechanical amplitudes

$$A = \int Dx(t)\, e^{\frac{i}{\hbar}S[x(t)]} \tag{28}$$

with all paths contributing, as in fig. 2.

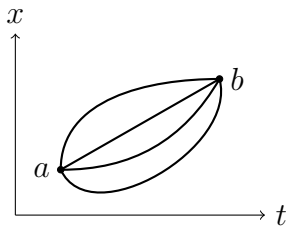


Figure 2: All paths $x(t)$ contribute to the path integral.

3.1 Free particle

For a free particle ($V(x) = 0$) one may use repeatedly gaussian integration and calculate from eq. (26) the exact transition amplitude

$$A(x_i, x_f; T) = \sqrt{\frac{m}{2\pi i\hbar T}}\, e^{\frac{i}{\hbar}\frac{m(x_f - x_i)^2}{2T}}\,. \tag{29}$$

It satisfies the free Schrödinger equation

$$i\hbar\frac{\partial}{\partial T}A(x_i,x_f;T) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_f^2}A(x_i,x_f;T) \tag{30}$$

with initial conditions

$$A(x_i,x_f;0) = \delta(x_f - x_i)\ . \tag{31}$$

The result is very suggestive: up to a prefactor it is given by the exponential of $\frac{i}{\hbar}$ times the classical action evaluated on the classical path, i.e. the path that satisfies the classical equations of motion. This is typical for those cases in which the semiclassical approximation is actually exact. One may interpret the prefactor as due to quantum ("one-loop") corrections to the classical ("tree-level") result. The free particle case is also quite special: the exact final result is valid for any N, and there is no need to take the limit $N \to \infty$. The case $N = 1$, which carries no integration at all, is already exact.

A formal but useful way of calculating gaussian path integrals is achieved by working directly in the continuum limit. One does not consider the precise definition of the path integral measure, but uses only its formal properties, in particular its translational invariance. The calculation is formal in the sense that one assumes properties of the path integral measure (that eventually must be proven by an explicit regularization and construction, as the one given earlier). The calculation goes as follows. The action is $S[x] = \int_0^T dt\,\frac{m}{2}\dot{x}^2$, and the classical equations of motion with the boundary conditions are solved by

$$x_{cl}(t) = x_i + (x_f - x_i)\frac{t}{T}\ . \tag{32}$$

One can represent a generic path $x(t)$ as the classical path $x_{cl}(t)$ plus quantum fluctuations $q(t)$

$$x(t) = x_{cl}(t) + q(t) \tag{33}$$

where the fluctuations $q(t)$ must vanish at $t = 0$ and $t = T$ to preserve the boundary conditions. One may interpret $x_{cl}(t)$ as the origin in the *space of functions*. Then, one computes the path integral as follows

$$\begin{aligned} A(x_i,x_f;T) &= \int Dx\, e^{\frac{i}{\hbar}S[x]} = \int D(x_{cl}+q)\, e^{\frac{i}{\hbar}S[x_{cl}+q]} \\ &= \int Dq\, e^{\frac{i}{\hbar}(S[x_{cl}]+S[q])} = e^{\frac{i}{\hbar}S[x_{cl}]}\int Dq\, e^{\frac{i}{\hbar}S[q]} \\ &= Ne^{\frac{i}{\hbar}S[x_{cl}]} = Ne^{\frac{i}{\hbar}\frac{m(x_f-x_i)^2}{2T}} \end{aligned} \tag{34}$$

where translational invariance of the path integral measure has been used in the form $Dx = D(x_{cl}+q) = Dq$. There is no linear term in $q(t)$ in the action because the function $x_{cl}(t)$ solves the classical equations of motion: for quadratic actions one has $S[x_{cl}+q] = S[x_{cl}] + S[q]$. The normalization factor $N = \int Dq\, e^{\frac{i}{\hbar}S[q]}$ is undetermined by this method, but it is a constant that does not depend on x_i and x_f. Very often its precise value is not needed, but one can fix it by requiring that the final result satisfies the Schrödinger equation, finding $N = \sqrt{\frac{m}{2\pi i\hbar T}}$.

3.2 Euclidean time and statistical mechanics

Quantum mechanics can be related to statistical mechanics by an analytic continuation. We introduce this relation by considering the free particle just described.

Continuing analytically the time parameter to purely imaginary values by $T \to -i\beta$ with real β, and setting $\hbar = 1$, the free Schrödinger equation (30) turns into the heat equation

$$\frac{\partial}{\partial \beta} A = \frac{1}{2m} \frac{\partial^2}{\partial x_f^2} A \, . \tag{35}$$

Its fundamental solution, i.e. the solution with boundary condition $A \overset{\beta \to 0}{\longrightarrow} \delta(x_f - x_i)$, is

$$A = \sqrt{\frac{m}{2\pi\beta}} \, e^{-\frac{m(x_f - x_i)^2}{2\beta}} \, , \tag{36}$$

and can be obtained from (29) by the same analytic continuation. This continuation is called "Wick rotation", see fig. 3.

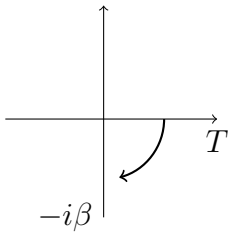


Figure 3: Wick rotation to euclidean times.

The Wick rotation can be performed directly on the path integral to obtain euclidean path integrals. Analytically continuing the time variable as $t \to -i\tau$, one finds that the action with "minkowskian" time (i.e. with a real time t) turns into an "euclidean" action S_E defined by

$$iS[x] = i \int_0^T dt \, \frac{m}{2} \dot{x}^2 \quad \to \quad -S_E[x] = - \int_0^\beta d\tau \, \frac{m}{2} \dot{x}^2 \tag{37}$$

where in the euclidean action one defines $\dot{x} = \frac{dx}{d\tau}$, with τ usually called "euclidean time". The euclidean action thus defined is positive definite. It appears in the path integral that, after the Wick rotation, takes the form

$$\int Dx \, e^{-S_E[x]} \, . \tag{38}$$

For a free theory it is truly gaussian, with exponential damping rather than with increasingly rapid phase oscillations. In this form it coincides with the functional integral introduced by Wiener in the 1920's to study brownian motion and the heat equation.

Such euclidean path integrals are quite useful in statistical mechanics, where β is related to the inverse temperature Θ by $\beta = \frac{1}{k\Theta}$, where k is the Boltzmann's constant. To understand this, let us consider the trace of the evolution operator $e^{-\frac{i}{\hbar}\hat{H}T}$. It can be written using energy eigenstates (labeled by n if the spectrum is discrete), or equivalently using position eigenstates (labeled by x), as

$$Z \equiv \mathrm{Tr}\, e^{-\frac{i}{\hbar}\hat{H}T} = \sum_n e^{-\frac{i}{\hbar}E_n T} = \int dx \, \langle x | e^{-\frac{i}{\hbar}\hat{H}T} | x \rangle \, . \tag{39}$$

It can be Wick rotated $Z \to Z_E$ (with $T \to -i\beta$) to obtain the statistical partition function Z_E of the quantum system with hamiltonian $\hat{H}$. Setting $\hbar = 1$, it reads

$$Z_E \equiv \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_n e^{-\beta E_n} = \int dx\, \langle x | e^{-\beta \hat{H}} | x \rangle \ . \tag{40}$$

At this stage it is immediate to find a path integral representation of the statistical partition function: one performs a Wick rotation of the path integral action, sets the initial state (at euclidean time $\tau = 0$) equal to the final state (at euclidean time $\tau = \beta$), and sums over all possible states, as indicated in (40). The paths become closed, $x(0) = x(\beta)$, and the partition function becomes

$$Z_E = \mathrm{Tr}\, e^{-\beta \hat{H}} = \int_{PBC} Dx\, e^{-S_E[x]} \tag{41}$$

where PBC stands for "periodic boundary conditions", indicating the sum over all paths that close onto themselves in an euclidean time β.

Introduced here for the free theory, the Wick rotation is supposed to be of more general value, relating quantum mechanics to statistical mechanics in the interacting case as well. Even if one is interested in the theory with a real time, nowadays one often works in the euclidean version of the theory, where factors of the imaginary unit i are absent, and path integral convergence is more easily kept under control. Only at the very end one performs the inverse Wick rotation to read off the result for the minkowskian theory.

The Wick rotation procedure is better appreciated by considering the usual time as corresponding to the real line of a complex plane: denoting the complex time by $t_\theta = te^{-i\theta}$, the usual real time appears at $\theta = 0$, while the euclidean time τ appears at $\theta = \frac{\pi}{2}$ as $t_{\frac{\pi}{2}} = -i\tau$. The analytical continuation of all physical quantities is achieved by continually increasing θ form 0 to $\frac{\pi}{2}$, a clockwise rotation of the real axis into the imaginary one. The generalized partition function $Z_\theta \equiv \mathrm{Tr}\, e^{-\frac{i}{\hbar}\hat{H}t_\theta}$ with a complex time $t_\theta = te^{-i\theta}$ with positive t has a damping factor for all $0 < \theta \leq \frac{\pi}{2}$ and for all hamiltonians that are bounded from below (up to an overall factor due to the value of the ground state energy, if that happens to be negative).

Similar considerations can be made for path integrals in minkowskian and euclidean times with other boundary conditions. Path integrals in euclidean times are mathematically better defined (one may develop a mathematically well-defined measure theory on the space of functions), at least for quadratic actions and perturbations thereof. Path integral with a minkowskian time are more delicate, and physicists usually use the argument of rapid phase oscillations to deduce that unwanted terms vanish. The Wick rotation suggests a way of defining the path integral in real time starting from the euclidean time one. These points of mathematical rigor are not needed for the applications that we are going to consider, and the derivation of path integrals described previously is enough for our purposes.

3.3 Comments

We have seen that the quantization of a classical system with action $S[x]$ is achieved by the path integral $\int Dx\, e^{\frac{i}{\hbar}S[x]}$ that computes the transition amplitude.

In the path integral formulation the classical limit is intuitive: macroscopic systems have large values of action in $\hbar$ units. Macroscopically small variations of paths can still make the phase variations $\frac{\delta S[x]}{\hbar}$ much bigger than π, so that amplitudes of nearby paths cancels by destructive interference. This is true except for variations that make $\delta S[x] = 0$, which is the

condition that identifies the classical path. Nearby paths have amplitudes that sum coherently with the classical one, and the path integral is dominated by the classical trajectory.

The notation $\int Dx$ is symbolic and indicates the formal integration over the space of functions $x(t)$. To make it precise one has to regulate the functional space by making it finite dimensional ("regularization"). Then one integrates over the regulated finite-dimensional space, and eventually takes the continuum limit by removing the regularization parameters. If this procedure is done with care, the limit exists and gives the correct transition amplitude. In the previous derivation we have seen that the space of paths is regulated by approximating the functions $x(t)$ by their $N-1$ values computed at intermediate points, the x_k's with $k = 1, ..., N-1$. This makes the space of functions finite dimensional. The action is discretized and evaluated using the approximated functions. At this stage the integration over the regulated functional space is well defined. Eventually one takes the continuum limit ($N \to \infty$): if the integration measure is chosen appropriately, as in eq. (26), this limit exists and gives a viable definition of the path integral.

We started from canonical quantization and derived the above discretized form of the space of functions. This regularization is often called Time Slicing (TS). Viceversa, one can start directly with the path integral, regulate it in a suitable way, and use it to construct the quantum theory (Feynman originally started this way). The path integral is used to evaluate a transition amplitude that is seen to satisfy a Schördinger wave equation. This can be viewed as an alternative approach to quantization. In the regularization procedure of the path integral one must make several choices, and they may produce different transition amplitudes. For example, in a TS regularization one may discretize the potential term $V(x(t))$ in the action to $V(x_k)$ or $V(x_{k-1})$ or $V(\frac{1}{2}(x_k + x_{k-1})$. In the present case this makes no difference, and one obtains the same continuum limit. For more complicated interactions, such as those arising from coupling to gauge fields, or in the presence of a nontrivial background metric, different discretizations may produce different final answers. These ambiguities are the path integral counterparts of the *ordering ambiguities* of canonical quantization, where choosing different orderings produce different quantum hamiltonians, and thus different quantum theories.

We have introduced path integrals by considering a single degree of freedom. Extension to a finite number of degrees of freedom is immediate, so that quantizing the motion of one or more particles in a finite dimensional space does not pose any new conceptual problem. For example, the motion of a nonrelativistic particle in $\mathbb{R}^3$ with cartesian coordinates $\vec{x}$, in the presence of a scalar potential $V(\vec{x})$, is quantized by the following discretized path integral

$$\int Dx\, e^{\frac{i}{\hbar}S[x]} \;=\; \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} d^3x_k\Big)\Big(\frac{m}{2\pi i\hbar\epsilon}\Big)^{\frac{3N}{2}} e^{\frac{i}{\hbar}\sum_{k=1}^{N}\epsilon\left[\frac{m}{2}\frac{(\vec{x}_k-\vec{x}_{k-1})^2}{\epsilon^2}-V(\vec{x}_{k-1})\right]} \tag{42}$$

where, of course, the classical action is

$$S[x] = \int_0^T dt\,\Big(\frac{m}{2}\dot{\vec{x}}^2 - V(\vec{x})\Big) \;=\; \lim_{N\to\infty}\sum_{k=1}^{N}\epsilon\Big[\frac{m}{2}\frac{(\vec{x}_k-\vec{x}_{k-1})^2}{\epsilon^2} - V(\vec{x}_{k-1})\Big]\,. \tag{43}$$

Formally, one can also consider the case of an infinite number of degrees of freedom, as appropriate for a field theory. In this case convergence is not guaranteed, and the removal of the regularization may lead to infinite results. In the class of theories called renormalizable, the infinites can be removed consistently by a *renormalization* procedure that redefines the dynamical variables and the coupling constants, and allows to obtain finite results, at least at the level of perturbation theory.

4 Correlation functions

Correlation functions are quantities used to describe several physical observables in the quantum theory. They are also useful to develop the perturbative expansion around the solvable gaussian path integral that corresponds to a "free" theory (we include the harmonic oscillator in this class).

Correlation functions are normalized averages of the product of n dynamical variables, evaluated at different times and weighted by $e^{\frac{i}{\hbar}S}$. In our one dimensional example, the normalized "n-point correlation function" is defined by

$$\langle x(t_1)x(t_2)\dots x(t_n)\rangle = \frac{1}{Z}\int Dx\, x(t_1)x(t_2)\dots x(t_n)\, e^{\frac{i}{\hbar}S[x]} \tag{44}$$

where $Z = \int Dx\, e^{\frac{i}{\hbar}S[x]}$ provides the normalization to guarantee that $\langle 1\rangle = 1$. Of particular importance is the 2-point function $\langle x(t_1)x(t_2)\rangle$, often called the propagator. It is understood that correlation functions depend implicitly on the boundary conditions that specify the initial and final states. Very often, especially in quantum field theory, one chooses the initial and final states to be the vacuum state (the state with lowest energy) and considers an infinite propagation time. We have mostly considered amplitudes between positions eigenstates, but one can insert any desired state as boundary state using eq. (11).

