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Gauge theories

(Lecture notes - 2020/21)

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1 Abelian gauge theories and QED

Let us consider the free lagrangian of a Dirac field of mass m

$$\mathcal{L}_{Dirac} = -\overline{\psi}\gamma^\mu \partial_\mu \psi - m\,\overline{\psi}\psi. \tag{1}$$

It is invariant under symmetry transformations belonging to the $U(1)$ group

$$\psi(x) \quad \rightarrow \quad \psi'(x) = \mathrm{e}^{i\alpha}\psi(x). \tag{2}$$

These are rigid (or global) transformations as the parameter α is constant (spacetime independent).

Let us now see how to extend the symmetry to a local one, i.e. with arbitrary functions $\alpha(x)$,

$$\psi(x) \quad \rightarrow \quad \psi'(x) = \mathrm{e}^{i\alpha(x)}\psi(x) \tag{3}$$

that implies for the Dirac conjugate

$$\overline{\psi}(x) \quad \rightarrow \quad \overline{\psi'}(x) = \mathrm{e}^{-i\alpha(x)}\overline{\psi}(x). \tag{4}$$

The mass term in the lagrangian is invariant

$$m\,\overline{\psi}\psi \quad \rightarrow \quad m\,\overline{\psi'}\psi' = m\,\overline{\psi}\,\mathrm{e}^{-i\alpha(x)}\mathrm{e}^{i\alpha(x)}\psi = m\,\overline{\psi}\psi, \tag{5}$$

but the term with the derivative is not invariant

$$\overline{\psi}\gamma^\mu\partial_\mu\psi \quad \rightarrow \quad \overline{\psi'}\gamma^\mu\partial_\mu\psi' = \overline{\psi}\,\mathrm{e}^{-i\alpha(x)}\gamma^\mu\partial_\mu(\mathrm{e}^{i\alpha(x)}\psi) = \overline{\psi}\gamma^\mu\partial_\mu\psi + i\,\overline{\psi}\gamma^\mu\psi\,\partial_\mu\alpha(x)\ . \tag{6}$$

There appears an extra term $i\,\overline{\psi}\gamma^\mu\psi\,\partial_\mu\alpha(x)$ that vanishes only for constant $\alpha(x)$. The lagrangian is not invariant, and we have to modify it to achieve gauge invariance (invariance with arbitrary functions $\alpha(x)$).

To construct gauge invariant actions it is useful to introduce a formalism based on the definition of tensors of the gauge group and covariant derivatives. The latter are constructed in such a way to produce tensors out of tensors.

We say that $\psi(x)$ is a tensor under the gauge group $U(1) = \{\mathrm{e}^{i\alpha(x)}\}$ if it transforms as in (3). Then $\partial_\mu\psi(x)$ is not a tensor, as it transforms in a more complicated way. The covariant derivative is defined by

$$D_\mu = \partial_\mu - iA_\mu(x) \tag{7}$$

where $A_\mu(x)$ is a vector field that is required to transform under the gauge group, so that the "tensorial" transformation rule would remain valid

$$D_\mu\psi(x) \to D'_\mu\psi'(x) = \mathrm{e}^{i\alpha(x)}D_\mu\psi(x) \tag{8}$$

where $D'_\mu = \partial_\mu - iA'_\mu(x)$. A short calculation shows that we must have the following rule

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x). \tag{9}$$

With the use of the covariant derivative it is simple to obtain a gauge invariant lagrangian out of (1):

$$\boxed{\mathcal{L} = -\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi}\,. \tag{10}$$

