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Additional notes on group theory

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1 Lie groups and algebras

Given a simple and compact Lie group G, we indicate its elements using the exponential parametrization $U(\alpha) = \exp(i\alpha_a T^a)$, where T^a are the infinitesimal hermitian generators that satisfy the Lie algebra

$$[T^a, T^b] = i f^{ab}{}_c T^c \ . \tag{1}$$

In general, considering an irreducible representations R of G, we get an irreducible representations of its Lie algebra with traceless hermitian matrices T^a_R

$$[T^a_R, T^b_R] = i f^{ab}{}_c T^c_R \ . \tag{2}$$

The matrices T^a_R act on a vector space of dimensions $D(R)$, and thus are $D(R)\times D(R)$ matrices. $D(R)$ is called the dimension of the representation. We will mostly consider $SU(N)$, whose most used representations are:
• the fundamental (or defining) representation N, with $D(N) = N$
• its complex conjugate representation $\bar{N}$, with $D(\bar{N}) = N$
• the adjoint representation Adj, with $D(\text{Adj}) = N^2 - 1$.

Given a representation R with generators T^a_R, the generators of its complex conjugate representation $\bar{R}$ are given by

$$T^a_{\bar{R}} = -(T^a_R)^* \tag{3}$$

as seen from taking the complex conjugate of the original representation

$$(\exp(i\alpha_a T^a_R))^* = \exp(-i\alpha_a (T^a_R)^*) \equiv \exp(i\alpha_a T^a_{\bar{R}}) \ . \tag{4}$$

The generators are normalized so that in the fundamental representation one has

$$\text{tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{5}$$

which normalizes the so-called Killing metric $\gamma^{ab} = 2\,\text{tr}(T^a T^b)$ to $\gamma^{ab} = \delta^{ab}$. This matrix is used to define scalar products and to raise/lower the indices that label the generators. In particular, it is used to define the structure constants with all upper indices

$$f^{abc} = f^{ab}{}_d \delta^{dc} \tag{6}$$

(more generally $f^{abc} = f^{ab}{}_d \gamma^{dc}$). This is proven to be totally antisymmetric. The antisymmetry of f^{abc} is obvious on the first two indices, as seen from the definition of the Lie algebra. Then using (1) and (5) one can compute

$$\begin{aligned}\text{tr}([T^a, T^b]T^c) &= i f^{ab}{}_d\, \text{tr}(T^d T^c) = \frac{i}{2} f^{abc} = \text{tr}(T^a T^b T^c) - \text{tr}(T^b T^a T^c) \\ &= \text{tr}(T^c T^a T^b) - \text{tr}(T^a T^c T^b) = -\text{tr}([T^a, T^c]T^b) = -\frac{i}{2} f^{acb}\end{aligned} \tag{7}$$

so that $f^{abc} = -f^{acb}$, which implies complete antisymmetry. In the above manipulations we have used the cyclic property of the trace.

The structure constants can be used to define the adjoint representation 'Adj' by

$$(T^a_{\rm Adj})^b{}_c = -i f^{ab}{}_c \tag{8}$$

since the relation

$$[T^a_{\rm Adj}, T^b_{\rm Adj}] = i f^{ab}{}_c T^c_{\rm Adj} \tag{9}$$

reduces to the Jacobi identity and is thus satisfied.

One defines the *index* $T(R)$ of a representation R by

$$\mathrm{tr}(T^a_R T^b_R) = T(R)\,\delta^{ab} \ . \tag{10}$$

with the index of the fundamental representation N normalized by (5) to $T(N) = \frac{1}{2}$.

