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Gauge theories

(Lecture notes - A.A. 2024/25)

Fiorenzo Bastianelli

1 Introduction

Gauge theories are building blocks of the Standard Model of particle physics. They arise from the requirement that massless spin-one particles, which mediate some of the fundamental forces of nature, should carry only two independent polarizations[1] even when described in terms of equations of motion that are manifestly Lorentz invariant: the photon is conveniently described by the vector field $A_\mu(x)$, which has four independent components as any four-vector, though the expected physical degrees of freedoms are only two. The other two degrees of freedom are redundant because of the gauge symmetry: one can eliminate them completely by performing gauge transformations to reach a unitary gauge, such as the Coulomb gauge used in electrodynamics. However, in such a gauge the Lorentz symmetry is not manifest and it becomes difficult to exploit its consequences in a useful manner. It is often more convenient to keep the redundancy associated with the gauge symmetry and maintain the Lorentz invariance as manifest as possible. Then, the principle of gauge invariance can be used to fix in a simple way all possible interactions mediated by massless particles of spin one in a way that is consistent with Lorentz invariance. Here we introduce the principle of gauge invariance, following the construction of the QED and QCD lagrangians, the main examples of abelian and non-abelian gauge theories, respectively.

2 Abelian gauge theories and QED

Let us first review how starting from the theory of free electrons described by the free Dirac equation, one finds the complete QED lagrangian using the principle of gauge invariance (and, of course, the fundamental requirement of Lorentz invariance). Let us consider the lagrangian of a free Dirac field of mass m

$$\mathcal{L}_{Dirac} = -\overline{\psi}\gamma^\mu\partial_\mu\psi - m\,\overline{\psi}\psi \tag{1}$$

which is consistent with special relativity, i.e. it is Lorentz invariant. It is invariant also under symmetry transformations belonging to the group $U(1)$

$$\begin{aligned} \psi(x) \quad &\to \quad \psi'(x) = \mathrm{e}^{i\alpha}\psi(x)\;, \qquad\qquad \mathrm{e}^{i\alpha} \in U(1) \\ \overline{\psi}(x) \quad &\to \quad \overline{\psi'}(x) = \mathrm{e}^{-i\alpha}\,\overline{\psi}(x)\;. \end{aligned} \tag{2}$$

This is a global symmetry as the parameter α is constant (spacetime independent).

Let us see how to extend the symmetry to a local one with arbitrary functions $\alpha(x)$

$$\psi(x) \quad \to \quad \psi'(x) = \mathrm{e}^{i\alpha(x)}\psi(x) \tag{3}$$

$$\overline{\psi}(x) \quad \to \quad \overline{\psi'}(x) = \mathrm{e}^{-i\alpha(x)}\,\overline{\psi}(x)\;. \tag{4}$$

[1]As implied by Wigner's classification of the irreducible unitary representations of the Poincaré group.

The mass term in the lagrangian is already invariant as the phases cancel out

$$m\overline{\psi}\psi \quad \to \quad m\overline{\psi}'\psi' = m\overline{\psi}\,\mathrm{e}^{-i\alpha(x)}\mathrm{e}^{i\alpha(x)}\psi = m\overline{\psi}\psi\ , \tag{5}$$

but the term with the derivative is not

$$\overline{\psi}\gamma^\mu\partial_\mu\psi \quad \to \quad \overline{\psi}'\gamma^\mu\partial_\mu\psi' = \overline{\psi}\,\mathrm{e}^{-i\alpha(x)}\gamma^\mu\partial_\mu(\mathrm{e}^{i\alpha(x)}\psi) = \overline{\psi}\gamma^\mu\partial_\mu\psi + i\,\overline{\psi}\gamma^\mu\psi\,\partial_\mu\alpha(x)\ . \tag{6}$$

There is an extra term that vanishes only for constant $\alpha(x)$. The lagrangian is not invariant, and one has to modify it to achieve gauge invariance, i.e. invariance for arbitrary functions $\alpha(x)$. Note that the term multiplying the derivative of $\alpha(x)$ is the Noether current $J^\mu = i\,\overline{\psi}\gamma^\mu\psi$ associated with the global symmetry (2).

To construct gauge invariant actions it is useful to introduce a formalism based on the definition of *tensors* of the gauge group and related *covariant derivatives*. The latter are constructed in such a way as to produce tensors out of tensors.

We say that $\psi(x)$ is a tensor under the gauge group $U(1) = \{\mathrm{e}^{i\alpha(x)}\}$ if it transforms as in eq. (3). Then $\partial_\mu\psi(x)$ is not a tensor, as it transforms in a more complicated way. The covariant derivative on ψ is obtained by introducing a gauge field A_μ (also known as "connection"). It is defined it by

$$D_\mu = \partial_\mu - iA_\mu(x)\ . \tag{7}$$

Then, one requires $A_\mu(x)$ to transform in such a way so that the "tensorial" transformation rule remains valid

$$D_\mu\psi(x) \quad \to \quad D'_\mu\psi'(x) = \mathrm{e}^{i\alpha(x)}D_\mu\psi(x) \tag{8}$$

where $D'_\mu = \partial_\mu - iA'_\mu(x)$. A short calculation shows that the following transformation rule must hold

$$A_\mu(x) \quad \to \quad A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x). \tag{9}$$

With covariant derivatives, it is simple to obtain a gauge invariant lagrangian from eq. (1):

$$\boxed{\mathcal{L} = -\overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi}\ . \tag{10}$$

Comment: a "tensor" for the gauge group $U(1)$ is more generally defined as a field $\psi_q(x)$ that transforms as

$$\psi_q(x) \to \psi'_q(x) = \mathrm{e}^{iq\alpha(x)}\psi_q(x) \tag{11}$$

where the integer $q \in \mathbb{Z}$ is called the "charge" of $\psi_q(x)$. Thus, $\psi_q(x)$ is a tensor of charge q: in mathematical terms q identifies an irreducible representation of the group $U(1)$. Then, the general definition of covariant derivative is extended to

$$D_\mu = \partial_\mu - iA_\mu(x)Q \tag{12}$$

where Q is an operator that measures the charge of the tensor on which it acts, i.e. it is the generator of the $U(1)$ group in the same representation of the tensor it acts upon. We can now recognize that the transformation in (4) corresponds to that of a tensor of charge -1, so that the covariant derivative of $\overline{\psi}$ is

$$D_\mu\overline{\psi} = \big(\partial_\mu + iA_\mu(x)\big)\overline{\psi} \tag{13}$$

consistent with taking the Dirac conjugate of $D_\mu\psi$. The covariant derivative has the property that it does not destroy the tensorial character of the object on which it acts: it generates tensors out of tensors. Indeed, one may verify that

$$D_\mu \psi_q(x) = \partial_\mu \psi_q(x) - iqA_\mu(x)\psi_q(x) \tag{14}$$

is again a tensor of charge q. Another property of this definition is the validity of the Leibniz rule for covariant derivatives: as product of tensors are tensors, one may verify that

$$D_\mu(\psi_{q_1}\psi_{q_2}) = (D_\mu\psi_{q_1})\psi_{q_2} + \psi_{q_1}(D_\mu\psi_{q_2}). \tag{15}$$