Comment: a "tensor" for the gauge group $U(1)$ is a field $\psi_q(x)$ that transforms as

$$\psi_q(x) \to \psi_q(x)' = \mathrm{e}^{iq\alpha(x)}\psi_q(x) \tag{11}$$

where $q \in Z$ is called the "charge" of $\psi_q(x)$ (it is a tensor of charge q: in mathematical terms q identifies an irreducible representation of the group $U(1)$). Then, the general definition of the covariant derivative is written as

$$D_\mu = \partial_\mu - iA_\mu(x)Q \tag{12}$$

where Q is an operator that measures the charge of the tensor on which it acts. The covariant derivative has the property that it does not destroy the tensorial character of the object on which it acts: it generates tensors out of tensors. Indeed, one may verify that

$$D_\mu\psi_q(x) = \partial_\mu\psi_q(x) - iqA_\mu(x)\psi_q(x) \tag{13}$$

is again a tensor of charge q (see eq. (8)). Another property of this definition is the validiy of the Leibniz rule for the covariant derivatives

$$D_\mu(\psi_{q_1}\psi_{q_2}) = (D_\mu\psi_{q_1})\psi_{q_2} + \psi_{q_1}(D_\mu\psi_{q_2}). \tag{14}$$

Therefore, local invariance can be obtained by introducing the new field $A_\mu(x)$, recognized as the potential of the electromagnetic field (the gauge potential). Having introduced a new field, we have to give it a suitable dynamics, i.e. we have to add a kinetic term for $A_\mu(x)$ to the lagrangian. This term will have to be gauge invariant, because the rest of the Lagrangian is: the gauge symmetry is the guiding principle for building the action! It is useful to proceed using tensors. We can calculate the commutator of two covariant derivatives acting on a tensor

$$[D_\mu, D_\nu]\psi = -iF_{\mu\nu}\psi \tag{15}$$

that defines the quantity $F_{\mu\nu}$. Since we have only tensorial quantities, the right-hand side must also be built out of tensor. Thus, we recognize that $F_{\mu\nu}$ is a tensor of charge $q = 0$, i.e. an invariant under gauge transformations (to recognize this, just set $q = 0$ in (11)).

Computing explicitly the left-hand side of (15) we find

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{16}$$

easily recognized as the electromagnetic field.

Now, it is immediate to construct a gauge invariant lagrangian with at most two derivatives on A_μ. It is enough to use as a building block the field strength $F_{\mu\nu}$, which we know to be gauge invariant. We have to require also Lorentz invariance (to have a relativistic theory), and we just find the free Maxwell lagrangian that we write with the standard normalization

$$\mathcal{L}_{Maxwell} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{17}$$

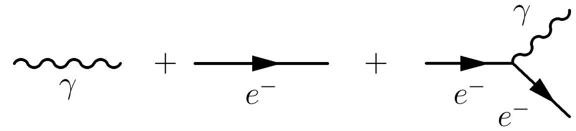
Summing all the pieces which are separately gauge invariant (eqs. (10) and (17)) we obtain the QED lagrangian

$$\boxed{\mathcal{L}_{QED} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \overline{\psi}\gamma^\mu D_\mu \psi - m\overline{\psi}\psi} \tag{18}$$

where we have introduced a free multiplicative parameter e^{-2} to account for a relative weight between the different terms that are separately gauge invariant.

Let us analyze the terms contained in (18): we redefine $A_\mu \to eA_\mu$ (to obtain the standard nomalization of the free Maxwell action) and recognize that e is the coupling constant (now it appears in the covariant derivative $D_\mu = \partial_\mu - ieA_\mu(x)$)

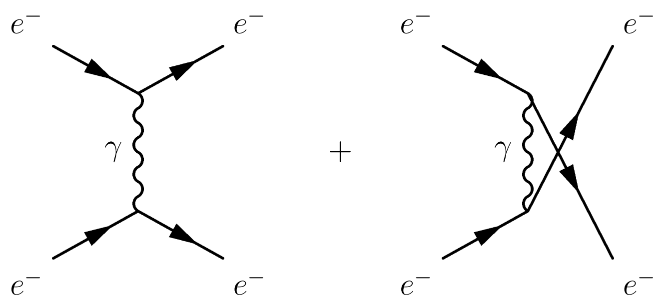
$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \overline{\psi}(\gamma^\mu \partial_\mu + m)\psi + ieA_\mu \overline{\psi}\gamma^\mu \psi$$

$=$  (19)