Casimir operators are operators built from the generators which commute with all the generators of the group. In particular, the quadratic Casimir operator constructed using the Killing metric

$$C_2 = T^a T^b \gamma_{ab} = T^a T^a \tag{11}$$

is such an operator. The proof is simple

$$\begin{aligned} [C_2, T^b] &= [T^a T^a, T^b] = T^a[T^a, T^b] + [T^a, T^b]T^a = T^a i f^{abc} T^c + i f^{abc} T^c T^a \\ &= i f^{abc}(T^a T^c + T^c T^a) = 0 \end{aligned} \tag{12}$$

that follows since the structure constants are completely antisymmetric[1]. Since C_2 commutes with all the generators, it must be proportional to the identity in any given irreducible representation. This defines the number $C(R)$, the quadratic Casimir in the irrep R, by

$$T^a_R T^a_R = C(R)\,\mathbb{1} \ . \tag{13}$$

Setting $a = b$ in (10) and summing (i.e. taking the scalar product with the Killing metric) gives the relation

$$T(R) D(\mathrm{Adj}) = C(R) D(R) \ . \tag{14}$$

For the simplest representation one finds

$$D(N) = D(\bar N) = N \qquad T(N) = T(\bar N) = \frac{1}{2} \qquad C(N) = C(\bar N) = \frac{N^2-1}{2N} \tag{15}$$

$$D(\mathrm{Adj}) = N^2 - 1 \qquad T(\mathrm{Adj}) = N \qquad C(\mathrm{Adj}) = N \ . \tag{16}$$

Finally, it is useful to recall the concept of *invariant tensors*. They are defined to be tensors that remain invariant after group transformations. For example, denoting by ψ^i the vectors transforming in the defining representation of $SU(N)$, so that the upper index i is transformed by the defining matrices $U^i{}_j$ of $SU(N)$, then the Kronecker symbol δ^i_j is an invariant tensor

$$\delta^i_j \ \to \ \delta'^i_j = U^i{}_k (U^{-1,T})_j{}^l \delta^k_l = U^i{}_k (U^*)_j{}^l \delta^k_l = U^i{}_k (U^*)_j{}^k = \delta^i_j \ . \tag{17}$$

It tells that in combining the representation N with $\bar N$ there appears a scalar

$$N \otimes \bar N = 1 \otimes + \cdots \tag{18}$$

[1] We have used that $[AB, C] = A[B,C] + [A,C]B$ for arbitrary operators.

i.e. one can form the scalar $\psi^i \chi_i$ out of ψ^i and χ_i. Similarly, the completely antisymmetric tensor with N upper indices, $\epsilon^{i_1 i_2 \ldots i_N}$, normalized to one, $\epsilon^{12\ldots N} = 1$, is an invariant tensor

$$\epsilon'^{i_1 i_2 \ldots i_N} = \epsilon^{i_1 i_2 \ldots i_N} \tag{19}$$

known also as the Levi-Civita symbol. Indeed, one computes

$$\epsilon^{i_1 i_2 \ldots i_N} \;\to\; \epsilon'^{i_1 i_2 \ldots i_N} = U^{i_1}{}_{j_1} U^{i_2}{}_{j_2} \ldots U^{i_N}{}_{j_N} \epsilon^{j_1 j_2 \ldots j_N} = (\det U) \epsilon^{i_1 i_2 \ldots i_N} \tag{20}$$

but $\det U = 1$ for $SU(N)$, and the invariant property follows. Same thing for $\epsilon_{i_1 i_2 \ldots i_N}$.

Other invariant tensors are the generators in any given representation R, which we may write as $(T^a_R)^\alpha{}_\beta$, where the upper index α belongs to (the vectors of) the representation R and the lower index β to the conjugate representation $\bar{R}$ (see note[2]) This statement follows from the Lie algebra (2) by recognizing that the structure constants $f^{ab}{}_c$ give rise to the generators in the adjoint representation, that transforms the index a in $(T^a_R)^\alpha{}_\beta$. This also means that

$$R \otimes \bar{R} \otimes \mathrm{Adj} = 1 \oplus \cdots \;. \tag{21}$$

Moreover, since the adjoint is a real representation (the $f^{ab}{}_c$ are real numbers and thus the group elements $e^{i\alpha_a T^a_{\mathrm{Adj}}}$ are real) one may understand that