We have seen that local invariance is achieved by introducing the gauge field $A_\mu(x)$, readily interpreted as the potential of the electromagnetic field. Having introduced a new field, one has to give it suitable dynamics by adding to the lagrangian a kinetic term for $A_\mu(x)$. This term has to be gauge invariant, because the rest of the Lagrangian already is: gauge symmetry is the guiding principle for building the action. It is useful to proceed using tensors. We start by calculating the commutator of two covariant derivatives acting on the tensor ψ of charge 1

$$[D_\mu, D_\nu]\psi = -iF_{\mu\nu}\psi \tag{16}$$

that defines the quantity $F_{\mu\nu}$. Since we have only tensorial quantities on the left-hand side, the right-hand side must also be built out of tensors. This way we see that $F_{\mu\nu}$ is a tensor of charge $q = 0$, i.e. a quantity that is invariant under gauge transformations (i.e. a transformation with $q = 0$ in eq. (11)). Computing explicitly the left-hand side of (16) one finds

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{17}$$

readily interpreted as the electromagnetic field tensor.

Now, it is immediate to construct a gauge invariant lagrangian with at most two derivatives on A_μ. It is enough to use as a building block the field strength $F_{\mu\nu}$ which is gauge invariant, and thus makes it easier to construct gauge invariant quantities. One must require also Lorentz invariance to have a relativistic theory, so that one is led to the free Maxwell lagrangian, which in a standard normalization reads

$$\mathcal{L}_{Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{18}$$

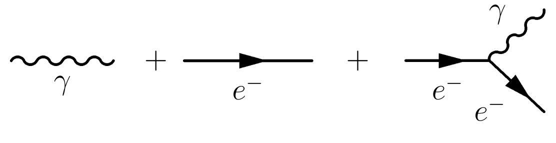
Summing together all the pieces that are separately gauge invariant (i.e. eqs. (10) and (18)) one finds the QED lagrangian

$$\boxed{\mathcal{L}_{QED} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \overline{\psi}\gamma^\mu D_\mu\psi - m\overline{\psi}\psi} \tag{19}$$

where a free multiplicative parameter $1/e^2$ accounts for a relative weight between different terms that are separately gauge invariant.

Let us analyze the various terms contained in (19). It is useful to redefine $A_\mu \to eA_\mu$ (to obtain the standard nomalization of the free Maxwell action) and recognize that e is the coupling constant (now it appears in the covariant derivative $D_\mu = \partial_\mu - ieA_\mu(x)$)

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \overline{\psi}(\gamma^\mu\partial_\mu + m)\psi + ieA_\mu\overline{\psi}\gamma^\mu\psi$$

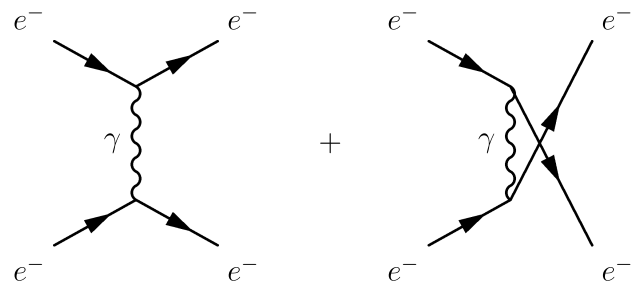
$=$ 

. (20)

The first term describes the free propagation of photons, the second one the free propagation of electrons, and the third one the elementary interaction between photons and electrons. The constant e is the coupling constant, identified with the elementary charge of the electron: the gauge principle has allowed us to discover the interaction between fields of spin 1/2 and 1. Let us summarize again the rules of gauge transformations for the lagrangian in (20): rescaling for simplicity also the angle $\alpha(x) \to e\alpha(x)$ we have

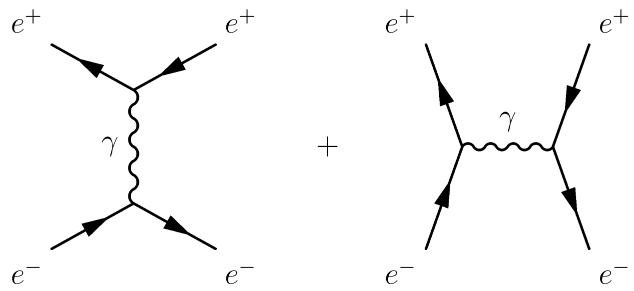
$$\begin{aligned} \psi \to \psi' &= \mathrm{e}^{ie\alpha}\psi \\ \overline{\psi} \to \overline{\psi'} &= \mathrm{e}^{-ie\alpha}\overline{\psi} \\ A_\mu \to A'_\mu &= A_\mu + \partial_\mu \alpha. \end{aligned} \tag{21}$$

If the coupling constant e is small enough it can be treated perturbatively, and the amplitudes for the various physical processes in QED can be associated to the Feynman diagrams built with the elementary vertex in (20). For example, the electron-electron scattering (Möller scattering) at the lowest order is given by

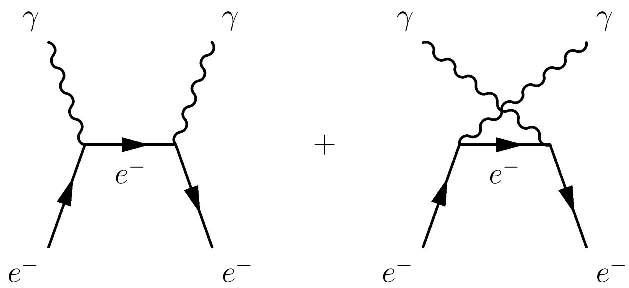


where time runs along the horizontal axis in our Feynman diagrams.

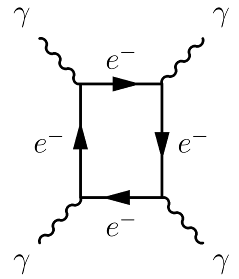
Other processes are the electron-positron scattering (Bhabha scattering)



and the electron-photon scattering (Compton scattering)



Also photon-photon scattering is possible: there is no elementary vertex and the first perturbative term that one finds is given by the graph



together with similar graphs where the external photon lines are attached to the vertices with different orderings. In general, loop corrections can be divergent and must be cured by renormalization. However, for the photon-photon scattering the calculation of the Feynman graph depicted above is finite, and there is no need to renormalize it. This fact may be interpreted as a consequence of gauge invariance.