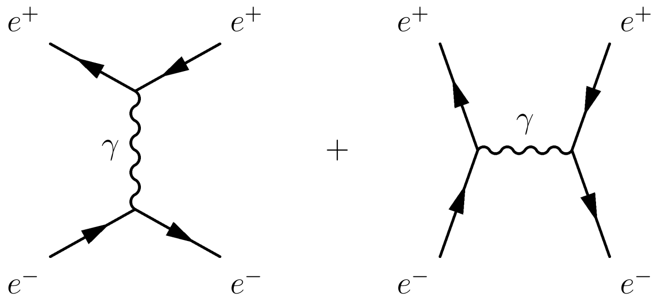
The first term describes the free propagation of the photons, the second term the free propagation of electrons (and positrons), the third term the elementary interaction between photons and electrons . The constant e represents the coupling constant, identified with the elementary charge of the electron: the gauge principle has allowed us to derive the interaction between spin fields 1/2 and 1. Let us summarize the rules of gauge transformations, rescaling for simplicity also the angle $\alpha(x) \to e\alpha(x)$,

$$\begin{aligned} &\psi \to \psi' = \mathrm{e}^{ie\alpha}\psi \\ &\overline{\psi} \to \overline{\psi'} = \mathrm{e}^{-ie\alpha}\overline{\psi} \\ &A_\mu \to A'_\mu = A_\mu + \partial_\mu \alpha. \end{aligned} \tag{20}$$

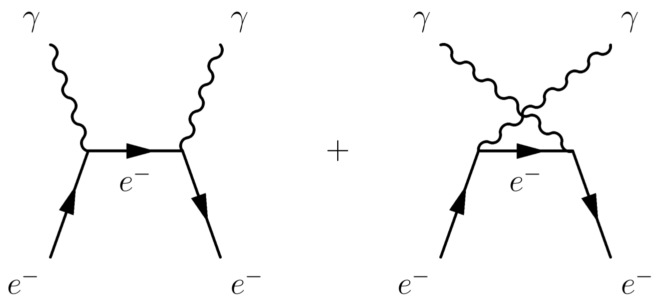
When the coupling constant e can be treated perturbatively, the amplitudes for the various physical processes dictated by QED can be constructed with the Feynman diagrams built with the elementary vertex in (19). For example, the electron-electron scattering (Möller scattering) at the lowest order is given by (time runs along the horizontal axis)



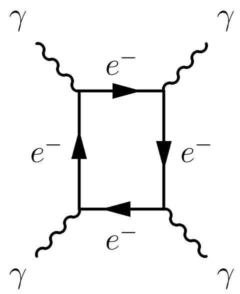
The electron/positron scattering (Bhabha scattering):



The electron/photon scattering (Compton scattering):



Also photon/photon scattering is possible: there is no elementary vertex, but the first perturbative term one can find is given by the graph



together with similar graphs where the external photon lines are attached to the vertices with different orderings. In general, loop corrections can be divergent and must be cured by renormalization. However, for the photon-photon scattering the calculation of the Feynman graph is finite, and there is no need to renormalize it: this can be seen as a consequence of gauge invariance.

2 Non-abelian gauge theories and QCD

The procedure for constructing gauge invariant actions can be extended to compact non-abelian groups. These theories are at the basis of the "Standard Model" of the fundamental interactions.

2.1 Lie groups

Let us briefly recall some properties of non-abelian Lie groups, keeping in mind the group $SU(N)$ as an example. An element of the non-abelian Lie group G, which is connected to the identity, can be parametrized with coordinates α_a (the parameters) associated with the hermitian generators T^a. Here is a lost of the main properties:

$$\begin{array}{ll}
(i) & U = \exp(i\alpha_a T^a) \in G \qquad a = 1, .., \dim G \\
(ii) & [T^a, T^b] = i f^{ab}{}_c T^c \\
(iii) & \mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \\
(iv) & f^{abc} = f^{ab}{}_d \delta^{dc} \qquad \text{antisymmetric tensor} \\
(v) & [[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0 \\
 & \Rightarrow\ f^{ab}{}_d f^{dc}{}_e + f^{bc}{}_d f^{da}{}_e + f^{ca}{}_d f^{db}{}_e = 0 \\
(iv) & (T^a_{\rm Adj})^b{}_c = -i f^{ab}{}_c \ .
\end{array}$$