It is useful to collect all correlation functions into a single object $Z[J]$, called the generating functional of correlation functions. One uses an arbitrary function $J(t)$, called "source", and defines $Z[J]$ by

$$Z[J] = \int Dx\, e^{\frac{i}{\hbar}(S[x]+\int dt\, J(t)x(t))} \tag{45}$$

Clearly, a functional derivative over $J(t)$ provides an insertion inside the path integral of $\frac{i}{\hbar}x(t)$, so that taking n functional derivartives (at different times), setting then $J(t) = 0$, and normalizing properly produces the n-point correlation function

$$\langle x(t_1)x(t_2)\cdots x(t_n)\rangle = \frac{1}{Z}\left(\frac{\hbar}{i}\right)^n \frac{\delta^n Z[J]}{\delta J(t_1)\delta J(t_2)\cdots\delta J(t_n)}\bigg|_{J=0} \tag{46}$$

where $Z = Z[0]$. Alternatively, one may expand the exponential $e^{\frac{i}{\hbar}\int dt\, J(t)x(t)}$ to obtain

$$\begin{aligned} Z[J] &= \int Dx\, e^{\frac{i}{\hbar}(S[x]+\int dt\, J(t)x(t))} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{\hbar}\right)^n \int dt_1 dt_2\dots dt_n\ \langle x(t_1)x(t_2)\cdots x(t_n)\rangle_U\ J(t_1)J(t_2)\cdots J(t_n) \end{aligned} \tag{47}$$

where the subscript "U" indicates un-normalized correlation functions, i.e. correlation functions obtained without dividing by Z, which can again be used to prove (46).

We are going to use mostly path integrals. However, it is useful to compare with the corresponding definition of correlation functions given in canonical quantization. We have employed the Schrödinger picture to evaluate the transition amplitude. In this picture operators are time independent and states acquire the time dependence by the Schrödinger equation. To state the equivalent definition of the n-point correlation function, given the times $t_1, t_2, ..., t_n$, one has to reorder them from the earliest to the latest one, i.e. use the permutation $T(1), T(2), ..., T(n)$

of the numbers $1, 2, ..., n$ such that $t_{T(1)} < t_{T(2)} < ... < t_{T(n)}$. Then one defines

$$\begin{aligned}\langle x(t_1)x(t_2)\cdots x(t_n)\rangle &= \frac{1}{Z}\langle x_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_{T(n)})}\hat{x}\, e^{-\frac{i}{\hbar}\hat{H}(t_{T(n)}-t_{T(n-1)})}\cdots \\ &\cdots\ e^{-\frac{i}{\hbar}\hat{H}(t_{T(3)}-t_{T(2)})}\hat{x}\, e^{-\frac{i}{\hbar}\hat{H}(t_{T(2)}-t_{T(1)})}\hat{x}\, e^{-\frac{i}{\hbar}\hat{H}(t_{T(1)}-t_i)}|x_i\rangle\end{aligned} \tag{48}$$

where $Z = \langle x_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|x_i\rangle$ is the transition amplitude. The time ordering guarantees that in the path integral derivation, due to the time slicing procedure, each position operator is substituted by the eigenvalue of the eigenstate carried by the resolution of the identity inserted next to the operator under consideration. This is always the case as, for very large N, the time discretization is sufficiently fine to have a resolution of the identity next to the position of one of the operators $\hat{x}$, which is then substituted by an eigenvalue.

Equivalently, in the Heisenberg picture, one assigns the time evolution to the operators while states are time independent. Heisenberg's equations of motion (Heisenberg's "matrix mechanics") read as

$$i\hbar\frac{d\hat{x}_H}{dt} = [\hat{x}_H, \hat{H}] \tag{49}$$

where the subscript H refers to operators in the Heisenberg picture. They correspond to the quantum version of Hamilton's equations, with the Poisson bracket substituted by a commutator that takes the value of $i\hbar$ times the classical Poisson bracket. For a time independent hamiltonian the solution can formally be written as

$$\hat{x}_H(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{x}_H(0)e^{-\frac{i}{\hbar}\hat{H}t} \tag{50}$$

where the value of $\hat{x}_H(0)$ can be identified with the time independent Schrödinger operator $\hat{x}$. Eigenstates of $\hat{x}_H(t)$ can be written as $|x,t\rangle_H$

$$\hat{x}_H(t)|x,t\rangle_H = x|x,t\rangle_H\ . \tag{51}$$

The relation to the Schrödinger picture is simply given in terms of the unitary operator $e^{\frac{i}{\hbar}\hat{H}t}$. This operator relates the two pictures which are then unitarily equivalent, as guaranteed by the Stone–von Neumann theorem. Correlation functions in the Heisenberg picture are defined by

$$\langle x(t_1)x(t_2)...x(t_n)\rangle = \frac{1}{Z}{}_H\langle x_f, t_f|T\Big(\hat{x}_H(t_1)\hat{x}_H(t_2)\cdots\hat{x}_H(t_n)\Big)|x_i,t_i\rangle_H \tag{52}$$

where the symbol T indicates time ordering, i.e. the prescription of ordering the operators in such a way that they have an increasing value of time when going form right to left. The value of the transition amplitude that normalizes the expression is written, in such a picture, as $Z = {}_H\langle x_f, t_f|x_i, t_i\rangle_H$. In extensions to QFT one normally chooses $t_i = -\infty$, $t_f = \infty$, and picks as final states the vacuum state, rather than position eigenstates.

4.1 Digression over gaussian integrals

Gaussian integrals in one or more variables are easily computed. For a real variable $\phi \in \mathbb{R}$ they are given by

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}\ e^{-\frac{1}{2}K\phi^2} &= \frac{1}{\sqrt{K}} \\ \int_{-\infty}^{\infty}\frac{d\phi}{\sqrt{2\pi}}\ e^{-\frac{1}{2}K\phi^2+J\phi} &= \frac{1}{\sqrt{K}}\, e^{\frac{1}{2}\frac{1}{K}J^2}\end{aligned} \tag{53}$$

with K a real positive number. The first one is the standard gaussian integral, whose square is easily computed in polar coordinates. The second one is obtained by square completion, i.e. writing $-\frac{1}{2}K\phi^2 + J\phi = -\frac{1}{2}K(\phi - \frac{J}{K})^2 + \frac{1}{2}\frac{1}{K}J^2$, and then shifting the measure from ϕ to $\phi - \frac{J}{K}$ to obtain the desired result.

They are straightforwardly extended to n real variables

$$\int \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}}\; e^{-\frac{1}{2}\phi^i K_{ij}\phi^j} = (\det K_{ij})^{-\frac{1}{2}}$$

$$\int \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}}\; e^{-\frac{1}{2}\phi^i K_{ij}\phi^j + J_i\phi^i} = (\det K_{ij})^{-\frac{1}{2}}\; e^{\frac{1}{2}J_i G^{ij} J_j} \tag{54}$$

where K_{ij} is a real, symmetric, positive definite matrix (all eigenvalues must be strictly positive), and G^{ij} its inverse (so that $K_{ij}G^{jk} = \delta_i^k$). The first integral is immediate if K_{ij} is diagonal, and valid in full generality by noting that K_{ij} is diagonalizable by an orthogonal transformation which leaves the measure invariant. The last integral is obtained again by square completion.

These gaussian integrals are suitable for euclidean path integrals. Moreover, in a hypercondensed notation to be explained shortly, path integrals look very much like ordinary integrals. Of course the definition of determinants for infinite dimensional matrices is delicate, and requires a regularization procedure.

By analytical extension one obtains gaussian integrals suitable for quantum mechanics

$$\int \frac{d^n\phi}{(-2\pi i)^{\frac{n}{2}}}\; e^{-\frac{i}{2}\phi^i K_{ij}\phi^j + iJ_i\phi^i} = (\det K_{ij})^{-\frac{1}{2}}\; e^{\frac{i}{2}J_i G^{ij} J_j} \tag{55}$$

where again G^{ij} is the inverse of K_{ij}. Convergence to the given values is guaranteed if K_{ij} has a small negative imaginary part. This can be obtained by the Feynman $i\epsilon$ trick: one replaces the real matrix K_{ij} by $K_{ij} - i\epsilon\delta_{ij}$ with $\epsilon > 0$, the ϵ part assures a gaussian damping for $|\phi| \to \infty$, thus guaranteeing the convergence of the integral, and at the end of the calculation one sets $\epsilon \to 0$ (in quantum field theory this corresponds to the causal $i\epsilon$ Feynman prescription). In a hypercondensed notation, to be explained shortly, these formulae give the formal solution of path integrals of free theories (meaning theories with quadratic actions, in this context) without gauge invariances, in either quantum mechanics or quantum field theory. Gauge invariance would produce a vanishing $\det K_{ij}$, and one must apply a gauge fixing procedure to obtain a finite answer. This situation will not be described on these notes.

4.2 Hypercondensed notation and generating functionals

To proceed swiftly, it is useful to introduce a hypercondensed notation. It allows to treat path integrals, including those for field theories, formally as ordinary integrals. The hypercondensed notation is defined by lumping together discrete and continuous indices into a single index, so that a variable ϕ^i can be used as a shorthand notation for the position $x(t)$ of the particle, identifying

$$x(t) \;\to\; \phi^i \quad\Longrightarrow\quad \begin{cases} x \to \phi \\ t \to i \end{cases} . \tag{56}$$

Similarly for fields, as for example the vector quadripotential $A_\mu(x^\nu)$, the hypercondensed notation is obtained by denoting

$$A_\mu(x^\nu) \;\to\; \phi^i \quad\Longrightarrow\quad \begin{cases} A \to \phi \\ \mu, x^\nu \to i \end{cases} \tag{57}$$

where now the index i contains a discrete part (the discrete index $\mu = 0, 1, 2, 3$) and a continuous part (the spacetime coordinates $x^\nu = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$). Indices may be lowered and raised with a metric (given by the identity matrix in many cases, though one may consider more general situations). Repeated indices are understood to be summed over (the Einstein summation convention). Thus, the notation $\phi^i \phi_i$ stands in the above cases for $\int dt\, x(t)x(t)$ and $\int d^4x\, A_\mu(x)A^\mu(x)$, respectively. Or equivalently, to make explicit the presence of a metric, as $\int dt \int dt'\, x(t)\delta(t-t')x(t')$ and $\int d^4x \int d^4y\, A_\mu(x)\eta^{\mu\nu}\delta^4(x-y)A_\nu(y)$, respectively (the metric would be $\delta(t-t')$ and $\eta^{\mu\nu}\delta^4(x-y)$, respectively, so that one could also write $\phi^i\phi_i = \phi^i g_{ij}\phi^j$)

$$\phi^i\phi_i \quad \Longrightarrow \quad \begin{cases} = \int dt\ x(t)x(t) & = \int dt \int dt'\ x(t)\delta(t-t')x(t') \\ = \int d^4x\ A_\mu(x)A^\mu(x) & = \int d^4x \int d^4y\ A_\mu(x)\eta^{\mu\nu}\delta^4(x-y)A_\nu(y) \end{cases} . \tag{58}$$

One must pay attention to simple looking expressions, as they include integrations or infinite sums, and might not converge.

With such a notation at hand, we are ready to review quickly the definition of correlation functions, introduce *generating functionals*, and present gaussian path integration formulae. We will also describe the *Wick's theorem*, that gives a simple way of computing all correlation functions in a free theory in terms of the 2-point function only (the propagator).

The path integrals in (26) and (42), after denoting the variables in a hypercondensed notation by ϕ^i, can be written as

$$\int D\phi\, e^{\frac{i}{\hbar}S[\phi]} \tag{59}$$

and the correlation functions as

$$\langle \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}\rangle = \frac{1}{Z}\int D\phi\ \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n} e^{\frac{i}{\hbar}S[\phi]} \tag{60}$$

where $Z = \int D\phi\, e^{\frac{i}{\hbar}S[\phi]}$. The generating functional takes the form

$$Z[J] = \int D\phi\ e^{\frac{i}{\hbar}(S[\phi]+J_i\phi^i)} \tag{61}$$

and generates all correlation functions by differentiation (in hypercondensed notation functional derivatives look like as usual derivatives, though we keep using the symbol δ of functional derivative)

$$\langle \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}\rangle = \frac{1}{Z[0]}\left(\frac{\hbar}{i}\right)^n \frac{\delta}{\delta J_{i_1}}\frac{\delta}{\delta J_{i_2}}\cdots\frac{\delta}{\delta J_{i_n}} Z[J]\Big|_{J=0} . \tag{62}$$

We can now define the generating functional of *connected correlation functions* $W[J]$ by

$$Z[J] = e^{\frac{i}{\hbar}W[J]} \qquad \Rightarrow \qquad W[J] = \frac{\hbar}{i}\ln Z[J] . \tag{63}$$

One can prove that it generates "connected" correlation functions by

$$\langle \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}\rangle_c = \left(\frac{\hbar}{i}\right)^{n-1} \frac{\delta}{\delta J_{i_1}}\frac{\delta}{\delta J_{i_2}}\cdots\frac{\delta}{\delta J_{i_n}} W[J]\Big|_{J=0} . \tag{64}$$

We will check this statement and its meaning in the free theory.

It is also useful to define the effective action $\Gamma[\varphi]$ as the Legendre transform of $W[J]$

$$\Gamma[\varphi] = \min_J \Big\{ W[J] - J_i \varphi^i \Big\} \tag{65}$$

which is considered as a classical action that includes all quantum corrections. It generates the so called *one-particle irreducible* (1PI) correlation functions, though we will not investigate further this definition. The minimum in J is obtained at $\varphi^i = \frac{\delta W[J]}{\delta J_i}$, a relation that must be inverted to obtain $J_i = J_i(\varphi)$ and inserted back into the right hand side of (65) to obtain the effective action indeed as a functional of the variable φ^i only.

The last two functionals, $W[J]$ and $\Gamma[\varphi]$, find their main applications in quantum field theory. Equivalent definitions can be given for euclidean path integrals.

4.3 Free theory

It is useful to study free theories, here meaning theories which have a quadratic action. They provide a simple application of the previous formulae, giving at the same time additional intuition. A free theory is described by a quadratic action

$$S[\phi] = -\frac{1}{2} \phi^i K_{ij} \phi^j \tag{66}$$

which produces the linear equations of motion $K_{ij}\phi^j = 0$. We assume K_{ij} invertible, which translates to the fact that there are no gauge symmetries in our model.

As an example, we may take the harmonic oscillator whose action is

$$\begin{aligned} S[x] &= \int_{-\infty}^{\infty} dt \left(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2} x^2 \right) = -\frac{1}{2} \int_{-\infty}^{\infty} dt\, x(t) \Big(\frac{d^2}{dt^2} + \omega^2 \Big) x(t) \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, x(t) \Big(\frac{d^2}{dt^2} + \omega^2 \Big) \delta(t-t') x(t') \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, x(t) K(t,t') x(t') \\ &\rightsquigarrow \quad S[\phi] = -\frac{1}{2} \phi^i K_{ij} \phi^j \end{aligned} \tag{67}$$

where we integrated by parts, and introduced the Dirac delta function $\delta(t-t')$ to expose the "kinetic matrix" $K(t,t') = (\frac{d^2}{dt^2} + \omega^2)\delta(t-t')$.