3 Non-abelian gauge theories and QCD

The construction of gauge invariant actions can be extended to compact non-abelian groups. For that, we need to review the main properties of Lie groups, leaving appendix A for some more detailed considerations.

3.1 Lie groups

Let us briefly review simple and compact Lie groups, having in mind $SU(N)$ as the main example. An element U of a compact, non-abelian Lie group G can be parametrized by coordinates α_a (the parameters) associated with hermitian generators T^a. Here is a list of the main definitions and properties:

$$
\begin{aligned}
&(i) \quad U = U(\alpha) = \exp(i\alpha_a T^a) \in G \qquad a = 1, .., \dim G \\
&(ii) \quad [T^a, T^b] = i f^{ab}{}_c T^c \\
&(iii) \quad \mathrm{tr}(T^a_F T^b_{\mathrm{F}}) = \frac{1}{2}\delta^{ab} \\
&(iv) \quad f^{abc} = f^{ab}{}_d \delta^{dc} \qquad \text{is totally antisymmetric} \\
&(v) \quad [[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0 \\
&\qquad \Rightarrow \; f^{ab}{}_d f^{dc}{}_e + f^{bc}{}_d f^{da}{}_e + f^{ca}{}_d f^{db}{}_e = 0 \\
&(iv) \quad (T^a_{\mathrm{Adj}})^b{}_c = -i f^{ab}{}_c \; .
\end{aligned}
$$

Point (i) describes the exponential representation of an arbitrary element U of the group G connected to the identity. This structure is verified considering the defining representation, i.e. the matrix representation that is used to define the group, which for $SU(N)$ is given by unitary $N \times N$ matrices with unit determinant. The index a takes as many values as the dimension of the group. Therefore, an element of the group is parameterized by the "angles" α_a that multiply the generators T^a, matrices that in the defining (also called fundamental) representation are

written as T_F^a, but are then interpreted as abstract generators T^a that could have also other representations.
Point (ii) gives the Lie algebra satisfied by the hermitian generators T^a. The constants $f^{ab}{}_c$ are real and are known as the structure constants. They characterize the group G and are manifestly antisymmetric in the indices a and b.
Point (iii) defines a standard choice for the normalization of the generators in the fundamental representation T_F^a, also called the defining representation. It identifies the so-called "Killing metric". More generally, one could have defined the Killing metric γ^{ab} by $\text{tr}(T_F^a T_{\text{F}}^b) = \frac{1}{2}\gamma^{ab}$, and prove that the Killing metric has to be positive definite for compact Lie groups, such as $SU(N)$. The normalization chosen above for the generators T_F^a give the Kronecker delta δ^{ab} as the Killing metric for the compact group G.
Point (iv) makes use of the Killing metric to raise an index in the structure constants. Then, one can show that the symbols f^{abc} are completely antisymmetric: antisymmetry on the indices a and b is obvious from (ii); antisymmetry on the indices b and c is deduced by multiplying the Lie algebra with an additional generator, taking the trace, and using point (iii) together with the cyclic property of the trace.
Point (v) gives the Jacobi identities satisfied by the structure constants.
Point (vi) defines the adjoint representation. It is proven to be a representation by using the Jacobi identities. Note that the indices in $(T_{\text{Adj}}^a)^b{}_c$ all run from 1 to dim G, so that the dimension of the adjoint representation coincides with the dimension of the group, i.e. dim G. In the case of $SU(N)$, these indices run from 1 to N^2-1 and the dimension of the adjoint representation is thus N^2-1.

3.2 Action with rigid $SU(N)$ symmetry

Let us now consider N free Dirac fields, assembled in column and raw vectors as follows

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ . \\ . \\ \psi^N \end{pmatrix} , \qquad \overline{\psi} = \left(\overline{\psi}_1, \overline{\psi}_2, ., ., \overline{\psi}_N\right) \tag{22}$$

so that the scalar product

$$\overline{\psi}\psi = \overline{\psi}_1\psi^1 + \overline{\psi}_2\psi^2 + \cdots + \overline{\psi}_N\psi^N \tag{23}$$

is manifestly $SU(N)$ invariant. If these fields have identical masses m, then there is a global symmetry. The free lagrangian is given by

$$\mathcal{L}_{Dirac} = -\overline{\psi}\gamma^\mu\partial_\mu\psi - m\,\overline{\psi}\psi \tag{24}$$

and is invariant under the $SU(N)$ symmetry transformations given by

$$\begin{aligned} \psi(x) &\;\to\; \psi'(x) = U\psi(x) \\ \overline{\psi}(x) &\;\to\; \overline{\psi'}(x) = \overline{\psi}(x)U^\dagger = \overline{\psi}(x)U^{-1} \end{aligned} \tag{25}$$

where $U \in SU(N)$, and $U^\dagger = U^{-1}$ since U is unitary. These are global transformations, as the α^a parameters contained in $U = U(\alpha) = \exp(i\alpha^a T^a)$ are constant (indices in α^a are raised and lowered with the Killing metric, that coincides with the identity in our conventions). Note that all the fermion masses must be identical, otherwise the mass term would break the $SU(N)$ symmetry. There is also an abelian $U(1)$ symmetry, but its gauging goes as described previously.

3.3 Covariant derivative

We would like to extend the theory to achieve a local $SU(N)$ symmetry. It is again convenient to introduce the concept of covariant derivatives for this purpose. By definition, the covariant derivative when applied to tensors produces new tensors. To start with, we recall that a tensor $\psi(x)$ in the fundamental representation of the gauge group $SU(N)$, i.e. the representation usually indicated by its dimension $\mathbf{N}$, is a field defined by the transformation

$$\psi(x) \quad \to \quad \psi'(x) = U(x)\psi(x) \tag{26}$$

where $U(x)$ is a $N \times N$ matrix of $SU(N)$ for any spacetime point x. More generally, fields transforming in any given representation $R(U(x))$ of the original matrices $U(x)$ are said to be tensors in the representation R. For example, the field $\overline{\psi}(x)$ is a tensor in the anti-fundamental representation, usually indicated by $\overline{\mathbf{N}}$ as it correspoinds to the complex conjugate of the representation $\mathbf{N}$, and its transformation rule is written as follows

$$\overline{\psi}(x) \quad \to \quad \overline{\psi'}(x) = \overline{\psi}(x)U^{-1}(x) \ . \tag{27}$$

Evidently, the term $\overline{\psi}(x)\psi(x)$ is a scalar under the gauge transformation.