Property (i) describes the exponential representation of an arbitrary element of the group connected to the identity. The index a takes as many values as the dimension of the group. An element of the group is therefore parameterized by the "angles" α_a.
Property (ii) is the Lie algebra satisfied by the infinitesimal generators T^a. The real constants $f^{ab}{}_c$ are the structure constants, and characterize the group G.
Item (iii) defines a choice for the normalization of the generators in the fundamental representation, and identifies the so-called "Killing metric ". This metric is positive definite for compact Lie groups (such as $SU(N)$). The normalization chosen here produces the Kronecker delta δ^{ab} as the Killing metric for G.
In (iv) we used the Killing metric to raise an index in the structure constants. The f^{abc} symbols are completely antisymmetric in all indices: this property can be deduced by multiplying the Lie algebra with an additional generator, taking the trace, and using (iii) and the cyclic property of the trace. The antisymmetry in the indices a and b is obvious from (ii).
Item (v) gives the Jacobi identities.
Item (vi) defines the adjoint representation. It is shown to be a representation of the Lie algebra by using the Jacobi identities.

2.2 Action with rigid $SU(N)$ symmetry

Let us now consider N free Dirac fields with identical masses m, assembled in column and raw vectors

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ . \\ . \\ \psi^N \end{pmatrix} \qquad \overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2, ., ., \overline{\psi}_N) \ . \tag{21}$$

The free lagrangian is given by

$$\mathcal{L}_{Dirac} = -\overline{\psi}\gamma^\mu \partial_\mu \psi - m\,\overline{\psi}\psi. \tag{22}$$

and is invariant under the $SU(N)$ symmetry transformations given

$$\begin{aligned} \psi(x) &\to \psi'(x) = U\psi(x) \\ \overline{\psi}(x) &\to \overline{\psi}'(x) = \overline{\psi}(x)U^\dagger = \overline{\psi}(x)U^{-1} \end{aligned} \tag{23}$$

where $U \in SU(N)$, and $U^\dagger = U^{-1}$ since U is unitary. These are rigid transformations, as the α^a parameters contained in $U = \exp(i\alpha^a T^a)$ are constant (indices in α^a are raised and lowered with the Killing metric, that coincides with the identity in our conventions).

2.3 Covariant derivative

To make local the $SU(N)$ symmetry it is again convenient to introduce covariant derivatives. By definition, the covariant derivative when applied to tensors produces new tensors. It is defined by

$$D_\mu = \partial_\mu + W_\mu(x) \tag{24}$$

where $W_\mu(x)$ is a matrix valued gauge field, also known as the connection (geometrically, it defines a parallel transport in a certain space). The gauge field W_μ mixes the N Dirac fermions contained in ψ, and thus is formed by $N \times N$ matrices for any μ. It performs infinitesimal group transformations (to define a sort of parallel transport) and thus can be expanded in terms of the generators as follows

$$W_\mu(x) = -iW_\mu^a(x)T^a \ . \tag{25}$$

This relation defines the gauge fields $W_\mu^a(x)$, and there are $N^2 - 1$ of them for the gauge group $SU(N)$. From the requirement of covariance

$$\begin{aligned} \psi(x) &\to \psi'(x) = U(x)\psi(x) \\ D_\mu\psi(x) &\to D'_\mu\psi'(x) = U(x)D_\mu\psi(x) \end{aligned} \tag{26}$$

one obtains the following transformation rule for the gauge potentials

$$W_\mu(x) \ \to \ W_\mu(x)' = U(x)W_\mu(x)U^{-1}(x) + U(x)\partial_\mu U^{-1}(x) \ . \tag{27}$$

Indeed, requiring that $D'_\mu\psi' = UD_\mu\psi$, one can compute

$$\begin{aligned} D'_\mu\psi' &\equiv (\partial_\mu + W'_\mu)\psi' \\ &= UD_\mu\psi = U(\partial_\mu + W_\mu)\psi = U(\partial_\mu + W_\mu)U^{-1}U\psi \\ &= U(\partial_\mu + W_\mu)U^{-1}\psi' = \partial_\mu\psi' + U[(\partial_\mu + W_\mu)U^{-1}]\psi' \end{aligned} \tag{28}$$

from which we get the above transformation rule for W_μ. To be more precise, as the ψ transform in the fundamental representation of $SU(N)$, the N representation, the T^a contained in the W_μ of eq. (28) will be the generators in the fundamental representation.