Denoting $D\phi \equiv \frac{d^n \phi}{(-2\pi i)^{\frac{n}{2}}}$, setting $\hbar = 1$ for simplicity, and using the gaussian result in eq. (55), one calculates formally the path integral with sources

$$Z[J] = \int D\phi\; e^{i(S[\phi] + J_i \phi^i)} = (\det K_{ij})^{-\frac{1}{2}}\; e^{\frac{i}{2} J_i G^{ij} J_j} \; . \tag{68}$$

Then, using eq. (62), one obtains the following correlation functions

$$\begin{aligned} &\langle 1 \rangle = 1 \\ &\langle \phi^i \rangle = 0 \\ &\langle \phi^i \phi^j \rangle = -i G^{ij} \; . \end{aligned} \tag{69}$$

The first one is a consequence of the normalization, the second one reflects the symmetry $\phi^i \to -\phi^i$, and the third one is known as the propagator, which we find proportional to the inverse of the kinetic matrix K_{ij}.

Continuing with the calculation of higher point functions, we see that all correlation functions with an odd number of points vanish, again consequence of the symmetry $\phi^i \to -\phi^i$. Those with an even number n factorize into a sum of $(n-1)!!$ terms, given by the product of the 2-point functions which connect two points in all possible ways. This fact is known as the "Wick's theorem". For example, the 4-point correlation function is given by

$$\langle \phi^1 \phi^2 \phi^3 \phi^4 \rangle = \langle \phi^1 \phi^2 \rangle \langle \phi^3 \phi^4 \rangle + \langle \phi^1 \phi^3 \rangle \langle \phi^2 \phi^4 \rangle + \langle \phi^1 \phi^4 \rangle \langle \phi^2 \phi^3 \rangle \tag{70}$$

that indeed contains the sum of 3!! terms. This correlation function is not connected, as it disconnects into the sum of products of correlation functions of lower order. This is true for all higher point correlation functions of the free theory.

The generating functional of connected correlation functions $W[J]$ is obtained from eq. (68) using the definition (63)

$$W[J] = \frac{1}{2} J_i G^{ij} J_j - \Lambda \tag{71}$$

where $\Lambda = -\frac{i}{2} \ln \det(K_{ij}) = -\frac{i}{2} \operatorname{tr} \ln(K_{ij})$ is a constant, whose precise value is often not needed. One verifies that it generates a 2-point correlation functions that is connected (it generates also a 0-point function, that can be shown to be connected as well).

Let us also calculate the effective action. The minimum in J of eq. (65) is achieved for

$$\frac{\delta W}{\delta J_i} = \varphi^i \quad \Longrightarrow \quad \varphi^i = G^{ij} J_j \quad \Longrightarrow \quad J_i = K_{ij} \varphi^j \tag{72}$$

so that

$$\Gamma[\varphi] = -\frac{1}{2} \varphi^i K_{ij} \varphi^j - \Lambda \, . \tag{73}$$

We see that for a free theory the effective action $\Gamma[\varphi]$ reproduces the original action $S[\varphi]$ with an additive constant $-\Lambda$, that could be interpreted as (minus) a vacuum energy of quantum origin. The latter can be disregarded if gravitational interactions are neglected. In general, the effective action is considered as a classical action that contains the effects of quantization in its couplings (and thus the effective actions should not be quantized again).

Reinserting $\hbar$ by a simple rescaling, we collect here the formulae for a free (gaussian) theory

$$\begin{aligned} S[\phi] &= -\frac{1}{2} \phi^i K_{ij} \phi^j \\ Z[J] &= (\det K_{ij})^{-\frac{1}{2}} \, e^{\frac{i}{2\hbar} J_i G^{ij} J_j} \\ W[J] &= \frac{1}{2} J_i G^{ij} J_j - \hbar \Lambda \\ \Gamma[\varphi] &= -\frac{1}{2} \varphi^i K_{ij} \varphi^j - \hbar \Lambda = S[\varphi] + \hbar \text{ corrections.} \end{aligned} \tag{74}$$

with connected 2-pt function

$$\langle \phi^i \phi^j \rangle = -i \hbar G^{ij} \, . \tag{75}$$

4.4 Harmonic oscillator

Let us work out in more explicit terms the case of a harmonic oscillator with unit mass

$$S[x] = \int_{-\infty}^{\infty} dt \Big(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2} x^2 \Big) \,, \qquad Z[J] = \int Dx \, e^{\frac{i}{\hbar}(S[x] + \int dt \, J(t)x(t))} \tag{76}$$

formally solved in the previous section. We repeat the deduction without using the hypercondensed notation. We consider an infinite propagation time and a transition amplitude between the ground state, classically achieved for $x = 0$. The action in the exponent can be manipulated with an integration by parts without producing boundary terms (imposing that $x(t)$ is in its classical vacuum at initial and final times gives a vanishing boundary term, another justification will be given later on when treating the euclidean version of the problem). Thus the action takes the form (67)

$$\begin{aligned} S[x] &= -\frac{1}{2} \int dt \, x(t) \Big(\frac{d^2}{dt^2} + \omega^2 \Big) x(t) = -\frac{1}{2} \int \int dt dt' \, x(t) \Big(\frac{d^2}{dt^2} + \omega^2 \Big) \delta(t - t') x(t') \\ &\equiv -\frac{1}{2} \int \int dt dt' \, x(t) K(t, t') x(t') \end{aligned} \tag{77}$$

where $K(t,t') = (\frac{d^2}{dt^2} + \omega^2)\delta(t - t')$ is the differential "kinetic" operator of the harmonic oscillator. The inverse of this matrix (i.e. the Green function of the differential operator) is conveniently written in a Fourier transform

$$G(t, t') = \int \frac{dp}{2\pi} \frac{e^{-ip(t-t')}}{-p^2 + \omega^2} \tag{78}$$

which is verified to satisfy the defining equation

$$\int dt'' \, K(t, t'') G(t'', t') = \Big(\frac{d^2}{dt^2} + \omega^2 \Big) G(t, t') = \delta(t - t') \tag{79}$$

that in a hypercondensed notation would have been written as $K_{ij} G^{jl} = \delta_i^l$. Adding the Feynman $i\epsilon$ prescription for specifying how to integrate around the poles $p = \pm\omega$ ($\omega^2 \to \omega^2 - i\epsilon$ with $\epsilon \to 0^+$) one computes

$$G(t, t') = \int \frac{dp}{2\pi} \frac{e^{-ip(t-t')}}{-p^2 + \omega^2 - i\epsilon} = \frac{i}{2\omega} e^{-i\omega|t-t'|} \,. \tag{80}$$

The computation is standard. The integration region is seen as the real axis of the complex plane, and one adds a semicircle at infinity (the one that gives no contibution) to close the contour and use the Cauchy's residue theorem to evaluate the integral. For positive $t - t'$ one closes the contour in the lower half plane, for negative $t - t'$ one closes the contour in the upper half plane. In formulae,

$$\begin{aligned} G(t, t') &= \int \frac{dp}{2\pi} \frac{e^{-ip(t-t')}}{-p^2 + \omega^2 - i\epsilon} \\ &= -\int \frac{dp}{2\pi} \frac{e^{-ip(t-t')}}{(p - \omega + i\epsilon')(p + \omega - i\epsilon')} \\ &= \theta(t - t') \frac{i}{2\omega} e^{-i\omega(t-t')} + \theta(t' - t) \frac{i}{2\omega} e^{i\omega(t-t')} = \frac{i}{2\omega} e^{-i\omega|t-t'|} \end{aligned} \tag{81}$$

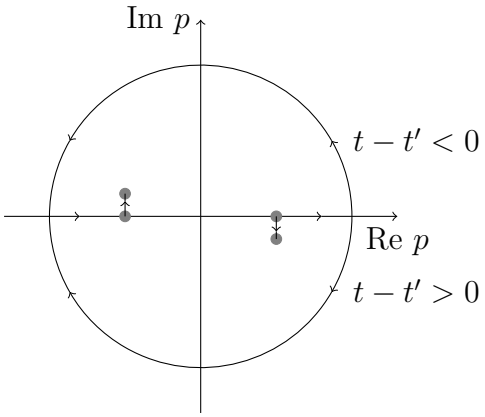


Figure 4: Contour integration around the poles with the Feynman prescription.

where $\epsilon \sim \epsilon' \to 0^+$ displaces the poles infinitesimally out of the real axis, as in Figure 4.

With the Green function at hand, one completes the square in (76) and finds

$$\begin{aligned} Z[J] &= \int Dx\, e^{\frac{i}{\hbar}(S[x]+\int dt\, Jx)} \\ &= \int Dx\, \exp\Big[-\frac{i}{\hbar}\iint dtdt' \Big(\frac{1}{2}x(t)K(t,t')x(t') - J(t)\delta(t-t')x(t') \\ &\qquad +\frac{1}{2}J(t)G(t,t')J(t') - \frac{1}{2}J(t)G(t,t')J(t')\Big)\Big] \\ &= e^{\frac{i}{2\hbar}\iint dt\, dt'\, J(t)G(t,t')J(t')} \underbrace{\int D\tilde{x}\, \exp\Big(-\frac{i}{\hbar}\iint dt\, dt'\, \frac{1}{2}\tilde{x}(t)K(t,t')\tilde{x}(t')\Big)}_{N\sim \det{}^{-1/2}[K(t,t')]} \\ &= N\exp\Big(\frac{i}{2\hbar}\iint dt\, dt'\, J(t)G(t,t')J(t')\Big) \end{aligned} \tag{82}$$

where $\tilde{x}(t) = x(t) - \int dt' G(t,t')J(t')$ is the shifted variable that completes the square.

Having found the complete generating functional, one may compute the 2-point function (the propagator)

$$\begin{aligned} \langle x(t)x(t')\rangle &= \frac{\int Dx\, x(t)x(t')e^{\frac{i}{\hbar}S[x]}}{\int Dx\, e^{\frac{i}{\hbar}S[x]}} = \frac{1}{Z[0]}\left(\frac{\hbar}{i}\right)^2 \frac{\delta^2 Z[J]}{\delta J(t)\delta J(t')}\bigg|_{J=0} = -i\hbar G(t,t') \\ &= \frac{\hbar}{2\omega}e^{-i\omega|t-t'|}\,. \end{aligned} \tag{83}$$

4.4.1 The Klein-Gordon propagator

The quantum field theory of a free Klein-Gordon scalar field can be viewed as an higher dimensional analogue of the harmonic oscillator. The action of a real scalar field $\phi(x)$ is given by

$$S[\phi] = \int d^4x \Big(-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2\Big)\,. \tag{84}$$

In hypercondensed notation

$$\phi(x) \to \phi^i \,, \qquad (-\Box_x + m^2)\delta^{(4)}(x-y) \to K_{ij} \tag{85}$$

it takes the standard form of a free theory given in eq. (66). Its path integral quantization is readily performed. Setting $\hbar = 1$ for simplicity, one finds the two-point function (see eq. (69))

$$\langle \phi(x)\phi(y)\rangle = -iG(x-y) \tag{86}$$

where $G(x)$ is the Green function of the Klein-Gordon operator

$$(-\Box + m^2)G(x) = \delta^{(4)}(x) \,. \tag{87}$$

Its Fourier transform is

$$G(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip_\mu x^\mu}}{p^2 + m^2 - i\epsilon} \tag{88}$$

where $d^4p \equiv dp^0 d^3p$ and $p^2 \equiv p^\mu p_\mu = -(p^0)^2 + \vec{p}^{\,2}$. There are poles corresponding to the solutions of the mass shell condition, $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$ (positive energies) and $p^0 = -\sqrt{\vec{p}^{\,2} + m^2}$ (negative energies). The Feynman $i\epsilon$ prescription sends positive energies forwards in time and negative energies backwards in time. It corresponds to the physical interpretation of particle and antiparticles with positive energies and always propagating forwards in time. Setting $E_p = \sqrt{\vec{p}^{\,2} + m^2}$ one finds

$$\begin{aligned}
\langle \phi(x)\phi(y)\rangle &= -iG(x-y) = -i \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip\cdot(x-y)}}{p^2 + m^2 - i\epsilon} \\
&= i \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \int \frac{dp^0}{2\pi} \frac{e^{-ip^0(x^0-y^0)}}{(p^0 - E_p + i\epsilon')(p^0 + E_p - i\epsilon')} \\
&= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \left[\theta(x^0-y^0) \frac{e^{-iE_p(x^0-y^0)}}{2E_p} + \theta(y^0-x^0) \frac{e^{-iE_p(y^0-x^0)}}{2E_p} \right] \\
&= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \frac{e^{-iE_p|x^0-y^0|}}{2E_p}
\end{aligned} \tag{89}$$

where $\epsilon \sim \epsilon' \to 0^+$ displaces the poles as in Figure 4. Comparing with the two-point function of the harmonic oscillator ($\sim \frac{e^{-i\omega|t-t'|}}{2\omega}$) one is led to interpret the Klein-Gordon field as a collection of an infinite number of harmonic oscillators of frequency E_p and parametrized by $\vec{p}$.

4.4.2 Harmonic oscillator in euclidean time

The statistical partition function in the limit of vanishing temperature ($\Theta \to 0$), corresponding to an infinite euclidean propagation time ($\beta \to \infty$), takes a simple form

$$Z_E = \mathrm{Tr}\, e^{-\beta\hat{H}} = \sum_n e^{-\beta E_n} \quad \overset{\beta\to\infty}{\longrightarrow} \quad e^{-\beta E_0} + \text{subleading terms.} \tag{90}$$

This is true even in the presence of a source J if one assumes that the source is nonvanishing for a finite interval of time only: the remaining infinite time is sufficient to project the operator $e^{-\beta\hat{H}}$

onto the ground state. This allows to rewrite the generating functional $Z[J]$ in the euclidean case in a simpler way, justifying the dropping of boundary terms in the integration by parts in the classical action. The statistical partition function is obtained by using periodic boundary conditions, and for large β one gets the projection onto the ground state

$$\begin{aligned} Z_E[J] &= \int_{PBC} Dx\, e^{-S_E[x]+\int d\tau\, Jx} \quad \stackrel{\beta\to\infty}{\longrightarrow} \quad \lim_{\beta\to\infty} e^{-\beta E_0(J)} \\ S_E[x] &= \int_{-\infty}^{\infty} d\tau \Big(\frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2\Big) \end{aligned} \tag{91}$$

where $E_0(J)$ is the ground state energy in the presence of the source J. We can now repeat the previous calculation in the present context. We integrate by parts without encountering boundary terms, as the paths are closed, and the path integral is strictly gaussian

$$\begin{aligned} Z_E[J] &= \int_{PBC} Dx\, \exp\Big[-\int d\tau \Big(\frac{1}{2}x(\tau)\Big(-\frac{d^2}{d\tau^2}+\omega^2\Big)x(\tau) - J(\tau)x(\tau)\Big)\Big] \\ &= N \exp\Big[\frac{1}{2}\int d\tau d\tau'\, J(\tau)G_E(\tau,\tau')J(\tau')\Big] \end{aligned} \tag{92}$$

where the euclidean Green function G_E is given by

$$G_E(\tau,\tau') = \Big[-\frac{d^2}{d\tau^2}+\omega^2\Big]^{-1} = \int \frac{dp_E}{2\pi}\,\frac{e^{-ip_E(\tau-\tau')}}{p_E^2+\omega^2} = \frac{1}{2\omega}e^{-\omega|\tau-\tau'|} \tag{93}$$

and is unique: there are no poles and related prescriptions to specify how to perform the integration.