The covariant derivative acting on tensors produces new tensors. It is defined by

$$D_\mu = \partial_\mu + W_\mu(x) \tag{28}$$

where $W_\mu(x)$ is a matrix-valued gauge field, also known as the connection (geometrically, it defines a parallel transport in a certain space, known as fiber bundle). When applied to the tensor $\psi(x)$, the term with the gauge field W_μ mixes the N Dirac fermions contained in ψ, and thus W_μ is given by $N \times N$ matrices (a matrix for any value of the index μ). It performs infinitesimal group transformations and defines a sort of parallel transport. It can be expanded in terms of the generators as follows

$$W_\mu(x) = -iW^a_\mu(x)T^a \ . \tag{29}$$

This relation defines the gauge fields $W^a_\mu(x)$, and there are N^2-1 of them for the gauge group $SU(N)$. From the requirement of covariance

$$\begin{aligned} \psi(x) \quad &\to \quad \psi'(x) = U(x)\psi(x) \\ D_\mu\psi(x) \quad &\to \quad D'_\mu\psi'(x) = U(x)D_\mu\psi(x) \end{aligned} \tag{30}$$

one obtains the following transformation rule for the gauge potential

$$W_\mu(x) \quad \to \quad W'_\mu(x) = U(x)W_\mu(x)U^{-1}(x) + U(x)\partial_\mu U^{-1}(x) \ . \tag{31}$$

Let us prove this. Requiring that $D'_\mu\psi' = UD_\mu\psi$, one computes

$$\begin{aligned} D'_\mu\psi' &\equiv (\partial_\mu + W'_\mu)\psi' \\ &= UD_\mu\psi = U\partial_\mu\psi + UW_\mu\psi = U\partial_\mu(U^{-1}U\psi) + UW_\mu U^{-1}U\psi \\ &= (U\partial_\mu U^{-1})\psi' + \partial_\mu\psi' + UW_\mu U^{-1}\psi' \\ &= [\partial_\mu + (UW_\mu U^{-1} + U\partial_\mu U^{-1})]\psi' \end{aligned} \tag{32}$$

which proves the above transformation rule for W_μ. A comment: as ψ transforms in the fundamental representation of $SU(N)$, the T^a contained in the W_μ of eqs. (29) and (32) are the generators in the fundamental representation. In general, one must use the generators in the representation of the tensors on which the covariant derivative acts upon.

Covariant derivatives do not commute. This fact allows us to define the "curvature" tensor (or "field strength") the following way

$$[D_\mu, D_\nu]\psi = F_{\mu\nu}\psi \tag{33}$$

so that

$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu] \ . \tag{34}$$

It is immediate to check that the field strength transform covariantly as

$$F_{\mu\nu} \quad \to \quad F'_{\mu\nu} = U F_{\mu\nu} U^{-1} \tag{35}$$

which follows from the covariance of (33). This rule corresponds to the adjoint representation (the representation of dimension $N^2 - 1$).

3.4 Gauge invariant action

It is now simple to construct a gauge invariant lagrangian from (24): it is enough to substitute derivatives with gauge covariant derivatives (this is also called "minimal coupling"). One finds

$$\mathcal{L}_1 = -\overline{\psi}(\gamma^\mu D_\mu + m)\psi \tag{36}$$

that depends on the new field W_μ contained in the covariant derivative D_μ. Now, one must introduce dynamics for W_μ as well. Considering the simplest gauge and Lorentz invariant lagrangian with at most two derivatives one is led to

$$\mathcal{L}_2 = \frac{1}{2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} \tag{37}$$

where we used the generators in the fundamental representation normalized by $\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$. We have chosen here a canonical normalization. More generally, one should consider a relative weight between the different gauge-invariant pieces, so that introducing a coupling constant g one writes the final lagrangian as

$$\boxed{\mathcal{L} = \frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \overline{\psi}(\gamma^\mu D_\mu + m)\psi} \tag{38}$$

which is gauge invariant under the following transformation rules

$$\begin{aligned}
\psi(x) &\to \psi'(x) = U(x)\psi(x) \\
\overline{\psi}(x) &\to \overline{\psi}'(x) = \overline{\psi}(x)U^{-1}(x) \\
W_\mu(x) &\to W'_\mu(x) = U(x)W_\mu(x)U^{-1}(x) + U(x)\partial_\mu U^{-1}(x) \\
F_{\mu\nu}(x) &\to F'_{\mu\nu}(x) = U(x)F_{\mu\nu}(x)U^{-1}(x) \ .
\end{aligned} \tag{39}$$

Let us report the infinitesimal transformations as well. Defining the matrix $\alpha \equiv -i\alpha_a T^a$ with Lie parameters $\alpha_a \ll 1$, one describes infinitesimal transformation in the form

$$U = e^{i\alpha_a T^a} = e^{-\alpha} = 1 - \alpha + O(\alpha^2) \tag{40}$$

so that for infinitesimal variations of fields ($\delta\psi(x) \equiv \psi'(x) - \psi(x)$) one finds

$$\begin{aligned} \delta\psi &= -\alpha\psi \\ \delta\overline{\psi} &= \overline{\psi}\alpha \\ \delta W_\mu &= \partial_\mu\alpha + [W_\mu, \alpha] = D_\mu\alpha \\ \delta F_{\mu\nu} &= [F_{\mu\nu}, \alpha] \end{aligned} \tag{41}$$

where in the last-but-one line we recognize the covariant derivative acting on the adjoint representation. The infinitesimal form of the gauge transformations is needed when studying the gauge-fixing procedure required to quantize the theory.

Now, let us rewrite the lagrangian by considering a field redefinition defined by the shift $W_\mu \to gW_\mu$. It is used to get a canonical normalization for the gauge field, which amounts to pushing the coupling constant in front of the interaction vertices. The shift induces a redefinition of the field strength $F_{\mu\nu} \to gF_{\mu\nu}$, with the new field strength given by

$$F_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + g\,[W_\mu, W_\nu]\ . \tag{42}$$

In components,

$$\begin{aligned} W_\mu(x) &= -iW^a_\mu(x)T^a \\ F_{\mu\nu}(x) &= -iF^a_{\mu\nu}(x)T^a \end{aligned} \tag{43}$$

and

$$F^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + gf^{abc}W^b_\mu W^c_\nu \tag{44}$$

and the lagrangian takes the more explicit form

$$\boxed{\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \overline{\psi}[\gamma^\mu(\partial_\mu - igW^a_\mu T^a) + m]\psi}\ . \tag{45}$$