Covariant derivatives do not commute. This fact allows to define the "curvature" tensor (or "field strength") the following way

$$[D_\mu, D_\nu] = F_{\mu\nu} \tag{29}$$

so that

$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu] \ . \tag{30}$$

It is immediate to check that the field strength transform covariantly as

$$F_{\mu\nu} \ \to \ F'_{\mu\nu} = UF_{\mu\nu}U^{-1} \tag{31}$$

which follows from the covariance of (29). This is the adjoint representation.

2.4 Gauge invariant action

It is now simple to construct a gauge invariant lagrangian from (22): it is enough to substitute derivatives with gauge covariant derivatives (this is also called "minimal coupling"). One obtains

$$\mathcal{L}_1 = -\overline{\psi}(\gamma^\mu D_\mu + m)\psi \tag{32}$$

that depends on the new field W_μ contained in D_μ. Now, one must give a dynamics to W_μ by using the simplest gauge and Lorentz invariant action with at most two derivatives: the lagrangian is

$$\mathcal{L}_2 = \frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu\, a} \ . \tag{33}$$

Now, using a coupling constant g to define a relative weight between the different gauge invariant pieces one obtains the final lagrangian

$$\boxed{\mathcal{L} = \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \overline{\psi}(\gamma^\mu D_\mu + m)\psi} \tag{34}$$

with gauge symmetries recapitulated as follows

$$\begin{aligned} \psi(x) \quad &\to \quad \psi'(x) = U(x)\psi(x) \\ W_\mu(x) \quad &\to \quad W'_\mu(x) = U(x)W_\mu(x)U^{-1}(x) + U(x)\partial_\mu U^{-1}(x) \ . \end{aligned} \tag{35}$$

Let us report also the infinitesimal transformations. Defining the matrix $\alpha \equiv -i\alpha_a T^a$ with $\alpha_a \ll 1$, we can write an infinitesimal transformation in the form $U = e^{-\alpha} = 1 - \alpha + O(\alpha^2)$, from which we get

$$\begin{aligned} \delta\psi &= -\alpha\psi \\ \delta\overline{\psi} &= \overline{\psi}\alpha \\ \delta W_\mu &= \partial_\mu\alpha + [W_\mu, \alpha] = D_\mu\alpha \end{aligned} \tag{36}$$

where in the last line the covariant derivaties is the one that acts in the adjoint representation.

We can rewrite the action by a field redefinition, $W_\mu \to \bar{W}_\mu = gW_\mu$, to get the canonical normalization of the gauge field action. In components

$$\begin{aligned} W_\mu(x) &= -iW^a_\mu(x)T^a \\ F_{\mu\nu}(x) &= -iF^a_{\mu\nu}(x)T^a \end{aligned} \tag{37}$$

so that we get

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + gf^{abc}W^b_\mu W^c_\nu \tag{38}$$

and the complete action takes the form

$$\boxed{\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \overline{\psi}(\gamma^\mu(\partial_\mu - igW^a_\mu T^a) + m)\psi} \ . \tag{39}$$

The infinitesimal gauge transformation are (redefining for commodity the parameters $\alpha^a \to g\alpha^a$)

$$\begin{aligned} \delta\psi(x) &= ig\alpha^a(x)T^a\psi(x) \\ \delta W^a_\mu(x) &= \partial_\mu\alpha^a(x) + gf^{abc}W^b_\mu(x)\alpha^c(x) = (D_\mu\alpha(x))^a \ . \end{aligned} \tag{40}$$