We now verify again the relation between quantum mechanics and statistical mechanics, realized by the analytic continuation in time, the Wick rotation. The inverse Wick rotation implies $\tau \equiv t_E \to it_M \equiv it$ and $p_E \to -ip_M \equiv -ip$, with the latter arising form the requirement that the correct Fourier transform is kept during the analytic deformation. For the two-point function one finds

$$\begin{aligned} \langle x(\tau)x(\tau')\rangle &= G_E(\tau,\tau') = \int \frac{dp_E}{2\pi}\,\frac{e^{-ip_E(\tau-\tau')}}{p_E^2+\omega^2} \quad \to \\ &\to \quad -i\int \frac{dp_M}{2\pi}\,\frac{e^{-ip_M(t-t')}}{-p_M^2+\omega^2} = -iG_M(t,t') = \langle x(t)x(t')\rangle \end{aligned} \tag{94}$$

which is the Feynman propagator in eq. (83) (with $\hbar = 1$), that is

$$\frac{1}{2\omega}e^{-\omega|\tau-\tau'|} \quad \to \quad \frac{1}{2\omega}e^{-i\omega|t-t'|}\,. \tag{95}$$

We recognize that the Feynman propagator is the unique analytical extension of the euclidean two-point function. All other Green functions, such as the retarded or advanced ones, correspond to different boundary conditions implemented with different prescriptions for performing the integration around the poles. They cannot be Wick rotated, as one would encounter poles in the analytic continuation.

5 Perturbative expansion

The free theory corresponds to a gaussian path integral which is exactly solvable. With interactions one is often unable to compute exactly the path integral, and one must resort to some sort of approximation. The simplest one is the perturbative expansion around a free theory, which consists in expanding the solution in power series of the coupling constants that parametrize the interactions. If the couplings are small enough, the perturbative expansion might give a good approximation of the solution.

We describe the perturbative expansion taking as guiding example the anharmonic oscillator

$$S[x] \;=\; \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2 - \frac{g}{3!}x^3 - \frac{\lambda}{4!}x^4\right) . \tag{96}$$

When the coupling constants g and λ vanish, the theory is exactly solvable. Thus one may try to include perturbatively the corrections that arise when g and λ are small enough. It is convenient to split the action as sum of two terms, a free part S_0 and an interacting one S_{int}

$$\begin{aligned} S[x] &= S_0[x] + S_{int}[x] \\ S_0[x] &= \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2\right) \\ S_{int}[x] &= \int dt \left(-\frac{g}{3!}x^3 - \frac{\lambda}{4!}x^4\right) . \end{aligned} \tag{97}$$

Including a source term, one considers the path integral and expands in a Taylor series the exponential of the interaction term

$$\begin{aligned} Z[J] &= \int Dx\, e^{\frac{i}{\hbar}(S[x]+\int dt\, Jx)} \qquad (98) \\ &= \int Dx\, e^{\frac{i}{\hbar}(S_0[x]+S_{int}[x]+\int dt\, Jx)} \\ &= \int Dx\, e^{\frac{i}{\hbar}S_{int}[x]}\, e^{\frac{i}{\hbar}(S_0[x]+\int dt\, Jx)} \\ &= \int Dx\, \Big[1 + \frac{i}{\hbar}S_{int}[x] + \frac{1}{2}\Big(\frac{i}{\hbar}S_{int}[x]\Big)^2 + \cdots \\ &\qquad \cdots + \frac{1}{n!}\Big(\frac{i}{\hbar}S_{int}[x]\Big)^n + \cdots\Big]\, e^{\frac{i}{\hbar}(S_0[x]+\int dt\, Jx)} . \end{aligned}$$

Written in the last form, one may proceed in computing it term by term, with the use of the Wick's theorem. In an obvious notation, the path integral can also be written as

$$Z[J] = \Big\langle e^{\frac{i}{\hbar}S_{int}[x]} \Big\rangle_{U,0,J} \tag{99}$$

where the subscripts $U, 0, J$ denote un-normalized averaging (U) with the free theory (0) and in the presence of a source (J). This last expression is sometimes called the "Dyson formula". It generates the perturbative expansion in terms of Feynman diagrams, as we shall see.

An alternative way of writing the perturbative series is the following one

$$\begin{aligned} Z[J] &= \int Dx\; e^{\frac{i}{\hbar}(S[x]+\int dt\, Jx)} = \int Dx\; e^{\frac{i}{\hbar}S_{int}[x]}\, e^{\frac{i}{\hbar}(S_0[x]+\int dt\, Jx)} \\ &= e^{\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}]} \int Dx\; e^{\frac{i}{\hbar}(S_0[x]+\int dt\, Jx)} \\ &= e^{\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}]}\, Z_0[J] \qquad (100) \end{aligned}$$

which presents the solution as a (quite complicated) differential operator acting on the solution of the free theory $Z_0[J]$. In particular, all vacuum diagrams are generated by

$$Z[0] = \int \mathcal{D}x \; e^{\frac{i}{\hbar}S[x]} = e^{\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}]} \; Z_0[J]\Big|_{J=0} \; . \tag{101}$$

The perturbative expansion, depicted in terms of Feynman diagrams, is obtained by expanding the interactions term inside the path integral, and using the Wick theorem to compute the correlation functions of the free theory: the vertices generated by the interactions potential contain a coupling constant plus quantum variables that are tied together two by two in all possible ways with the free propagators (graphically each *vertex* is denoted by a dot and *propagators* are denoted by lines). This is exemplified next in the case of vacuum diagrams for the anharmonic oscillator.

5.1 Vacuum diagrams

As an example we compute perturbatively the corrections to the ground state energy of the harmonic oscillator due to the anharmonic potential terms. Also, as mentioned, it is often the case that one computes using the euclidean version of the theory and only at the very end performs the inverse Wick rotation to obtain the results in minkowskian time. Thus we will proceed with the euclidean version of the theory, which is the one used later on in presenting worldline applications.

We consider

$$\begin{aligned} Z_E[J] &= \int Dx \; e^{-S_E[x]+\int d\tau Jx} \\ S_E[x] &= \lim_{\beta\to\infty}\int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2 + \frac{g}{3!}x^3 + \frac{\lambda}{4!}x^4\right) \end{aligned} \tag{102}$$

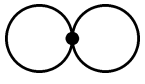
with $\beta \to \infty$. The corrections to the ground state energy can be recognized from

$$\begin{aligned} Z_E[0] &= \langle 1\rangle_U = \lim_{\beta\to\infty}\langle 0|e^{-\beta\hat{H}}|0\rangle = \lim_{\beta\to\infty} e^{-\beta E_0} \\ &= \left\langle e^{-S_{E,int}[x]}\right\rangle_{U,0} = \lim_{\beta\to\infty} e^{-\beta(E_0^{(0)}+\Delta E_0)} \end{aligned} \tag{103}$$

where the exact energy E_0 of the ground state $|0\rangle$ differs from the ground state energy of the harmonic oscillator $E_0^{(0)}$ by the term ΔE_0 due to the anharmonic potential. The latter can be computed perturbatively. We consider the first non vanishing corrections only to exemplify the perturbative expansion with path integrals, and the use of Feynman diagrams.

Let us look first at the case with $g = 0$ and focus on the first correction in λ

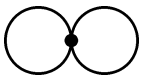
$$\begin{aligned} Z_E[0] &= \langle 1\rangle_U = \left\langle e^{-S_{E,int}[x]}\right\rangle_{U,0} = \left\langle (1 - S_{E,int}[x] + \cdots)\right\rangle_{U,0} \\ &= \langle 1\rangle_{U,0} - \frac{\lambda}{4!}\int_{-\beta/2}^{\beta/2} d\tau \, \langle x^4(\tau)\rangle_{U,0} + \cdots \\ &= \langle 1\rangle_{U,0}\left[1 - \frac{\lambda}{4!}\int_{-\beta/2}^{\beta/2} d\tau \, \langle x^4(\tau)\rangle_0 + \cdots\right] \\ &= \langle 1\rangle_{U,0}\left[1 - \frac{\lambda}{4!}\Big[3 \times \quad \Big] + \cdots\right] . \end{aligned} \tag{104}$$



In the last line we have used Wick contractions to calculate normalized correlations functions in the free theory, and then introduced a graphical representation in terms of Feynman diagrams. In this graphical representation a line denotes a propagator that joins two points in time, while vertices arising form the interactions are denoted by dots. The correction we computed contains just one vertex where four lines can enter or exit, corresponding to the power four of the dynamical variable $x(\tau)$ belonging to the interaction under consideration. Recalling the euclidean propagator, computed in eq. (93),

$$\langle x(\tau)x(\tau')\rangle_0 = G_E(\tau-\tau') = \frac{1}{2\omega}e^{-\omega|\tau-\tau'|} = \quad \tau \text{ ——— } \tau' \tag{105}$$

one immediately finds



$$= \int_{-\beta/2}^{\beta/2} d\tau\, G_E^2(0) = \frac{\beta}{4\omega^2}\ . \tag{106}$$

Thus, to this perturbative order one gets

$$Z_E[0] = \langle 1\rangle_{U,0}\left[1 - \frac{\lambda}{4!}\left[3\frac{\beta}{4\omega^2}\right] + \cdots\right] = \langle 1\rangle_{U,0}\ e^{-\frac{\beta\lambda}{32\omega^2}+\cdots} \tag{107}$$

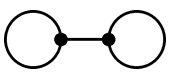
and comparing with eq. (103) one finds

$$\Delta E_0 = \frac{1}{32}\frac{\lambda}{\omega^2}\ . \tag{108}$$

Similarly one may consider the case with $g \neq 0$ and $\lambda = 0$. The first non vanishing correction is obtained from

$$\begin{aligned} Z_E[0] &= \langle 1\rangle_U = \left\langle \left(1 - S_{E,int} + \frac{1}{2}S_{E,int}^2 + \cdots\right)\right\rangle_{U,0} \qquad (109)\\ &= \langle 1\rangle_{U,0} - \frac{g}{3!}\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau\, \langle x^3(\tau)\rangle_{U,0} \\ &\quad + \frac{1}{2}\left(\frac{g}{3!}\right)^2 \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau'\, \langle x^3(\tau)x^3(\tau')\rangle_{U,0} + \cdots \\ &= \langle 1\rangle_{U,0}\left[1 + 0 + \frac{1}{2}\left(\frac{g}{3!}\right)^2\left[3!\times \right.\right. \end{aligned}$$

$$\left.\left. + 3^2 \times \right.\right.$$

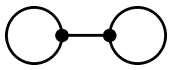


$$\left.\left.\right]\right] + \cdots\Bigg]\ .$$

Now

$$\begin{aligned} &= \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau'\, G_E^3(\tau-\tau') = \frac{1}{8\omega^3}\int_{-\beta/2}^{\beta/2} d\tau \int_{-\infty}^{\infty} d\sigma\, e^{-3\omega|\sigma|} \\ &= \frac{\beta}{8\omega^3}\frac{2}{3\omega} \end{aligned} \tag{110}$$

and



$$= \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' \, G_E(0) G_E(\tau - \tau') G_E(0)$$

$$= \frac{1}{8\omega^3} \int_{-\beta/2}^{\beta/2} d\tau \int_{-\infty}^{\infty} d\sigma \, e^{-\omega|\sigma|} = \frac{\beta}{8\omega^3} \frac{2}{\omega} \tag{111}$$

where the limit $\beta \to \infty$ has been used suitably to calculate the integrals. Therefore

$$Z_E[0] = \langle 1 \rangle_{U,0} \left[1 + \frac{1}{2} \Big(\frac{g}{3!} \Big)^2 \Big(3! \frac{\beta}{12\omega^4} + 3^2 \frac{\beta}{4\omega^4} \Big) + \cdots \right] = \langle 1 \rangle_{U,0} \; e^{\beta \frac{11}{8(3!)^2} \frac{g^2}{\omega^4} + \cdots} \tag{112}$$

and one finds

$$\Delta E_0 = -\frac{11}{288} \frac{g^2}{\omega^4} \; . \tag{113}$$

6 Path integral for fermions

We now discuss how to extend the path integral method to fermionic systems. Fermions at the classical level can be described by Grassmann variables, also known as anticommuting numbers or fermionic variables. Grassmann variables allow to define "classical" models whose quantization produces degrees of freedom that satisfy the *Pauli exclusion principle*. Models with Grassmann variables are often called "pseudoclassical", as the spin at the classical level is just a formal construction (the value of any finite spin vanishes for $\hbar \to 0$, and thus cannot be measured classically).

In the following we first exemplify the use of Grassmann variables in mechanical models. The method extends to field theories as well, so that a Dirac field can be treated classically with Grassmann variables. Then we develop canonical quantization for mechanical models containing Grassmann variables. At last, we derive a path integral representation of the transition amplitude for fermionic systems starting from its operatorial expression and using a suitable definition of fermionic coherent states.

6.1 Grassmann algebras

A n-dimensional Grassmann algebra $\mathcal{G}_n$ is generated by a set of generators θ_i with $i = 1, ..., n$ that satisfy

$$\theta_i \theta_j + \theta_j \theta_i = 0 \tag{114}$$

or, equivalently, in terms of the anticommutator

$$\{\theta_i, \theta_j\} = 0 \; . \tag{115}$$

In particular any fixed generator squares to zero

$$\theta_i^2 = 0 \tag{116}$$

suggesting already at the classical level the essence of the Pauli exclusion principle, according to which one cannot put two identical fermions in the same quantum state. Physicists often call these generators *anticommuting numbers*.

Functions. One can multiply these generators and their products by real or complex numbers, and form polynomials that are used to define functions of the Grassmann variables (i.e. the elements of the Grassmann algebra). For example, for $n = 1$ there is only one Grassmann variable θ and an arbitrary function is given by

$$f(\theta) = f_0 + f_1\theta \tag{117}$$

where f_0 and f_1 are taken to be either real or complex numbers. Similarly, for $n = 2$ one has

$$f(\theta_1, \theta_2) = f_0 + f_1\theta_1 + f_2\theta_2 + f_3\theta_1\theta_2 \ . \tag{118}$$

A term with $\theta_2\theta_1$ is not written as it is not independent of $\theta_1\theta_2$, as $\theta_2\theta_1 = -\theta_1\theta_2$. Terms with an even number of θ's are called Grassmann even (or equivalently: even, commuting, bosonic). Terms with an odd number of θ's are called Grassmann odd (or equivalently: odd, anticommuting, fermionic). Generic functions are always defined in terms of their Taylor expansions, which contain a finite number of terms because of the Grassmann property. For example, the exponential function e^θ means just $e^\theta = 1 + \theta$ because $\theta^2 = 0$.