The coupling constant now appears in front of the interaction vertices. The infinitesimal gauge transformations, after redefining the parameters $\alpha^a \to g\alpha^a$, read

$$\begin{aligned} \delta\psi(x) &= ig\alpha^a(x)T^a\psi(x) \\ \delta W^a_\mu(x) &= \partial_\mu\alpha^a(x) + gf^{abc}W^b_\mu(x)\alpha^c(x) = D_\mu\alpha^a(x)\ . \end{aligned} \tag{46}$$

The first term in the action (45) describes the free propagation of the fields W^a_μ (the non-abelian spin 1 particles) along with cubic and quartic self-interactions with coupling constant g and g^2, respectively. A non positive-definite Killing metric would result in a term with kinetic energy that is not positive-definite, and this would not be acceptable: it is necessary to consider only compact groups to satisfy this request. The second term describes the free propagation of the ψ fields (spin 1/2 particles with non-abelian charges, i.e. “color” charges) together with an interaction with the gauge field W^a_μ with a strength measured by the coupling constant g. It can be treated perturbatively if g is small enough. The “non-abelian” or “color” charge corresponds to the representation of the gauge group chosen for the fields ψ (in our case we have taken the fundamental representation, but any other representation could have been chosen as well). Thus, we have seen how the principle of gauge invariance has allowed us to derive all the interaction vertices between fields of spin 1/2 and 1 in terms of the single coupling constant g.

It is useful to analyze in more detail the structure of the infinitesimal transformation laws that we have been deriving. Introducing indices, the infinitesimal transformation of the fermion field $\psi(x)$ in (41) is written as

$$\delta\psi^i(x) = ig\alpha^a(x)(T^a_F)^i{}_j\psi^j(x) \tag{47}$$

with $i, j = 1, ..., N$, and where by $(T^a_F)^i{}_j$ we now indicate the generators in the fundamental representation ($N \times N$ matrices, one for each value of the index a). Similarly, the transformation of the Dirac conjugate field (morally, the complex conjugate field) is written as

$$\delta\overline{\psi}_i(x) = -ig\alpha^a(x)(T^{a*}_F)_i{}^j\,\overline{\psi}_j(x) = ig\alpha^a(x)(T^a_{\overline{F}})_i{}^j\,\overline{\psi}_j(x) \tag{48}$$

where[2] $T^a_{\overline{F}} = -T^{a*}_F$ are the generators in the complex conjugate of the fundamental representation T^a_F. Also, we may check that the field strength $F^a_{\mu\nu}$ transforms in the adjoint representation

$$\delta F^a_{\mu\nu} = gf^{abc}F^b_{\mu\nu}\alpha^c = ig\alpha^c(T^c_{\text{Adj}})^{ab}F^b_{\mu\nu} \tag{49}$$

as the generators in the adjoint representation are given by

$$(T^c_{\text{Adj}})^{ab} = -if^{abc}\ . \tag{50}$$

Let us also verify explicitly that the transformation law of W^a_μ can be expressed in terms of the covariant derivative acting on a tensor in the adjoint representation

$$\delta W^a_\mu = \partial_\mu\alpha^a + gf^{abc}W^b_\mu\alpha^c = \partial_\mu\alpha^a - igW^b_\mu(T^b_{\text{Adj}})^{ac}\alpha^c = D_\mu\alpha^a\ . \tag{51}$$

Note also that the non-derivative part of this transformation can be written equivalently as

$$\delta W^a_\mu = ig\alpha^c(T^c_{\text{Adj}})^{ab}W^b_\mu + \cdots \tag{52}$$

which highlights the tensorial character of this part of the transformation, matching (49).

Finally, let us recall that the Jacobi identities for arbitrary operators, once applied to the covariant derivative $D_\mu = \partial_\mu + W_\mu$,

$$[D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] + [D_\lambda, [D_\mu, D_\nu]] = 0\ , \tag{53}$$

give rise to the so-called Bianchi identities for the field strength $F_{\mu\nu}$

$$D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0\ , \tag{54}$$

where the covariant derivative in the adjoint representation is written as

$$D_\mu F_{\nu\lambda} = \partial_\mu F_{\nu\lambda} + [W_\mu, F_{\nu\lambda}] \tag{55}$$

that, in components, takes the form

$$D_\mu F^a_{\nu\lambda} = \partial_\mu F^a_{\nu\lambda} + gf^{abc}\,W^b_\mu\,F^c_{\nu\lambda}\ . \tag{56}$$

[2]Given the matrices in the fundamental representation $U = e^{i\alpha^a T^a_F}$, the complex conjugate matrices also furnish a representation $U^* = e^{-i\alpha^a T^{a*}_F} = e^{i\alpha^a(-T^{a*}_F)} \equiv e^{i\alpha^a T^a_{\overline{F}}}$ so that $T^a_{\overline{F}} = -T^{a*}_F = -T^{a\,T}_F$, where the last relation follows because the generators are hermitian.

3.5 The action of quantum chromodynamics (QCD)

The action of quantum chromodynamics is based on the group $SU(3)$. In addition to the gluons (the eight spin 1 particles associated with the gauge field W_μ^a, which has an index in the adjoint representation, and thus belongs to the **8** of $SU(3)$), the lagrangian contains six fermion fields ψ_f corresponding to the six known flavors of quarks, $f = (u, d, c, s, t, b)$, namely up, down, charm, strange, top, bottom. Each quark flavor is degenerate, as it transforms in the **3** of the $SU(3)$ gauge group: the quark is said to be colored (with color red, green and blue, in the usual convention). The absence of color indicates a scalar, like the lagrangian (the scalar corresponds to the representation **1** of $SU(3)$). Of course, the corresponding antiparticles, the antiquarks $(\bar{u}, \bar{d}, \bar{c}, \bar{s}, \bar{t}, \bar{b})$, transform in the conjugate representation, the $\bar{\mathbf{3}}$ of $SU(3)$, which is the representation of $\overline{\psi}_f$, the Dirac conjugate of ψ_f.

The eight infinitesimal generators of $SU(3)$ in the fundamental representation are given by the Gell-Mann matrices λ^a, which generalize the Pauli matrices σ^i of $SU(2)$ to $SU(3)$,

$$T^a = \frac{\lambda^a}{2} \qquad a = 1, \ldots, 8 \tag{57}$$

where

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$
$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{58}$$

These generators are normalized so that $\mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

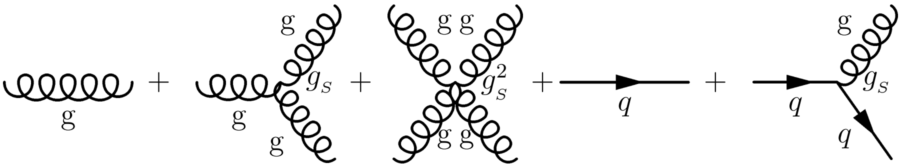
An arbitrary element of the $SU(3)$ group in the fundamental representation is therefore given by 3×3 matrices of the form $U = \exp(i\alpha_a \frac{\lambda^a}{2})$. By calculating the Lie algebra $[T^a, T^b] = i f^{abc} T^c$, one finds the explicit values of the structure constants f^{abc} of the $SU(3)$ group. They are totally antisymmetric and given by:

$$f^{123} = 1$$
$$f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}$$
$$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

while all other f^{abc} not related to these by permuting indices are zero. As an exercise, try to verify some of these numbers.