The first term in the action (39) describes the free propagation of the fields W_μ^a (the non-abelian spin 1 particles) along with cubic and quartic self-interactions. A positive non-definite Killing metric would result in a term with kinetic energy that is not positive-definite, and this would not be acceptable: it is necessary to consider compact groups only to satisfy this request. The second term describes the free propagation of the ψ fields (spin 1/2 particles with non-abelian charges, i.e. "color" charges) together with their interaction with the gauge field. The constant g is the coupling constant. It can be treated perturbatively if small enough. The "non-abelian" or color charge corresponds to the representation of the gauge group chosen for the ψ fields (in our case we have taken the fundamental representation, but any other representation could have been chosen). The gauge principle allowed us to derive all the interaction vertices between fields of spin 1/2 and 1 in terms of the single coupling constant g.

As a consequence of the transformation law (40), or directly from (31), one recognizes that the field $F_{\mu\nu}^a$ transforms in the adjoint representation

$$\delta F_{\mu\nu}^a = g f^{abc} F_{\mu\nu}^b \alpha^c = i g \alpha^c (T_{\text{Adj}}^c)^{ab} F_{\mu\nu}^b \tag{41}$$

with the generators in the adjoint representation given by the following matrices

$$(T_{\text{Adj}}^c)^{ab} = -i f^{abc} \ . \tag{42}$$

That this is indeed a representation follows from the Jacobi identities. Similarly, the transformation law of W_μ^a can be expressed in terms of the covariant dervative acting on a tensor in the adjoint representation

$$\delta W_\mu^a = \partial_\mu \alpha^a + g f^{abc} W_\mu^b \alpha^c = \partial_\mu \alpha^a - i g W_\mu^b (T^b)^{ac} \alpha^c = (D_\mu \alpha)^a \ . \tag{43}$$

2.5 The action of quantum crodmodynamics (QCD)

The action of quantum chromodynamics is based on the group $SU(3)$ and in addition to the gluons (the eight particles associated with the W_μ^a gauge fields, which are charged as the index of the adjoint representation appears, the $\underline{8}$ of $SU(3)$) contains six fermion fields that transform in the fundamental representation of $SU(3)$ and describe the six known flavors of quarks $q = (u, d, c, s, t, b)$, i.e. up, down, charm, strange, top, bottom. Each quark flavor is degenerate, as it transforms in the $\underline{3}$ of $SU(3)$: they are said to be colored (red, green and blue, in the usual convention) while the absence of color indicates a scalar, like the lagrangian (the $\underline{1}$ of $SU(3)$). Of course, the corresponding antiparticles, the antiquarks $\bar{q} = (\bar{u}, \bar{d}, \bar{c}, \bar{s}, \bar{t}, \bar{b})$, transform in the conjugate representation, the $\underline{\bar{3}}$ of $SU(3)$.

The eight infinitesimal generators of $SU(3)$ in the fundamental representation are given by the Gell-Mann matrices λ^a (which generalize the Pauli matrices σ^i of $SU(2)$)

$$T^a = \frac{\lambda^a}{2} \qquad a = 1, \ldots, 8 \tag{44}$$

where

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \qquad \lambda^5 = \begin{pmatrix} 0 & & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

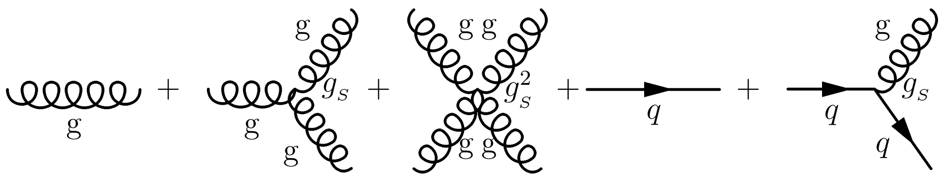
$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \qquad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , \qquad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} . \quad (45)$$