Derivatives. Derivatives with respect to Grassmann variables are very simple. As any function can be at most linear with respect to a fixed Grassmann variable, its derivative is straightforward, and one has to keep track just of signs. Left derivatives are defined by removing the variable from the left of its Taylor expansion: for example for the function $f(\theta_1, \theta_2)$ given above

$$\frac{\partial_{_L} f(\theta_1, \theta_2)}{\partial\theta_1} = f_1 + f_3\theta_2 \tag{119}$$

since θ_1 is removed from the left. Similarly, right derivatives are obtained by removing the variable from the right

$$\frac{\partial_{_R} f(\theta_1, \theta_2)}{\partial\theta_1} = f_1 - f_3\theta_2 \tag{120}$$

where a minus sign emerges because one has first to commute θ_1 past θ_2. Equivalently, using Grassmann increments $\delta\theta$, one may write

$$\delta f = \delta\theta \frac{\partial_{_L} f}{\partial\theta} = \frac{\partial_{_R} f}{\partial\theta}\delta\theta \tag{121}$$

which recall how to keep track of signs. If not specified otherwise, we use left derivatives and omit the corresponding subscript.

Integrals. Integration can be defined, according to Berezin, to be identical with differentiation

$$\int d\theta \equiv \frac{\partial_{_L}}{\partial\theta} \ . \tag{122}$$

This definition has the virtue of producing a translational invariant measure, that is

$$\int d\theta f(\theta + \eta) = \int d\theta f(\theta) \ . \tag{123}$$

This statement is easily proven by a direct calculation

$$\int d\theta f(\theta + \eta) = \int d\theta \ (f_0 + f_1\theta + f_1\eta) = f_1 = \int d\theta f(\theta) \ . \tag{124}$$

Then, the same result is obtained by using the property of translational invariance

$$\int d\theta f(\theta+\eta) = \int d(\theta+\eta) f(\theta+\eta) = \int d\tilde{\theta} f(\tilde{\theta}) \tag{125}$$

where $\tilde{\theta} = \theta + \eta$, as the measure $d\theta = d(\theta+\eta)$ is translational invariant,

Reality properties. Grassmann variables can be defined to be either real or complex. A real variable satisfies

$$\bar{\theta} = \theta \tag{126}$$

with the bar indicating complex conjugation. For products of Grassmann variables the complex conjugate is defined to include an exchange of their position

$$\overline{\theta_1\theta_2} = \bar{\theta}_2\bar{\theta}_1 \ . \tag{127}$$

Thus the complex conjugate of the product of two real variables is purely imaginary

$$\overline{\theta_1\theta_2} = -\theta_1\theta_2 \ . \tag{128}$$

It is the combination $i\theta_1\theta_2$ that is real, as the complex conjugate of the imaginary unit carries the additional minus sign to obtain a formally real object

$$\overline{i\theta_1\theta_2} = -i\theta_2\theta_1 = i\theta_1\theta_2 \ . \tag{129}$$

Complex Grassmann variables η and $\bar{\eta}$ can always be decomposed in terms of two real Grassmann variables θ_1 and θ_2 by setting

$$\eta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \ , \qquad \bar{\eta} = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2) \ . \tag{130}$$

These are the definitions that are most useful for physical applications, since one requires that real variables become hermitian operators upon quantization.

Gaussian integration. Having defined integration over Grassmann variables, we consider in more details the gaussian integration, which is at the core of fermonic path integrals. For the case of a single real Grassmann variable θ the gaussian function is trivial, $e^{-a\theta^2} = 1$, since $\theta^2 = 0$ as θ anticommutes with itself. One needs at least two real Grassmann variables θ_1 and θ_2 to have a nontrivial exponential function with an exponent quadratic in Grassmann variables

$$e^{-a\theta_1\theta_2} = 1 - a\theta_1\theta_2 \tag{131}$$

where a is either a real or complex number. With the above definitions the corresponding "gaussian integral" is computed straghtforwardly

$$\int d\theta_1 d\theta_2 \, e^{-a\theta_1\theta_2} = a \ . \tag{132}$$

Note that there is a precise sign defined by the chosen measure, as $\int d\theta_1 d\theta_2 = -\int d\theta_2 d\theta_1$. Defining the antisymmetric 2×2 matrix A^{ij} by

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \ , \qquad \det A = a^2 \tag{133}$$

one may rewrite the result of the Grassmann gaussian integration as

$$\int d\theta_1 d\theta_2 \, e^{-\frac{1}{2}\theta_i A^{ij}\theta_j} = \det{}^{\frac{1}{2}} A \, . \tag{134}$$

The square root of the determinant of an antisymmetric matrix A is called the pfaffian, and often indicated by 'Pfaff$A$'. Indeed the determinant is always positive definite for real antisymmetric matrices, and its square root is well-defined (by analytic extensions it is also well-defined for antisymmetric matrices with complex entries). It is easy to see that the above formula extends to an even number $n = 2m$ of real Grassmann variables, so that one may write in general

$$\int d^n\theta \, e^{-\frac{1}{2}\theta_i A^{ij}\theta_j} = \det{}^{\frac{1}{2}} A \tag{135}$$

with the measure normalized as $d^n\theta \equiv d\theta_1 d\theta_2 \dots d\theta_n$. To prove this, one notice that with an orthogonal transformation one can skew-diagonalize the antisymmetric matrix A^{ij} and put it in the form

$$\begin{pmatrix} 0 & a_1 & & & & & & \\ -a_1 & 0 & & & & & & \\ & & 0 & a_2 & & & & \\ & & -a_2 & 0 & & & & \\ & & & & . & & & \\ & & & & & . & & \\ & & & & & & 0 & a_m \\ & & & & & & -a_m & 0 \end{pmatrix} . \tag{136}$$

The orthogonal transformation leaves the integration measure invariant and thus one gets the above result with $\det{}^{\frac{1}{2}} A = a_1 a_2 \cdots a_m$. A cautionary note: to compute correctly the jacobian under a change of variables one should recall the definition of the integration in terms of derivatives (the Berezin integration), and thus find an inverse matrix with respect to the one associated with an analogous bosonic integral.

In a similar way, one finds that gaussian integration over complex Grassmann variables (η_i, $\bar{\eta}_i$) produce a determinant

$$\int d^n\bar{\eta} d^n\eta \, e^{-\bar{\eta}_i A^{ij}\eta_j} = \det A \tag{137}$$

where the measure is now defined by $d^n\bar{\eta} d^n\eta \equiv d\bar{\eta}_1 d\eta_1 d\bar{\eta}_2 d\eta_2 \dots d\bar{\eta}_n d\eta_n$.

For applications to dynamical models and subsequent path integral quantization, it is useful to consider infinite dimensional Grassmann algebras ($n \to \infty$). Then one may use Grassmann valued functions of time, i.e. $\theta_i \rightsquigarrow \theta(t)$. For different values of t one has different generators of the algebra, so that properties such as $\theta^2(t) = 0$ and $\theta(t_1)\theta(t_2) = -\theta(t_2)\theta(t_1)$ hold. They are used to introduce interesting mechanical systems at the classical level, that upon quantization produce systems satisfying the Pauli exclusion principle (and the Fermi-Dirac statistic).

6.2 Pseudoclassical models and canonical quantization

To gain some familiarity with the use of Grassmann numbers as dynamical variables we consider the fermionic harmonic oscillator, an example that by itself contains the essence of all fermionic systems.

Classically, it is described by the function $\psi(t)$ and its complex conjugate $\bar\psi(t)$ that take values in a Grassmann algebra (t denotes the time). The Grassmann property implies generic relations like $\psi(t)\psi(t)=0$, $\psi(t)\psi(t')=-\psi(t')\psi(t)$, $\dot\psi(t)\dot\psi(t)=0$, $\psi(t)\dot\psi(t)=-\dot\psi(t)\psi(t)$, etc., where dots denote time derivatives, $\dot\psi=\frac{d}{dt}\psi$. These relations can be used in extremizing the action, testing the presence of symmetries, and so on.

Before introducing the action of the fermionic harmonic oscillator, and gaining some intuition, let us rewrite the action of the usual bosonic harmonic oscillator in phase space by using complex combinations of the coordinate and momentum (x,p), defined by

$$a=\frac{1}{\sqrt{2\omega}}(\omega x+ip)\ ,\qquad \bar a=\frac{1}{\sqrt{2\omega}}(\omega x-ip)\ . \tag{138}$$

Up to boundary terms, one finds

$$S[x,p]=\int dt\Big(p\dot x-\frac{1}{2}(p^2+\omega^2x^2)\Big)\quad\to\quad S[a,\bar a]=\int dt\ (i\bar a\dot a-\omega\bar a a)\ . \tag{139}$$

Quantization of the complex variables $(a,\bar a)$ gives rise to the annhilation/creation operators $(\hat a,\hat a^\dagger)$ that satisfy the algebra $[\hat a,\hat a^\dagger]=\hbar$ (however we will use mostly $\hbar=1$). They are used in the Fock construction of the Hilbert space of the harmonic oscillator, which for convenience is reviewed later on in sec. 6.3.

The dynamics of the fermionic harmonic oscillator is similarly described by complex Grassmann valued functions $\psi(t)$ and $\bar\psi(t)$ and fixed by the action

$$S[\psi,\bar\psi]=\int dt\ \Big(i\bar\psi\dot\psi-\omega\bar\psi\psi\Big)\ . \tag{140}$$

The action is formally real (up to boundary terms), just like its bosonic cousin in (139). The equations of motion are obtained by extremizing the action, and easily solved

$$i\dot\psi-\omega\psi=0\qquad\Longrightarrow\qquad \psi(t)=\psi_0\ e^{-i\omega t} \tag{141}$$

where ψ_0 is a suitable initial datum. This equation may be called the Dirac equation in a 0+1 dimensional spacetime, as one may rewrite it as $(\gamma^0\partial_0+m)\psi=0$, with $\gamma^0=-i$, $x^0=t$, and $\omega=m$ playing the role of the Dirac mass[3].

Canonical quantization is achieved by considering the hamiltonian structure of the model. We sketch it first, postponing for a while a proper discussion of the phase space with Grassmann variables. The momentum π conjugate to ψ is defined by

$$\pi\equiv\frac{\partial_L L}{\partial\dot\psi}=-i\bar\psi \tag{142}$$

which shows that the systems is already in a hamiltonian form, the conjugate momenta being $\bar\psi$ up to a factor. The classical Poisson bracket $\{\pi,\psi\}_{PB}=-1$ is rewritten as $\{\psi,\bar\psi\}_{PB}=-i$, and has the property of being symmetric (this fact will be discussed in a short while).

Quantizing with anticommutators (fermionic system must be treated this way) one obtains

$$\{\hat\psi,\hat\psi^\dagger\}=\hbar\ ,\quad \{\hat\psi,\hat\psi\}=\{\hat\psi^\dagger,\hat\psi^\dagger\}=0 \tag{143}$$

[3]In higher dimensions one may write the Dirac equation as $(\gamma^\mu\partial_\mu+m)\psi$=0.

that is, the classical variables ψ and $\bar{\psi}$ are promoted to linear operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ satisfying anticommutation relations that are set to be equal to $i\hbar$ times the value of the classical Poisson brackets. Setting $\hbar = 1$ for simplicity, one finds the fermionic creation/annihilation algebra

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = 1 \ , \quad \{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0 \tag{144}$$

that can be realized in a two dimensional Hilbert space, and with the correct hermiticity properties. The Hilbert is explicitly constructed à la Fock, considering $\hat{\psi}$ as destruction operator and $\hat{\psi}^\dagger$ as creation operator[4]. One starts defining the Fock vacuum $|0\rangle$, fixed by the condition $\hat{\psi}|0\rangle = 0$. A second state is obtained acting with $\hat{\psi}^\dagger$

$$|1\rangle = \hat{\psi}^\dagger|0\rangle \ . \tag{145}$$

No other states can be obtained acting again with the creation operator $\hat{\psi}^\dagger$ as $(\hat{\psi}^\dagger)^2 = 0$. Normalizing the Fock vacuum to unity, $\langle 0|0\rangle = 1$, with $\langle 0| = |0\rangle^\dagger$, one finds that these two states are orthonomal

$$\langle m|n\rangle = \delta_{mn} \qquad m, n = 0, 1 \tag{146}$$

and span a two-dimensional Hilbert space, $\mathcal{F}_2 = \text{Span}\{|0\rangle, |1\rangle\}$. In terms of matrices one computes matrix elements and finds the realization

$$\begin{aligned} \hat{\psi} \ &\longrightarrow \ \begin{pmatrix} \langle 0|\hat{\psi}|0\rangle & \langle 0|\hat{\psi}|1\rangle \\ \langle 1|\hat{\psi}|0\rangle & \langle 1|\hat{\psi}|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & \langle 0|0\rangle \\ 0 & \langle 1|0\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \hat{\psi}^\dagger \ &\longrightarrow \ \begin{pmatrix} \langle 0|\hat{\psi}^\dagger|0\rangle & \langle 0|\hat{\psi}^\dagger|1\rangle \\ \langle 1|\hat{\psi}^\dagger|0\rangle & \langle 1|\hat{\psi}^\dagger|1\rangle \end{pmatrix} = \begin{pmatrix} \langle 0|1\rangle & 0 \\ \langle 1|1\rangle & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ . \end{aligned} \tag{147}$$

The Fock vacuum is indeed the ground state of the fermionic oscillator, whose quantum Hamiltonian $\hat{H} = \omega(\hat{\psi}^\dagger\hat{\psi} - \frac{1}{2})$ is obtained from the classical one by choosing a "symmetric" ordering of the operators upon quantization. Let us see this in more details. The classical hamiltonian is given by the Legendre transform

$$H = \dot{\psi}\pi - L = \omega\bar{\psi}\psi = \frac{\omega}{2}(\bar{\psi}\psi - \psi\bar{\psi}) \ . \tag{148}$$

The last form is a classically equivalent way of writing it, and is the one that is quantized to resolve the ordering ambiguities

$$\hat{H} = \frac{\omega}{2}(\hat{\psi}^\dagger\hat{\psi} - \hat{\psi}\hat{\psi}^\dagger) = \omega\Big(\hat{\psi}^\dagger\hat{\psi} - \frac{1}{2}\Big) \tag{149}$$

where the first relation in (144) has been used. Note also that in the Legendre transform the order of $\dot{\psi}$ and π matters, and we have used the one that follows from having defined the conjugate momentum (142) with left derivatives.

Hamiltonian structure and canonical quantization

Path integrals for fermions can be derived from the canonical formalism, just as in the bosonic case. For that let us first review the hamiltonian formalism and the canonical quantization of mechanical systems with Grassmann variables.