The QCD lagrangian is therefore

$$
\begin{aligned}
\mathcal{L}_{QCD} &= -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \sum_{f=1}^{6}\overline{\psi}_f\Big(\gamma^\mu D_\mu + m_f\Big)\psi_f \\
&= -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \sum_{f=1}^{6}\overline{\psi}_f\Big(\gamma^\mu \partial_\mu + m_f\Big)\psi_f + i\frac{g_S}{2}W^a_\mu\sum_{f=1}^{6}\overline{\psi}_f\gamma^\mu\lambda^a\psi_f
\end{aligned}
\tag{59}
$$

$=$ 

where the coupling constant is denoted by g_S, and the index $f \in (1, 2, \cdots, 6) = (u, d, c, s, t, b)$ indicates the flavor of the quark. Different flavors of quarks have different masses m_f. Note that to obtain the propagator of the gauge field from the first term, as indicated in the figure, one must implement a gauge-fixing procedure.

The QCD lagrangian possesses additional rigid symmetries. A well-known rigid symmetry is the $U(1)$ symmetry which rotates all fields of the quarks by the same phase: the associated conserved charge is the *baryon number*. It is a symmetry that is preserved by the other fundamental interactions as well.

Other $U(1)$ symmetries rotate the various fermionic fields separately. They give rise to conservation laws of the respective *fermion numbers* (e.g. *strangeness* $\mathcal{S}$, *charm* $\mathcal{C}$, etc ..). These flavor symmetries are exact for QCD (and QED), but the weak force violates them. There are six $U(1)$ independent conserved charges, one for each quark flavor, and the baryon number B is a particular linear combination of these six independent charges. Also, the *electric charge* Q is a linear combination of them, which is eventually gauged to obtain the electromagnetic couplings.

A summary of these U(1) symmetries is given in the following table, which reports the various $U(1)$ charges with a standard normalization:

Quarks	$\mathcal{U}$	$\mathcal{D}$	$\mathcal{C}$	$\mathcal{S}$	$\mathcal{T}$	$\mathcal{B}$	B	Q
u	1	0	0	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$
d	0	-1	0	0	0	0	$\frac{1}{3}$	$-\frac{1}{3}$
c	0	0	1	0	0	0	$\frac{1}{3}$	$\frac{2}{3}$
s	0	0	0	-1	0	0	$\frac{1}{3}$	$-\frac{1}{3}$
t	0	0	0	0	1	0	$\frac{1}{3}$	$\frac{2}{3}$
b	0	0	0	0	0	-1	$\frac{1}{3}$	$-\frac{1}{3}$

Note that we have indicated the baryon number by B, and the bottom (or beauty) quantum number by $\mathcal{B}$. For each symmetry, each quark flavor transforms with the charge indicated in the table, for example for the electric charge Q we have

$$
\psi_f \to \psi'_f = e^{i\alpha Q_f}\psi_f \; . \tag{60}
$$

By looking at the table, one recognizes the following relations

$$B = \frac{1}{3}(\mathcal{U} + \mathcal{C} + \mathcal{T}) - \frac{1}{3}(\mathcal{D} + \mathcal{S} + \mathcal{B})$$
$$Q = \frac{2}{3}(\mathcal{U} + \mathcal{C} + \mathcal{T}) + \frac{1}{3}(\mathcal{D} + \mathcal{S} + \mathcal{B}) \; . \tag{61}$$

There are also approximate symmetries of the QCD lagrangian. In the limit in which some of the quark masses are taken to be identical, there is a rigid additional non-abelian symmetry. For example, assuming identical masses for the up and down quarks, $m_u = m_d$, one can mix the fields ψ_u and ψ_d with a $SU(2)$ matrix to redefine what is meant by "up" and "down"

$$\begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \rightarrow \begin{pmatrix} \psi'_u \\ \psi'_d \end{pmatrix} = U \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \qquad U \in SU(2) \; . \tag{62}$$

This rigid $SU(2)$ symmetry corresponds to the *strong isospin* $\vec{I}$, used to group hadrons into families or mulitplets. From phenomenology, one knows that the strong force binds states of quarks together and show the phenomenon of color confinement: bound states of quarks are color singlets and correspond to the mesons and baryons. Examples of these families are: (*i*) the isospin doublet of the nucleons (proton and nucleon) composed of three confined up and down quarks; (*ii*) the triplet of π mesons, the pions $\pi^\pm$ and π^0, composed of a quark and an antiquark of the up and down types.

Considering identical the masses for the quarks up, down, and strange, $m_u = m_d = m_s$, one finds an even larger symmetry group, the $SU(3)$ *flavor group*, that mixes the three flavors up, down and strange:

$$\begin{pmatrix} \psi_u \\ \psi_d \\ \psi_s \end{pmatrix} \rightarrow \begin{pmatrix} \psi'_u \\ \psi'_d \\ \psi'_s \end{pmatrix} = U \begin{pmatrix} \psi_u \\ \psi_d \\ \psi_s \end{pmatrix} \qquad U \in SU(3) \; . \tag{63}$$

This $SU(3)$ flavor group is the one that is used in the static quark model (the "eightfold way" of Gell-Mann) to take care of the similarities observed between the various hadrons. It should not be confused with the color group, also an $SU(3)$ group. As already said, color is expected to confine inside the hadrons and leaves only composite colorless states. Examples of multiplets of hadronic particles described by the $SU(3)$ flavor group are:
the meson octet $(\pi^\pm, \pi^0, K^\pm, K^0, \bar{K}^0, \eta)$,
the baryon octet $(p, n, \Sigma^\pm, \Sigma^0, \Xi^\pm, \Lambda)$,
the baryon decuplet $(\Delta^-, \Delta^0, \Delta^+, \Delta^{++}, \Sigma^{*\pm}, \Sigma^{*0}, \Xi^{*\pm}, \Omega^-)$.
The existence of these families is compatible with group theory: the **8** and the **10** are representations of $SU(3)$. Let us consider the mesons in more detail. They consist of a quark-antiquark pair $(q\bar{q})$. The quarks q transform in the **3** of $SU(3)$, with $\mathbf{3} \sim (u, d, s)$, while antiquarks $\bar{q}$ transforms in the $\bar{\mathbf{3}}$ of $SU(3)$, with $\bar{\mathbf{3}} \sim (\bar{u}, \bar{d}, \bar{s})$. From this, it follows that possible bound states $(q\bar{q})$ must transform in the

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$$

and therefore both singlet and octets could in principle exist for the mesons.