These matrices are normalized according the convention

$$\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab} . \quad (46)$$

An arbitrary element of the $SU(3)$ group in the fundamental representation is therefore described by 3×3 matrices of the form $U = \exp(-i\alpha^a \frac{\lambda^a}{2})$. By calculating the Lie algebra one can find the explicit values of the structure constants that identify the $SU(3)$ group. The QCD lagrangian is therefore

$$\begin{aligned} \mathcal{L}_{\mathcal{QCD}} &= -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \sum_{f=1}^{6} \overline{\psi}_f \Big(\gamma^\mu D_\mu + m_f\Big)\psi_f \\ &= -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \sum_{f=1}^{6} \overline{\psi}_f \Big(\gamma^\mu \partial_\mu + m_f\Big)\psi_f + i\frac{g_S}{2}W^a_\mu \sum_{f=1}^{6} \overline{\psi}_f \gamma^\mu \lambda^a \psi_f \end{aligned} \quad (47)$$

$=$ 

where the coupling constant is denoted by g_S, and the index $f \in (1, 2, \cdots, 6) = (u, d, c, s, t, b)$ indicates the flavor of the quark. Different flavors of quarks have different masses m_f.

The QCD lagrangian also possesses various rigid symmetries in addition to those already mentioned. A rigid symmetry always present is the $U(1)$ symmetry which rotates all fields of the quarks by the same phase: the associated conserved charge is the *baryon number*. It is a symmetry that is also preserved by the other fundamental interactions.

There are also other $U(1)$ symmetries which rotate the various fermionic fields separately. They give rise to the conservation laws of the respective *fermion numbers* (e.g. *strangeness* S, *charm* C, etc ..). These flavor symmetries are exact only for QCD (and QED), but the weak force violates them. In total there are six $U(1)$ conserved charges, one for each quark flavor, and the baryon number is a particular linear combination of these six independent charges.

There are also other approximate symmetries of the QCD lagrangian. In the limit in which some of the quark masses are taken to be identical, there is a rigid additional non-abelian symmetry. For example, assuming identical the masses for the up and down quarks, $m_u = m_d$, one can rotate the fields ψ_u and ψ_d with each other with an $SU(2)$ matrix. This rigid $SU(2)$ symmetry corresponds to the strong *isospin*, used to group hadrons into families (states of quarks bound by the strong force show the phenomenon of color confinement: the bound states are color singlets, corresponding to the mesons and baryons). An example of these families are: (i) the isospin doublet of the nucleons (proton and nucleon) composed of three confined up and down quarks; (ii) the triplet of π mesons, the pions $\pi^{\pm}$ and π^{0}, composed of a quark and an antiquark of the up and down types.

Considering identical the masses for the up, down and strange quarks, $m_u = m_d = m_s$, we find an even larger symmetry group, the $SU(3)$ *flavor group*, that mixes the three flavors up, down and strange. It should not be confused with the color group, also an $SU(3)$. Examples of multiplets of hadronic particles described by the $SU(3)$ flavor group are:
the meson octet $(\pi^{\pm}, \pi^{0}, K^{\pm}, K^{0}, \bar{K}^{0}, \eta)$,
the baryon octet $(p, n, \Sigma^{\pm}, \Sigma^{0}, \Xi^{\pm}, \Lambda)$,
the baryon decuplet $(\Delta^{-}, \Delta^{0}, \Delta^{+}, \Delta^{++}, \Sigma^{*\pm}, \Sigma^{*0}, \Xi^{*\pm}, \Omega^{-})$.
The existence of these families is understandable from group theory: the $\underline{8}$ and the $\underline{10}$ are representations of $SU(3)$. Let us consider the mesons in more detail. They consist of a quark-antiquark pair $(q\bar{q})$. The quarks q transforms in the 3 of $SU(3)$, with $3 \sim (u, d, s)$, while antiquarks $\bar{q}$ transforms in the $\bar{3}$ of $SU(3)$, with $\bar{3} \sim (\bar{u}, \bar{d}, \bar{s})$. From this it follows that a bound state $(q\bar{q})$ must transform in the

$$3 \otimes \bar{3} = 1 \oplus 8$$

and therefore both singlet and octets can exist for the mesons.

On the other hand, baryons are bound states of three quarks (qqq), and since under $SU(3)$

$$3 \otimes 3 \otimes 3 = (6 \oplus \bar{3}) \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$$

octets and decuplets can exist for the baryons, as they do.