[4]Their role could also be reversed in fermionic systems, a property not shared by bosonic systems.

The hamiltonian formalism aims at producing equations of motion that are first order differential equations in time. For a simple bosonic model with phase space coordinates (x, p), the phase space action is usually written in the form

$$S[x,p] = \int dt \Big(p\dot{x} - H(x,p) \Big) \ . \tag{150}$$

The first term with derivatives (the $p\dot{x}$ term) is called the symplectic term, and fixes the Poisson bracket structure of phase space. Up to total derivatives it can be written in a more symmetrical form, with the time derivatives shared equally by x and p

$$S[x,p] = \int dt \Big(\frac{1}{2}(p\dot{x} - x\dot{p}) - H(x,p) \Big) = \int dt \Big(\frac{1}{2} z^a (\Omega^{-1})_{ab} \dot{z}^b - H(z) \Big) \tag{151}$$

where we have denoted collectively the phase space coordinates by $z^a = (z^1, z^2) = (x, p)$. The symplectic term contains the constant invertible matrix

$$(\Omega^{-1})_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{152}$$

with inverse

$$\Omega^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{153}$$

It is an antisymmetric matrix, as a symmetric matrix would give terms in the action that are total derivatives and can be dropped (they would not modify the equations of motion). It is used to define the Poisson bracket between two generic phase space functions F and G

$$\{F, G\}_{PB} = \frac{\partial F}{\partial z^a} \Omega^{ab} \frac{\partial G}{\partial z^b} \ . \tag{154}$$

In particular, one finds for the phase space coordinates

$$\{z^a, z^b\}_{PB} = \Omega^{ab} \ . \tag{155}$$

This coincides with the standard definitions. The Poisson bracket satisfies the following properties

$$\begin{aligned} &\{F, G\}_{PB} = -\{G, F\}_{PB} \qquad \text{(antisymmetry)} \\ &\{F, GH\}_{PB} = \{F, G\}_{PB} H + G\{F, H\}_{PB} \qquad \text{(Leibniz rule)} \\ &\{F, \{G, H\}_{PB}\}_{PB} + \{G, \{H, F\}_{PB}\}_{PB} + \{H, \{F, G\}_{PB}\}_{PB} = 0 \qquad \text{(Jacobi identity)} \ . \end{aligned} \tag{156}$$

These properties make it consistent to adopt the canonical quantization rules of substituting the fundamental variables z^a by linear operators $\hat{z}^a$ acting on a Hilbert space of physical states, with commutation relations fixed to be $i\hbar$ times the value of the classical Poisson brackets

$$[\hat{z}^a, \hat{z}^b] = i\hbar \Omega^{ab} \ . \tag{157}$$

This prescription[5] is consistent as both sides satisfy the same algebraic properties, listed in (156) for the Poisson brackets. At this stage, once one has found an irreducible representation

[5] More generally, phase space functions $F(z)$ are elevated to operators $\hat{F}(\hat{z})$ (after fixing ordering ambiguities) with commutation relations that take the form $[\hat{F}(\hat{z}), \hat{G}(\hat{z})] = i\hbar\{F, G\}_{PB} +$ higher order terms in $\hbar$.

of the operator algebra (157) on a well-defined vector space (Hilbert space), the process of canonical quantization has been achieved.

This set up can be extended to models with Grassmann variables. The basic structure remains unaltered, but one must take care of signs arising from the anticommuting sector. Let us show how this is done.

We denote collectively the phase space coordinates by $Z^A = (x^i, p_i, \theta^\alpha)$, with (x^i, p_i) the usual Grassmann even phase space variables and θ^α the Grassmann odd variables. We consider a phase space action of the form

$$S[Z^A] = \int dt \left(\frac{1}{2} Z^A (\Omega^{-1})_{AB} \dot{Z}^B - H(Z) \right) \tag{158}$$

where the symplectic term depends on a constant invertible matrix $(\Omega^{-1})_{AB}$ with inverse Ω^{AB}. Again this term must be written splitting the time derivatives democratically between all variables, as in (151). The symplectic term and the hamiltonian are taken to be Grassmann even (i.e. commuting objects), so that the whole action is of bosonic nature. Then, it is seen that Ω^{AB} is *antisymmetric* in the sector related to the bosonic coordinates, and *symmetric* in the sector belonging to the Grassmann variables (other off-diagonal entries vanish as the action is taken to be commuting). Denoting the variables by $Z^A = (z^a, \theta^\alpha)$ with z^a bosonic and θ^α fermionic, the matrix Ω^{AB} (as well as its inverse) has a block diagonal form

$$\Omega^{AB} = \begin{pmatrix} \Omega^{ab} & 0 \\ 0 & \Omega^{\alpha\beta} \end{pmatrix} \tag{159}$$

with Ω^{ab} antisymmetric and $\Omega^{\alpha\beta}$ symmetric.

The matrix Ω^{AB} is used to define the Poisson bracket by

$$\{F, G\}_{PB} = \frac{\partial_R F}{\partial Z^A} \Omega^{AB} \frac{\partial_L G}{\partial Z^B} \tag{160}$$

where both right and left derivatives are used. In particular one finds

$$\{Z^A, Z^B\}_{PB} = \Omega^{AB} \ . \tag{161}$$

Phase space functions are usually restricted to have a definite Grassmann parity. Given any such function F, we denote its Grassmann parity by $(-1)^{\epsilon_F}$, where $\epsilon_F = 0$ if F is Grassmann even (bosonic function) and $\epsilon_F = 1$ if F is Grassmann odd (fermionic function). Then, one finds that the definition (160) satisfies a graded generalization of the properties in (156), namely

$$\begin{aligned} &\{F, G\}_{PB} = (-1)^{\epsilon_F \epsilon_G + 1} \{G, F\}_{PB} \\ &\{F, GH\}_{PB} = \{F, G\}_{PB} H + (-1)^{\epsilon_F \epsilon_G} G \{F, H\}_{PB} \\ &\{F, \{G, H\}_{PB}\}_{PB} + (-1)^{\epsilon_F(\epsilon_G + \epsilon_H)} \{G, \{H, F\}_{PB}\}_{PB} + (-1)^{\epsilon_H(\epsilon_F + \epsilon_G)} \{H, \{F, G\}_{PB}\}_{PB} = 0 \ . \end{aligned} \tag{162}$$

The equations of motion are first order in time. They can be derived minimizing the action and can be expressed in term of the Poisson brackets

$$\dot{Z}^A = \Omega^{AB} \frac{\partial_L H}{\partial Z^B} \qquad \rightarrow \qquad \dot{Z}^A = \{Z^A, H\}_{PB} \ . \tag{163}$$

These are the Hamilton's equations of motion.

The properties of the Poisson brackets make it consistent to adopt the canonical quantization rules, that consist in promoting the phase space coordinates Z^A to operators $\hat{Z}^A$ with commutation/anticommutation relation fixed by their classical Poisson brackets

$$[\hat{Z}^A, \hat{Z}^B\} = i\hbar\{Z^A, Z^B\}_{PB} = i\hbar\Omega^{AB} \tag{164}$$

where we have employed the compact notation

$$[\cdot\,,\cdot\} = \begin{cases} \{\cdot\,,\cdot\} & \text{anticommutator if both variables are fermionic} \\ [\cdot\,,\cdot] & \text{commutator otherwise} \end{cases} \tag{165}$$

often called "graded commutator". The graded commutator satisfies identities similar to those for the Poisson brackets in (162), and makes it consistent to adopt the given quantization rules.

This quick exposition becomes clearer by working through simple examples.

Examples

(i) Single real Grassmann variable ψ ("single Majorana fermion in one dimension").
Taking as phase space lagrangian

$$L = \frac{i}{2}\psi\dot{\psi} - H(\psi) \tag{166}$$

which is formally real and produces equation of motion of the first order in time, one finds $\Omega^{-1} = i$, $\Omega = -i$, and Poisson bracket at equal times $\{\psi, \psi\}_{PB} = -i$. The dynamical variable $\psi(t)$ is often called a Majorana fermion in one dimension, as it satisfies the Dirac equation in one dimension plus a reality condition (akin to the Majorana condition used in four dimensions). One notices that the only possible Grassmann even hamiltonian is a constant, so that the model is rather trivial. One verifies in this example that the phase space can be odd dimensional if Grassmann variables are present. The model is quantized by introducing the hermitian operator $\hat{\psi}$ with anticommutator

$$\{\hat{\psi}, \hat{\psi}\} = \hbar\,. \tag{167}$$

The quantum theory is also trivial, as one represents irreducibly this algebra in a one dimensional Hilbert space, with the operator $\hat{\psi}$ acting as multiplication by the constant $\sqrt{\hbar/2}$. This Hilbert space has no room for any nontrivial dynamics, as there is only the vacuum state.

(ii) Several real Grassmann variables ψ^i ("Majorana fermions in one dimension").
For the case of several real Grassmann variables one may take as phase space lagrangian

$$L = \frac{i}{2}\psi^i\dot{\psi}^i - H(\psi^i) \qquad i = 1, ..., n \tag{168}$$

and one finds $(\Omega^{-1})_{ij} = i\delta_{ij}$, $\Omega^{ij} = -i\delta^{ij}$. The Poisson brackets at equal times read as $\{\psi^i, \psi^j\}_{PB} = -i\delta^{ij}$. Quantization is obtained by considering the anticommutation relations

$$\{\hat{\psi}^i, \hat{\psi}^j\} = \hbar\delta^{ij} \tag{169}$$

which is recognized to be proportional to the Clifford algebra of the gamma matrices, appearing in the Dirac equation in n euclidean dimensions. Indeed setting $\hat{\psi}^i = \sqrt{\hbar/2}\,\gamma^i$ turns the above anticommutation relations into the Clifford algebra

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \tag{170}$$

which is the defining properties of the gamma matrices of the Dirac equation

$$(\gamma^i \partial_i + m)\Psi(x) = 0 \ . \tag{171}$$

It is known that the algebra (170) is realized on a complex vector space of dimension $2^{[\frac{n}{2}]}$, where $[\frac{n}{2}]$ indicates the integer part of $\frac{n}{2}$. For example, for $n = 2$ and $n = 3$ the gamma matrices are 2 by 2, for $n = 4$ and $n = 5$ the gamma matrices are 4 by 4, for $n = 6$ and $n = 7$ the gamma matrices are 8 by 8, etc.. One concludes that the operators $\hat{\psi}^i$ are realised as hermitian operators in a Hilbert space of dimensions $2^{[\frac{n}{2}]}$.

(iii) Complex Grassmann variables ψ and $\bar{\psi}$ ("single Dirac fermion in one dimension"). Taking as phase space lagrangian

$$L = i\bar{\psi}\dot{\psi} - H(\psi, \bar{\psi}) \tag{172}$$

one finds $\{\psi, \bar{\psi}\}_{PB} = -i$ as the only nontrivial Poisson bracket between the phase space coordinates $(\psi, \bar{\psi})$. It is quantized by the anticommutator $\{\hat{\psi}, \hat{\psi}^\dagger\} = \hbar$, producing a fermionic annihilation/creation algebra. It is realized in a two dimensional Fock space, as anticipated earlier while discussing the fermionic harmonic oscillator. This model is equivalent to that with two real (Majorana) fermions, seen as the real and the imaginary part of the Dirac fermion. Also, one may straightforwardly consider the theory of a set of several Dirac fermions in one dimension.

These basic examples can be used to construct explicitly the irreducible representations of the gamma matrices in arbitrary dimensions, and check their dimensionality as anticipated above. One proceeds as follows. In even dimensions $n = 2m$ one combines the $2m$ Majorana worldline fermions, corresponding to the gamma matrices, into m pairs of worldline Dirac fermions, that generate a set of m copies of independent, anticommuting creation/annihilation operators. The latter act on the tensor products of m two-dimensional fermionic Fock spaces, each one realizing an independent set of fermionic creation/annihilation operators. This gives a total Hilbert space of 2^m dimensions: indeed for each set of creation/annihilation operators a state can only be empty or filled with the corresponding fermionic excitation. This is in accord with the assertion given above about the dimensionality of the gamma matrices. Adding an extra Majorana fermion corresponds to a Clifford algebra in odd dimensions (i.e. $2m + 1$ dimensions): the related dimension of the Hilbert space does not change as the last Majorana fermion can be realized as proportional to the chirality matrix of the $2m$ dimensional case, which always exists.

6.3 Coherent states

It is useful to introduce coherent states, an overcomplete basis of vectors for the fermionic Fock space described previously, for deriving a path integral for fermionic systems. They provide ket eigenstates of the fermionic operator $\hat{\psi}$ with Grassmann valued eigenvalues. Together with a resolution of the identity, they allow to convert the matrix elements of the quantum evolution operator (transition amplitudes) into a path integral where one sums over Grassmann valued functions. We first review the construction of bosonic coherent states, used in the theory of the harmonic oscillator, as a guide on the construction in the fermionic case.

In the theory of the harmonic oscillator one introduces coherent states defined as eigenstates of the annihilation operator $\hat{a}$. Let us recall the algebra of the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$

$$[\hat{a}, \hat{a}^\dagger] = 1 \ , \quad [\hat{a}, \hat{a}] = 0 \ , \quad [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \ . \tag{173}$$

It is realized by operators acting on an infinite dimensional Hilbert space, identified with a Fock space constructed as follows. A complete orthonormal basis of the Fock space is obtained by starting from the Fock vacuum $|0\rangle$, defined by the condition $\hat{a}|0\rangle = 0$. The other states of the basis are obtained by acting with the creation operator $\hat{a}^\dagger$ an arbitrary number of times on the Fock vacuum $|0\rangle$

$$\begin{aligned}
&|0\rangle \qquad \text{such that } \hat{a}|0\rangle = 0 \\
&|1\rangle = \hat{a}^\dagger|0\rangle \\
&|2\rangle = \frac{\hat{a}^\dagger}{\sqrt{2}}|1\rangle = \frac{(\hat{a}^\dagger)^2}{\sqrt{2!}}|0\rangle \\
&|3\rangle = \frac{\hat{a}^\dagger}{\sqrt{3}}|2\rangle = \frac{(\hat{a}^\dagger)^3}{\sqrt{3!}}|0\rangle \\
&\ldots\ldots \\
&|n\rangle = \frac{\hat{a}^\dagger}{\sqrt{n}}|n-1\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \\
&\ldots\ldots
\end{aligned} \tag{174}$$

Normalizing the Fock vacuum to unit norm, $\langle 0|0\rangle = 1$, where $\langle 0| = |0\rangle^\dagger$, one finds that these states are orthonomal

$$\langle m|n\rangle = \delta_{mn} \qquad m, n = 0, 1, 2, \ldots \; . \tag{175}$$

Now, choosing a complex number a, one builds a coherent state $|a\rangle$ defined by

$$|a\rangle = e^{a\hat{a}^\dagger}|0\rangle \; . \tag{176}$$

It is an eigenstate of the annihilation operator $\hat{a}$

$$\hat{a}|a\rangle = a|a\rangle \; . \tag{177}$$

A way of proving this is by expanding the exponential and viewing $|a\rangle$ as an infinite sum with suitable coefficients of the basis vectors of the Fock space

$$\begin{aligned}
|a\rangle &= e^{a\hat{a}^\dagger}|0\rangle \\
&= \Big(1 + a\hat{a}^\dagger + \frac{1}{2!}(a\hat{a}^\dagger)^2 + \frac{1}{3!}(a\hat{a}^\dagger)^3 + \cdots + \frac{1}{n!}(a\hat{a}^\dagger)^n + \cdots\Big)|0\rangle \\
&= |0\rangle + a|1\rangle + \frac{a^2}{\sqrt{2!}}|2\rangle + \frac{a^3}{\sqrt{3!}}|3\rangle + \cdots + \frac{a^n}{\sqrt{n!}}|n\rangle + \cdots \; .
\end{aligned} \tag{178}$$

In this form it is easy to calculate (using $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$)

$$\begin{aligned}
\hat{a}|a\rangle &= \hat{a}\Big(|0\rangle + a|1\rangle + \frac{a^2}{\sqrt{2!}}|2\rangle + \frac{a^3}{\sqrt{3!}}|3\rangle + \cdots + \frac{a^n}{\sqrt{n!}}|n\rangle + \cdots\Big) \\
&= 0 + a|0\rangle + a^2|1\rangle + \frac{a^3}{\sqrt{2!}}|2\rangle + \cdots + \frac{a^n}{\sqrt{(n-1)!}}|n-1\rangle + \cdots \\
&= a\Big(|0\rangle + a|1\rangle + \frac{a^2}{\sqrt{2!}}|2\rangle + \cdots + \frac{a^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle + \cdots\Big) \\
&= a|a\rangle \; .
\end{aligned} \tag{179}$$

A quick way of proving the same result is to recognize that the algebra (173) can be realized by

$$\hat{a}^\dagger \;\to\; \bar{a}\;, \qquad \hat{a} \;\to\; \frac{\partial}{\partial \bar{a}} \tag{180}$$

acting on functions of $\bar{a} \in \mathbb{C}$, and the result follows straighfowardly[6].