Baryons are bound states of three quarks (qqq), and since $SU(3)$ group theory tells us that

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = (\mathbf{6} \oplus \bar{\mathbf{3}}) \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$$

so that we understand that octets and decuplets are allowed possibilities for multiplets of baryons.

A Notes on group theory

A.1 Lie groups and algebras

Given a simple and compact Lie group G, we indicate its elements using the exponential parametrization $U(\alpha) = \exp(i\alpha_a T^a)$, where T^a are the infinitesimal hermitian generators that satisfy the Lie algebra

$$[T^a, T^b] = i f^{ab}{}_c T^c \ . \tag{64}$$

In general, considering an irreducible unitary representation R of G, we automatically get an irreducible representation of its Lie algebra with traceless hermitian matrices T^a_R

$$[T^a_R, T^b_R] = i f^{ab}{}_c T^c_R \ . \tag{65}$$

The matrices T^a_R act on a vector space of dimensions $D(R)$, and thus are $D(R) \times D(R)$ matrices. $D(R)$ is called the dimension of the representation. We will mostly consider $SU(N)$, whose main representations are:

- the fundamental (or defining) representation N, with $D(N) = N$
- its complex conjugate representation $\overline{N}$, with $D(\overline{N}) = N$
- the adjoint representation Adj, with $D(\text{Adj}) = N^2 - 1$.

Given a representation R with generators T^a_R, the generators of its complex conjugate representation $\overline{R}$ are given by

$$T^a_{\overline{R}} = -T^{a\,*}_R \tag{66}$$

as seen from taking the complex conjugate of the original representation

$$(\exp(i\alpha_a T^a_R))^* = \exp(-i\alpha_a T^{a\,*}_R) \equiv \exp(i\alpha_a T^a_{\overline{R}}) \ . \tag{67}$$

The generators are usually normalized so that in the fundamental representation one has

$$\text{tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{68}$$

which fixes the Killing metric $\gamma^{ab} \equiv 2\,\text{tr}(T^a T^b)$ to be $\gamma^{ab} = \delta^{ab}$. This metric defines scalar products and raises or lowers the indices that label the generators (the indices of the adjoint representation). In particular, it is used to define the structure constants with upper indices

$$f^{abc} = f^{ab}{}_d \delta^{dc} \tag{69}$$

(more generally $f^{abc} = f^{ab}{}_d \gamma^{dc}$). This is proven to be totally antisymmetric. The antisymmetry of f^{abc} is obvious on the first two indices, as seen from the definition of the Lie algebra. Then using (64) and (68) one can compute

$$\begin{aligned} \text{tr}([T^a, T^b]T^c) &= i f^{ab}{}_d \,\text{tr}(T^d T^c) = \frac{i}{2} f^{abc} = \text{tr}(T^a T^b T^c) - \text{tr}(T^b T^a T^c) \\ &= \text{tr}(T^c T^a T^b) - \text{tr}(T^a T^c T^b) = -\text{tr}([T^a, T^c]T^b) = -\frac{i}{2} f^{acb} \end{aligned} \tag{70}$$

so that $f^{abc} = -f^{acb}$, which implies complete antisymmetry. In the above manipulations, we have used the cyclic property of the trace.

The structure constants can be used to define the adjoint representation 'Adj' by

$$(T^a_{\text{Adj}})^b{}_c = -i f^{ab}{}_c \tag{71}$$

since the relation

$$[T^a_{\rm Adj}, T^b_{\rm Adj}] = i f^{ab}{}_c T^c_{\rm Adj} \tag{72}$$

is satisfied because of the Jacobi identity.

One defines the *index* $T(R)$ of a representation R by

$$\mathrm{tr}(T^a_R T^b_R) = T(R)\,\delta^{ab}\ . \tag{73}$$

The index of the fundamental representation N is usually normalized to $T(N) = \frac{1}{2}$, see (68). Then, the values of the indices of other representations are fixed unambiguously.

Casimir operators are operators built from the generators which commute with all the generators of the group. In particular, the quadratic Casimir operator that is constructed using the Killing metric

$$C_2 = T^a T^b \gamma_{ab} = T^a T^a \tag{74}$$

is such an operator. The proof is simple

$$\begin{aligned}[C_2, T^b] &= [T^a T^a, T^b] = T^a[T^a, T^b] + [T^a, T^b]T^a = T^a i f^{abc} T^c + i f^{abc} T^c T^a \\ &= i f^{abc}(T^a T^c + T^c T^a) = 0\end{aligned} \tag{75}$$

as the structure constants are completely antisymmetric[3]. Since C_2 commutes with all the generators, it must be proportional to the identity in any given irreducible representation (Schur's lemma). This defines the number $C(R)$, the quadratic Casimir in the irrep R, by

$$T^a_R T^a_R = C(R)\,\mathbb{1}\ . \tag{76}$$

Setting $a = b$ in (73) and summing (i.e. taking the scalar product with the Killing metric) gives the relation

$$C(R)D(R) = T(R)D(\mathrm{Adj})\ . \tag{77}$$

For the simplest representations one finds

$$D(N) = D(\overline{N}) = N \qquad T(N) = T(\overline{N}) = \frac{1}{2} \qquad C(N) = C(\overline{N}) = \frac{N^2-1}{2N} \tag{78}$$

$$D(\mathrm{Adj}) = N^2 - 1 \qquad T(\mathrm{Adj}) = N \qquad C(\mathrm{Adj}) = N\ . \tag{79}$$

Finally, it is useful to recall the concept of *invariant tensors*. They are defined as tensors that remain invariant after group transformations. For example, denoting by ψ^i the vectors transforming in the defining representation of $SU(N)$, so that the upper index i is transformed by the defining matrices $U^i{}_j$ of $SU(N)$, then the Kronecker symbol δ^i_j is an invariant tensor

$$\delta^i_j \;\to\; \delta'^i_j = U^i{}_k (U^{-1,T})_j{}^l \delta^k_l = U^i{}_k (U^*)_j{}^l \delta^k_l = U^i{}_k (U^*)_j{}^k = \delta^i_j\ . \tag{80}$$