A list of properties that can be proven with similar calculations are

$$\begin{aligned}
&(i) && \langle \bar{a}| = |a\rangle^\dagger = \langle 0| e^{\bar{a}\hat{a}} && \Longrightarrow && \langle \bar{a}|\hat{a}^\dagger = \langle \bar{a}|\bar{a} \\
&(ii) && \langle \bar{a}|a\rangle = e^{\bar{a}a} && && \text{(scalar product)} \\
&(iii) && \mathbb{1} = \int \frac{d\bar{a}da}{2\pi i}\, e^{-\bar{a}a}\, |a\rangle\langle \bar{a}| && && \text{(resolution of the identity)} \\
&(iv) && \mathrm{Tr}\, \hat{A} = \int \frac{d\bar{a}da}{2\pi i}\, e^{-\bar{a}a}\, \langle \bar{a}|\hat{A}|a\rangle && && \text{(trace of the operator } \hat{A}) \,.
\end{aligned} \tag{181}$$

One should note that the set of coherent states form an over-complete basis, in particular they are not orthonormal, in fact $\langle \bar{b}|a\rangle = e^{\bar{b}a} \neq 0$. However, it is useful to keep this redundancy. Coherent states may be used to rederive a form of the path integral in phase space in terms of the so-called holomorphic trajectories, corresponding to paths for the $a(t)$ and $\bar{a}(t)$ variables. We will not present it here, but consider only the corresponding fermionic construction which is, mutatis mutandis, analogous.

Thus, let us introduce coherent states for fermionic systems. As we have seen, the algebra of the anticommutators of the fermionic creation/annihilation operators $\hat{\psi}^\dagger$ and $\hat{\psi}$ reads as

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = 1\;, \quad \{\hat{\psi}, \hat{\psi}\} = 0\;, \quad \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0 \tag{182}$$

and can be realized by 2×2 matrices acting in the two-dimensional fermionic Fock space generated by the vectors $|0\rangle$ and $|1\rangle$, defined by

$$\hat{\psi}|0\rangle = 0\;, \quad |1\rangle = \hat{\psi}^\dagger|0\rangle\;. \tag{183}$$

One defines fermionic coherent states as eigenstates $|\psi\rangle$ of the annihilation operator $\hat{\psi}$, having the complex Grassmann number ψ as eigenvalue

$$\hat{\psi}|\psi\rangle = \psi|\psi\rangle\;. \tag{184}$$

The Grassmann numbers, such as ψ and its complex conjugate $\bar{\psi}$, anticommute between themselves, and we define them to anticommute also with the fermionic operators $\hat{\psi}^\dagger$ and $\hat{\psi}$. No confusion should arise between the operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ that have a hat, and the complex Grassmann variables ψ and $\bar{\psi}$, eigenvalues of the eigenstates $|\psi\rangle$ and $\langle\bar{\psi}|$ respectively, that carry no hat (similarly, in the previous chapter, we indicated position operator, eigenstates and eigenvalues so that $\hat{x}|x\rangle = x|x\rangle$).

[6] One represents $|a\rangle$ by $|a\rangle = e^{a\bar{a}}$ and computes $\hat{a}|a\rangle = \frac{\partial}{\partial \bar{a}} e^{a\bar{a}} = a e^{a\bar{a}} = a|a\rangle$.

One can prove the following statements

$$
\begin{aligned}
&(i) && |\psi\rangle = e^{\hat{\psi}^\dagger \psi}|0\rangle \\
&(ii) && \langle\bar{\psi}| = \langle 0|e^{\bar{\psi}\hat{\psi}} \quad \Longrightarrow \quad \langle\bar{\psi}|\hat{\psi}^\dagger = \langle\bar{\psi}|\bar{\psi} \\
&(iii) && \langle\bar{\psi}|\psi\rangle = e^{\bar{\psi}\psi} \\
&(iv) && \mathbb{1} = \int d\bar{\psi}d\psi\, e^{-\bar{\psi}\psi}\, |\psi\rangle\langle\bar{\psi}| \\
&(v) && \mathrm{Tr}\, \hat{A} = \int d\bar{\psi}d\psi\, e^{-\bar{\psi}\psi}\, \langle -\bar{\psi}|\hat{A}|\psi\rangle \\
&(vi) && \mathrm{STr}\, \hat{A} \equiv \mathrm{Tr}[(-1)^{\hat{F}}\hat{A}] = \int d\bar{\psi}d\psi\, e^{-\bar{\psi}\psi}\, \langle\bar{\psi}|\hat{A}|\psi\rangle
\end{aligned}
\tag{185}
$$

where $\hat{A}$ is an arbitrary bosonic operator, and $\hat{F} = \hat{\psi}^\dagger\hat{\psi}$ the fermion number operator.

The proofs are obtained by explicit calculation. Let us proceed systematically.

(i) One expands the exponential and write the coherent state as

$$
\begin{aligned}
|\psi\rangle &= e^{\hat{\psi}^\dagger\psi}|0\rangle \\
&= (1+\hat{\psi}^\dagger\psi)|0\rangle = |0\rangle - \psi\hat{\psi}^\dagger|0\rangle \\
&= |0\rangle - \psi|1\rangle
\end{aligned}
\tag{186}
$$

and computes

$$
\begin{aligned}
\hat{\psi}|\psi\rangle &= \hat{\psi}e^{\hat{\psi}^\dagger\psi}|0\rangle \\
&= \hat{\psi}\Big(|0\rangle - \psi|1\rangle\Big) = -\hat{\psi}\psi|1\rangle = \psi\hat{\psi}|1\rangle = \psi|0\rangle = \psi\Big(|0\rangle - \psi|1\rangle\Big) \\
&= \psi|\psi\rangle
\end{aligned}
\tag{187}
$$

which proves that $|\psi\rangle$ is a coherent state. Note that terms proportional to ψ^2 can be inserted or eliminated at wish, as they vanish due to Grassmann property $\psi^2 = 0$.

(ii) "Bra" coherent state. To prove this relation it is sufficient to take the hermitian conjugate of the ket coherent state $|\psi\rangle$. One must remember that the definition of hermitian conjugate reduces to complex conjugation for Grassmann variables and reverses the positions of both variables and operators. For example

$$
(\hat{\psi}^\dagger\psi)^\dagger = \bar{\psi}\hat{\psi}\ . \tag{188}
$$

(iii) Scalar product. A direct computation (recalling that $\psi^2 = 0$, $\bar{\psi}^2 = 0$ and $\psi\bar{\psi} = -\bar{\psi}\psi$) gives

$$
\begin{aligned}
\langle\bar{\psi}|\psi\rangle &= \Big(\langle 0| - \langle 1|\bar{\psi}\Big)\Big(|0\rangle - \psi|1\rangle\Big) \\
&= \langle 0|0\rangle + \bar{\psi}\psi\langle 1|1\rangle = 1 + \bar{\psi}\psi \\
&= e^{\bar{\psi}\psi}\ .
\end{aligned}
\tag{189}
$$

We point out that the Grassmann variables are here defined to commute with the Fock vacuum $|0\rangle$, so that they commute with the coherent states, but anticommute with $|1\rangle = \hat{\psi}^\dagger|0\rangle$ (as they anticommute with $\hat{\psi}^\dagger$).

(*iv*) Resolution of the identity. First of all one must recall that the definition of integration over Grassmann variables makes it identical with differentiation. In particular, we use left differentiation, that removes the variable from the left (one must pay attention to signs arising from this operation)

$$\int d\psi \equiv \frac{\partial_{\scriptscriptstyle L}}{\partial \psi} \ , \qquad \int d\bar{\psi} \equiv \frac{\partial_{\scriptscriptstyle L}}{\partial \bar{\psi}} \ . \tag{190}$$

Now a direct calculation shows that

$$\begin{aligned} \int d\bar{\psi} d\psi \, e^{-\bar{\psi}\psi} \, |\psi\rangle\langle\bar{\psi}| &= \int d\bar{\psi} d\psi \, (1 - \bar{\psi}\psi) \Big(|0\rangle - \psi|1\rangle \Big) \Big(\langle 0| - \langle 1|\bar{\psi} \Big) \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| \ . \end{aligned} \tag{191}$$

(*v*) Trace. Given a bosonic operator $\hat{A}$, that commutes with ψ and $\bar{\psi}$, one can verify that

$$\begin{aligned} \int d\bar{\psi} d\psi \, e^{-\bar{\psi}\psi} \, \langle -\bar{\psi}|\hat{A}|\psi\rangle &= \int d\bar{\psi} d\psi \, (1 - \bar{\psi}\psi) \Big(\langle 0| + \langle 1|\bar{\psi} \Big) \hat{A} \Big(|0\rangle - \psi|1\rangle \Big) \\ &= \int d\bar{\psi} d\psi \, (1 - \bar{\psi}\psi) \Big(\langle 0|\hat{A}|0\rangle - \bar{\psi}\psi\langle 1|\hat{A}|1\rangle + \cdots \Big) \\ &= \langle 0|\hat{A}|0\rangle + \langle 1|\hat{A}|1\rangle \\ &= \mathrm{Tr} \, \hat{A} \ . \end{aligned} \tag{192}$$

(*vi*) Supertrace. An analogous calculation gives

$$\begin{aligned} \int d\bar{\psi} d\psi \, e^{-\bar{\psi}\psi} \, \langle \bar{\psi}|\hat{A}|\psi\rangle &= \int d\bar{\psi} d\psi \, (1 - \bar{\psi}\psi) \Big(\langle 0| - \langle 1|\bar{\psi} \Big) \hat{A} \Big(|0\rangle - \psi|1\rangle \Big) \\ &= \int d\bar{\psi} d\psi \, (1 - \bar{\psi}\psi) \Big(\langle 0|\hat{A}|0\rangle + \bar{\psi}\psi\langle 1|\hat{A}|1\rangle + \cdots \Big) \\ &= \langle 0|\hat{A}|0\rangle - \langle 1|\hat{A}|1\rangle = \mathrm{Tr}[(-1)^{\hat{F}} \hat{A}] \\ &= \mathrm{STr} \, \hat{A} \ . \end{aligned} \tag{193}$$

Here $\hat{F} = \hat{\psi}^\dagger \hat{\psi}$ is the fermion number operator (with eigenvalues $F = 0$ for $|0\rangle$ and $F = 1$ for $|1\rangle$). The last line gives the definition of the supertrace.

The generalization to more fermionic degrees of freedom is straightforward.

6.4 Fermionic path integrals

We now have all the tools to find a path integral representation of the transition amplitude between coherent states $\langle \bar{\psi}_f | e^{-i\hat{H}T} | \psi_i \rangle$, where we set $\hbar = 1$ for notational simplicity. We consider an hamiltonian $\hat{H} = \hat{H}(\hat{\psi}^\dagger, \hat{\psi})$ written in such a way that all creation operators are on the left of the annihilation operators, something that is always possible to achieve using the fundamental anticommutation relations in (182). Note also that for a single pair of fermionic creation/annihilation operators the most general (bosonic) hamiltonian takes the simple form $\hat{H} = \omega \hat{\psi}^\dagger \hat{\psi} + h_0$, with ω and h_0 real constants.

To turn the transition amplitude into a path integral, one divides the total propagation time T into N steps of duration $\epsilon = \frac{T}{N}$, so that $T = N\epsilon$. Using $N - 1$ times the decomposition

of the identity in terms of coherent states, one gets the following equalites

$$\begin{aligned}
\langle\bar{\psi}_f|e^{-i\hat{H}T}|\psi_i\rangle &= \langle\bar{\psi}_f|\underbrace{e^{-i\hat{H}\epsilon}e^{-i\hat{H}\epsilon}\cdots e^{-i\hat{H}\epsilon}}_{N\ \text{times}}|\psi_i\rangle \\
&= \langle\bar{\psi}_f|e^{-i\hat{H}\epsilon}\,\mathbb{1}\,e^{-i\hat{H}\epsilon}\,\mathbb{1}\cdots\mathbb{1}\,e^{-i\hat{H}\epsilon}|\psi_i\rangle \\
&= \int\Big(\prod_{k=1}^{N-1}d\bar{\psi}_k d\psi_k\,e^{-\bar{\psi}_k\psi_k}\Big)\prod_{k=1}^{N}\langle\bar{\psi}_k|e^{-i\hat{H}\epsilon}|\psi_{k-1}\rangle
\end{aligned} \tag{194}$$

where we have defined $\psi_0 \equiv \psi_i$ and $\bar{\psi}_N \equiv \bar{\psi}_f$. For $\epsilon \to 0$ one can approximate the elementary transition amplitudes as

$$\begin{aligned}
\langle\bar{\psi}_k|e^{-i\hat{H}(\hat{\psi}^\dagger,\hat{\psi})\epsilon}|\psi_{k-1}\rangle &= \langle\bar{\psi}_k|\Big(1-i\hat{H}(\hat{\psi}^\dagger,\hat{\psi})\epsilon+\cdots\Big)|\psi_{k-1}\rangle \\
&= \langle\bar{\psi}_k|\psi_{k-1}\rangle - i\epsilon\langle\bar{\psi}_k|\hat{H}(\hat{\psi}^\dagger,\hat{\psi})|\psi_{k-1}\rangle+\cdots \\
&= \Big(1-i\epsilon H(\bar{\psi}_k,\psi_{k-1})+\cdots\Big)\langle\bar{\psi}_k|\psi_{k-1}\rangle \\
&= e^{-i\epsilon H(\bar{\psi}_k,\psi_{k-1})}e^{\bar{\psi}_k\psi_{k-1}}\ .
\end{aligned} \tag{195}$$

The substitution $\hat{H}(\hat{\psi}^\dagger,\hat{\psi}) \to H(\bar{\psi}_k,\psi_{k-1})$ follows from the ordering of the hamiltonian specified previously ($\hat{\psi}^\dagger$ on the left and $\hat{\psi}$ on the right). This allows one to act with the creation operator on a bra eigenstate, and with the annihilation operator on a ket eigenstate, so that all operators in the hamitonian gets substituted by the respective eigenvalues, producing a function of these Grassmann numbers. This way the hamiltonian operator $\hat{H}(\hat{\psi}^\dagger,\hat{\psi})$ gets substituted by the hamiltonian function $H(\bar{\psi}_k,\psi_{k-1})$. These approximations are valid for $N\to\infty$, i.e. $\epsilon\to 0$. Substituting (195) into (194) one finds

$$\begin{aligned}
\langle\bar{\psi}_f|e^{-i\hat{H}T}|\psi_i\rangle &= \lim_{N\to\infty}\int\Big(\prod_{k=1}^{N-1}d\bar{\psi}_k d\psi_k\,e^{-\bar{\psi}_k\psi_k}\Big)e^{i\sum_{k=1}^{N}[-i\bar{\psi}_k\psi_{k-1}-H(\bar{\psi}_k,\psi_{k-1})\epsilon]} \\
&= \lim_{N\to\infty}\int\Big(\prod_{k=1}^{N-1}d\bar{\psi}_k d\psi_k\Big)e^{i\sum_{k=1}^{N}[i\bar{\psi}_k\frac{(\psi_k-\psi_{k-1})}{\epsilon}-H(\bar{\psi}_k,\psi_{k-1})]\epsilon+\bar{\psi}_N\psi_N} \\
&= \int D\bar{\psi}D\psi\,e^{i\int_0^T dt[i\bar{\psi}\dot{\psi}-H(\bar{\psi},\psi)]+\bar{\psi}(T)\psi(T)} = \int D\bar{\psi}D\psi\,e^{iS[\bar{\psi},\psi]}\ .
\end{aligned} \tag{196}$$

This is the path integral for one complex fermionic degree of freedom. We recognize in the exponent the discretization of the classical action

$$\begin{aligned}
S[\bar{\psi},\psi] &= \int_0^T dt\,[i\bar{\psi}\dot{\psi}-H(\bar{\psi},\psi)]-i\bar{\psi}(T)\psi(T) \\
&\to \sum_{k=1}^{N}\epsilon\,[i\bar{\psi}_k\frac{(\psi_k-\psi_{k-1})}{\epsilon}-H(\bar{\psi}_k,\psi_{k-1})]-i\bar{\psi}_N\psi_N
\end{aligned} \tag{197}$$

where $T=N\epsilon$ is the total propagation time. The discrete values ψ_k and $\bar{\psi}_k$ are those corresponding to the values of the continuous functions evaluated at times $t=k\epsilon$, i.e. $\psi_k=\psi(k\epsilon)$ and $\bar{\psi}_k=\bar{\psi}(k\epsilon)$. The last way of writing the amplitude in (196) is symbolic and indicates the formal sum over all paths $\bar{\psi}(t),\psi(t)$ with boundary conditions $\psi(0)=\psi_0\equiv\psi_i$ and $\bar{\psi}(T)=\bar{\psi}_N\equiv\bar{\psi}_f$, weighed by the exponential of i times the classical action $S[\bar{\psi},\psi]$. Note that the action contains

the boundary term $-i\bar{\psi}(T)\psi(T)$. It is essential for formulating a variational principle where the boundary data are fixed by specifying the initial value of the function $\psi(t)$ and the final value of the function $\bar{\psi}(t)$ (i.e. $\psi(0) = \psi_i$ and $\bar{\psi}(T) = \bar{\psi}_f$).

Trace

One can now produce a path integral expression for the trace of the transition amplitude $e^{-i\hat{H}T}$. Using the expression of the trace in the coherent state basis, and the path integral representation of the transition amplitude, one finds

$$\begin{aligned}
\mathrm{Tr}[e^{-i\hat{H}T}] &= \int d\bar{\psi}_0 d\psi_0\, e^{-\bar{\psi}_0\psi_0}\, \langle -\bar{\psi}_0|e^{-i\hat{H}T}|\psi_0\rangle \\
&= \lim_{N\to\infty} \int \Big(\prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k\Big) e^{i\sum_{k=1}^{N}[i\bar{\psi}_k\frac{(\psi_k-\psi_{k-1})}{\epsilon} - H(\bar{\psi}_k,\psi_{k-1})]\epsilon} \\
&= \int_A D\bar{\psi}D\psi\, e^{iS[\bar{\psi},\psi]} \qquad (198)
\end{aligned}$$

where we have identified $\bar{\psi}_N = -\bar{\psi}_0$ and $\psi_N = -\psi_0$, and used that the exponential $e^{-\bar{\psi}_0\psi_0}$ from the trace cancels the boundary term $e^{\bar{\psi}_N\psi_N}$. Note that with this identification the path integral measure can be written also as a sum form 1 to N, $\prod_{k=1}^{N} d\bar{\psi}_k d\psi_k$. In the continuum limit one finds a sum on all antiperiodic paths i.e. with $\psi(T) = -\psi(0)$ and $\bar{\psi}(T) = -\bar{\psi}(0)$ (A stands for antiperiodic boundary conditions). This representation finds obvious applications in statistical mechanical problems involving fermions.

Supertrace

In a similar way the supertrace is calculated by

$$\begin{aligned}
\mathrm{STr}[e^{-i\hat{H}T}] &= \int d\bar{\psi}_0 d\psi_0\, e^{-\bar{\psi}_0\psi_0}\, \langle \bar{\psi}_0|e^{-i\hat{H}T}|\psi_0\rangle \\
&= \lim_{N\to\infty} \int \Big(\prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k\Big) e^{i\sum_{k=1}^{N}[i\bar{\psi}_k\frac{(\psi_k-\psi_{k-1})}{\epsilon} - H(\bar{\psi}_k,\psi_{k-1})]\epsilon} \\
&= \int_P D\bar{\psi}D\psi\, e^{iS[\bar{\psi},\psi]} \qquad (199)
\end{aligned}$$

where we have now identified $\bar{\psi}_N = \bar{\psi}_0$ and $\psi_N = \psi_0$. Again the term $e^{-\bar{\psi}_0\psi_0}$ from the supertrace cancels the boundary term $e^{\bar{\psi}_N\psi_N}$. In the continuum limit the sum is over all periodic trajectories defined by the boundary conditions $\psi(T) = \psi(0)$ and $\bar{\psi}(T) = \bar{\psi}(0)$ (P stands for periodic boundary conditions).

To conclude, we have derived the path integral for fermionic systems from the operatorial formulation using a time slicing of the total propagation time. Time slicing produces a discretization of the classical action and defines concretely the meaning of the path integral once written in the continuum notation (i.e. it provides a regularization). We have discussed a simple model with one complex degree of freedom $\psi(t)$ and its complex conjugate $\bar{\psi}(t)$ (it may be called a Dirac fermion in one dimension). The extension to several complex degrees of freedom is immediate.

6.4.1 Correlation functions

Correlation functions are defined as normalized averages of the dynamical variables. Again, one may introduce a generating functional by adding sources to the path integral. Using a hypercondensed notation, denoting all fermionic functions by ψ^i and corresponding sources by η_i (also taking values in a Grassmann algebra), one writes down the generating functional of correlation functions

$$Z[\eta] = \int D\psi \; e^{iS[\psi]+i\eta_i\psi^i} \; . \tag{200}$$

As an example, the two-point function is given by

$$\langle \psi^i \psi^j \rangle = \frac{\int D\psi \; \psi^i\psi^j \; e^{iS[\psi]}}{\int D\psi \; e^{iS[\psi]}} = \frac{1}{Z[0]} \Big(\frac{1}{i}\Big)^2 \frac{\delta^2 Z[\eta]}{\delta\eta_i \delta\eta_j}\Big|_{\eta=0} \; . \tag{201}$$

In a free theory, identified by a quadratic action of the form $S[\psi] = -\frac{1}{2}\psi^i K_{ij}\psi^j$ with K_{ij} an antisymmetric matrix, one formally computes the path integral with sources by gaussian integration (after completing squares in terms of $\psi^i + G^{ij}\eta_j$ and using the transitional invariance of the measure). The answer takes the form

$$Z[\eta] = \det{}^{\frac{1}{2}}(K_{ij}) \; e^{-\frac{i}{2}\eta_i G^{ij}\eta_j} \tag{202}$$

where G^{ij} is the inverse of K_{ij}, which is also an antisymmetric matrix. For the two-point function one finds

$$\langle \psi^i \psi^j \rangle = -iG^{ij} \tag{203}$$

where G^{ij} must be interpreted as a Green function in quantum mechanical applications. To check the overall normalization one must be careful with signs arising from the anticommuting character of the Grassmann variables and from the antisymmetric properties of K_{ij} and G^{ij}.

Similar formulae may be written down for complex fermions (they are actually contained in the above formula as well). As these are useful for the standard treatment of a Dirac fermion, let us write down the essential formulae

$$\begin{aligned} Z[\eta,\bar\eta] &= \int D\bar\psi D\psi \; e^{iS[\psi,\bar\psi]+i\bar\eta_i\psi^i+i\bar\psi_i\eta^i} \\ \langle \psi^i \bar\psi_j \rangle &= \frac{1}{Z}\frac{\delta}{\delta\bar\eta_i}\frac{\delta}{\delta\eta^j} Z[\eta,\bar\eta]\Big|_{\eta=\bar\eta=0} \end{aligned} \tag{204}$$

that for a free action

$$S[\psi,\bar\psi] = -\bar\psi_i K^i{}_j \psi^j \tag{205}$$

produce

$$\begin{aligned} Z[\eta,\bar\eta] &= \det(K^i{}_j) \, e^{i\bar\eta_i G^i{}_j \eta^j} \\ \langle \psi^i \bar\psi_j \rangle &= -iG^i{}_j \end{aligned} \tag{206}$$

where the value of $Z[\eta,\bar\eta]$ has been computed by completing squares (using the inverse $G^i{}_j$ of $K^i{}_j$). Of course, one must take into account the chosen boundary conditions on the path integral and use the corresponding Green functions $G^i{}_j$. The whole set of generating functionals described in the bosonic case may be introduced here as well, but we leave their derivation as an exercize.

To conclude, we consider the examples of the fermionic oscillator and the Dirac field. We identify their two-point functions using the general formula just derived. The *fermionic oscillator* has classical action (see eq. (140))

$$S[\psi,\bar\psi]=\int dt\,\bar\psi(i\partial_t-\omega)\psi \tag{207}$$

and the path integral is computed by completing squares

$$\begin{aligned} Z[\eta,\bar\eta] &= \int D\bar\psi D\psi\; e^{iS[\psi,\bar\psi]+i\int dt[\bar\eta(t)\psi(t)+\bar\psi(t)\eta(t)]} \\ &= N\exp\left(i\iint dtdt'\,\bar\eta(t)G(t-t')\eta(t')\right) \end{aligned} \tag{208}$$

where $G(t-t')$ is the Green function of the operator $K\equiv -i\partial_t+\omega$

$$(-i\partial_t+\omega)G(t-t')=\delta(t-t') \tag{209}$$

and reads

$$G(t-t')=\int\frac{dp}{2\pi}\,\frac{e^{-ip(t-t')}}{-p+\omega}=\int\frac{dp}{2\pi}\,e^{-ip(t-t')}\frac{p+\omega}{-p^2+\omega^2-i\epsilon}=i\theta(t-t')e^{-i\omega(t-t')}\,. \tag{210}$$

The propagator is then found to be

$$\langle\psi(t)\bar\psi(t')\rangle=-iG(t-t')=\theta(t-t')e^{-i\omega(t-t')}\,. \tag{211}$$

The *Dirac field* is described by the action

$$S[\psi,\bar\psi]=\int d^4x\,\left(-\bar\psi(\gamma^\mu\partial_\mu+m)\psi\right) \tag{212}$$

and the path integral becomes

$$\begin{aligned} Z[\eta,\bar\eta] &= \int D\bar\psi D\psi\; e^{iS[\psi,\bar\psi]+i\int d^4x[\bar\eta(x)\psi(x)+\bar\psi(x)\eta(x)]} \\ &= N\exp\left(i\iint d^4xd^4x'\,\bar\eta(x)G(x-x')\eta(x')\right) \end{aligned} \tag{213}$$

where $G(x-x')$ is the Green function of the operator $K\equiv\gamma^\mu\partial_\mu+m=\not\partial+m$

$$(\not\partial+m)G(x-y)=\delta^{(4)}(x-y) \tag{214}$$

and reads

$$G(x-y)=\int\frac{d^4p}{(2\pi)^4}\,e^{ip_\mu(x^\mu-y^\mu)}\frac{-i\not p+m}{p^2+m^2-i\epsilon}\,. \tag{215}$$

It is often indicated also by $S(x-y)$ (S for spinor). The two-point function (propagator) then becomes

$$\langle\psi(x)\bar\psi(y)\rangle=-iG(x-y)=-\int\frac{d^4p}{(2\pi)^4}\,e^{ip_\mu(x^\mu-y^\mu)}\frac{\not p+im}{p^2+m^2-i\epsilon} \tag{216}$$

which is interpreted as describing a particle propagating from y^μ to x^μ if $x^0 > y^0$, or an antiparticle propagating from x^μ to y^μ if $y^0 > x^0$.

To conclude, a last remark: a well-known property of perturbation theory written in terms of Feynman diagram (and related Feynman rules that translate them into equations) states that fermionic loops carry a minus sign. This is easily seen as arising for the Grassmann character of the fermionic variables. In a hypercondensed notation the free propagator is denoted by

$$\langle \psi^i \bar{\psi}_j \rangle = \; j \longrightarrow i \tag{217}$$

and considering an interaction of the form $S_{int} = \lambda \bar{\psi}_i \psi^i$ one finds in connected diagrams, included in the expansion of the Dyson formula $\langle e^{iS_{int}} \rangle$, terms of the form

$$\langle S_{int}^2 \rangle_c = \lambda^2 \langle \bar{\psi}_i \psi^i \bar{\psi}_j \psi^j \rangle_c = -\lambda^2 \langle \psi^i \bar{\psi}_j \rangle \langle \psi^j \bar{\psi}_i \rangle = \tag{218}$$

where we recognize the propagators forming the loop and the explicit minus sign.