It tells that in combining the representation N with $\overline{N}$ there must appear a scalar

$$N \otimes \overline{N} = 1 \otimes + \cdots \tag{81}$$

i.e. one can form the scalar $\psi^i \chi_i$ out of ψ^i and χ_i. Similarly, the completely antisymmetric tensor with N upper indices, $\epsilon^{i_1 i_2 \ldots i_N}$, normalized to $\epsilon^{12\ldots N} = 1$, is an invariant tensor

$$\epsilon'^{i_1 i_2 \ldots i_N} = \epsilon^{i_1 i_2 \ldots i_N} \tag{82}$$

[3]We have used that $[AB, C] = A[B, C] + [A, C]B$ for arbitrary operators.

known also as the Levi-Civita symbol. Indeed, one verifies that

$$\epsilon^{i_1 i_2 \dots i_N} \;\to\; \epsilon'^{i_1 i_2 \dots i_N} = U^{i_1}{}_{j_1} U^{i_2}{}_{j_2} \dots U^{i_N}{}_{j_N} \epsilon^{j_1 j_2 \dots j_N} = (\det U) \epsilon^{i_1 i_2 \dots i_N} \tag{83}$$

but $\det U = 1$ for $SU(N)$ and the invariant property follows. Same thing for $\epsilon_{i_1 i_2 \dots i_N}$.

Other invariant tensors are the generators in any given representation R, which we write as $(T^a_R)^\alpha{}_\beta$, where the upper index α is associated the representation R and the lower index β to the conjugate representation $\overline{R}$ (see note[4]) This statement follows from the Lie algebra (65) by recognizing that the structure constants $f^{ab}{}_c$ give rise to the generators in the adjoint representation, that transforms the index a of $(T^a_R)^\alpha{}_\beta$. This also means that

$$R \otimes \overline{R} \otimes \text{Adj} = 1 \oplus \cdots \ . \tag{84}$$

Moreover, since the adjoint is a real representation (the $f^{ab}{}_c$ are real numbers and thus the group elements $e^{i\alpha_a T^a_{\text{Adj}}}$ are real) one understands that

$$\text{Adj} \otimes \text{Adj} = 1 \oplus \cdots \tag{85}$$

that matches with the fact that the Killing metric δ^{ab} is an invariant tensor that is used to construct scalar products (more generally the tensor δ^α_β for the arbitrary representations R and $\bar{R}$ is an invariant tensor). Then (84) and (85) imply

$$R \otimes \overline{R} = \text{Adj} \oplus \cdots \tag{86}$$

which is interpreted by saying that $(T^a_R)^\alpha{}_\beta$ are Clebsch-Gordan coefficients: they combine the tensors in the representation R with those in the representation $\overline{R}$ to extract a tensor transforming in the adjoint. Said differently, Clebsch-Gordan coefficients are invariant tensors.

Finally, let us define another invariant tensor, the d^{abc} tensor, together with the anomaly coefficients $A(R)$ by

$$A(R) d^{abc} = \frac{1}{2} \text{tr} \left(T^a_R \{ T^b_R, T^c_R \} \right) \tag{87}$$

where the overall normalization may be fixed by setting $A = 1$ for the fundamental representation. It is totally symmetric and appears in the study of chiral anomalies. The only simple groups that have a non-vanishing d^{abc} tensor, and therefore a cubic Casimir operator $C_3 \sim d^{abc} T^a T^b T^c$, are $SU(N)$ for $N \geq 3$ and $SO(6)$.

A.2 Cartan-Weyl basis

It is often useful to rewrite the generators of a Lie algebra in the Cartan-Weyl basis. This is defined by first finding the maximal number of generators (or independent linear combination of generators) H_i that commute between themselves

$$[H_i, H_j] = 0 \ . \tag{88}$$

This maximal number is called the *rank* of the group. They are taken to be hermitian, and they define the Cartan subalgebra of the Lie algebra. Since they commute, they can be diagonalized

[4] One may recall that given a representation R, one finds that $R^{-1,T}$, R^* and $R^{-1,\dagger}$ are also representations. These four representations acts on vectors $v^\alpha, v_\alpha, v^{\dot\alpha}, v_{\dot\alpha}$ belonging to the appropriate vector spaces (the original vector space, its dual, its complex conjugate, and the dual complex conjugate). For unitary representations $v^{\dot\alpha} \sim v_\alpha$ and $v_{\dot\alpha} \sim v^\alpha$.

simultaneously in any given representation, and the eigenvalues are called the *weights*. This definition generalizes the angular momentum generator J_3 of $SU(2)$, which is a group of rank 1. J_3 is the generator that is usually diagonalized in quantum mechanics[5]. The particular weights of the adjoint representation are called *roots*.

The remaining generators are combined in complex combinations so that they correspond to the roots α_i

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{89}$$

which can be interpreted by saying that α_i are eigenvalues while E_α are eigenvectors (the root α is a vector with components α_i). The generators E_α cannot be hermitian, but rather one has that $E_\alpha^\dagger = E_{-\alpha}$, so that if α is a root then also $-\alpha$ is a root. They generalize the $J_\pm$ angular momentum operators of $SU(2)$. Finally, one has the remaining structure constants that appear in calculating

$$[E_\alpha, E_\beta] \ . \tag{90}$$

The Jacobi identity can be used to study them, and in particular one finds that

$$[E_\alpha, E_{-\alpha}] = \alpha_i H_i \ . \tag{91}$$

which also generalizes the $SU(2)$ case.

This basis (and a related one called the Chevalley basis) is very useful in deriving general properties of Lie algebras, in a close analogy with the theory of angular momentum in quantum mechanics. In particular, it helps in constructing the complete classification of simple Lie algebras, which is due to Killing and Cartan. This classification is encoded in the Dynkin diagrams of fig. 1. The algebras depicted there correspond to the following compact groups: $A_n = SU(n+1)$, $B_n = SO(2n+1)$, $C_n = Sp(2n)$, and $D_n = SO(2n)$, where n is the rank. The remaining algebras G_2, F_4, E_6, E_7, E_8 correspond to the so-called exceptional groups.

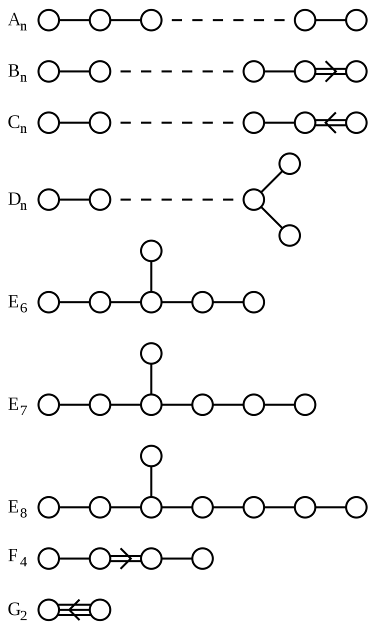


Figure 1: Dynkin diagrams

[5]Recall the SU(2) algebra: $[J_3, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_3$.