$$\mathrm{Adj} \otimes \mathrm{Adj} = 1 \oplus \cdots \tag{22}$$

that matches with the fact that the Killing metric δ^{ab} is an invariant tensor that can be used to construct scalar products (more generally the tensor δ^α_β for the arbitrary representations R and $\bar{R}$ is an invariant tensor). Then (21) and (22) imply

$$R \otimes \bar{R} = \mathrm{Adj} \oplus \cdots \tag{23}$$

which is interpreted by saying that $(T^a_R)^\alpha{}_\beta$ are Clebsch-Gordan coefficients: they combine the tensors in the representation R with those in the representation $\bar{R}$ to produce a tensor transforming in the adjoint. Said differently, Clebsch-Gordan coefficients are invariant tensors.

Finally, let us define another invariant tensor, the d^{abc} tensor, together with the anomaly coefficients $A(R)$ by

$$A(R) d^{abc} = \frac{1}{2} \mathrm{tr}\left(T^a_R \{T^b_R, T^c_R\}\right) \tag{24}$$

where the overall normalization may be fixed by setting $A = 1$ for the fundamental representation. It is totally symmetric and appears in the study of chiral anomalies. The only simple groups that have a non-vanishing d^{abc} tensor, and therefore a cubic Casimir operator $C_3 \sim d^{abc} T^a T^b T^c$, are $SU(N)$ for $N \geq 3$ and $SO(6)$.

1.1 Cartan-Weyl basis

It is often useful to rewrite the generators of a Lie algebra in the Cartan-Weyl basis. This is defined by first finding the maximal number of generators (or independent linear combination of generators) H_i that commute between themselves

$$[H_i, H_j] = 0 \;. \tag{25}$$

[2]One may recall that given a representation R, one finds that $R^{-1,T}$, R^* and $R^{-1,\dagger}$ are also representations. These four representations acts on vectors $v^\alpha, v_\alpha, v^{\dot\alpha}, v_{\dot\alpha}$ belonging to the appropriate vector space. For unitary representations $v^{\dot\alpha} \sim v_\alpha$ and $v_{\dot\alpha} \sim v^\alpha$.

This maximal number is called the *rank* of the group. They are taken to be hermitian, and they define the Cartan subalgebra of the Lie algebra. Since they commute, they can be diagonalized simultaneously in any given representation, and the eigenvalues are called the *weights*. This definition generalizes the angular momentum generator J_3 of $SU(2)$, which is a group of rank 1. J_3 is the generator that is usually diagonalized in quantum mechanics[3]. The particular weights of the adjoint representation are called *roots*.

The remaining generators are combined in complex combinations so that they correspond to the roots α_i

$$[H_i, E_\alpha] = \alpha_i E_\alpha \tag{26}$$

which can be interpreted by saying that α_i are eigenvalues and E_α are eigenvectors (the root α is a vector with components α_i). The generators E_α cannot be hermitian, but rather one has that $E_\alpha^\dagger = E_{-\alpha}$, so that if α is a root then also $-\alpha$ is a root. They generalize the $J_\pm$ angular momentum operators of $SU(2)$. Finally, one has the remaining structure constants that appear in calculating

$$[E_\alpha, E_\beta] \ . \tag{27}$$

The Jacobi identity can be used to study them, and in particular one finds that

$$[E_\alpha, E_{-\alpha}] = \alpha_i H_i \ . \tag{28}$$

which also generalizes the $SU(2)$ case.

This basis (and a related one called the Chevalley basis) are very useful in deriving general properties of the Lie algebras, in a close analogy with the theory of angular momentum in quantum mechanics. In particular, it is useful to prove the complete classification of simple Lie algebras, due to Killing and Cartan. This classification is often portrayed with the Dynkin diagrams of fig. 1. The algebras depicted there correspond to the following compact groups: $A_n = SU(n+1)$, $B_n = SO(2n+1)$, $C_n = Sp(2n)$, and $D_n = SO(2n)$, where n is the rank. The remaining algebras correspond to the so-called exceptional groups G_2, F_4, E_6, E_7, E_8.

[3]Recall the SU(2) algebra: $[J_3, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_3$.

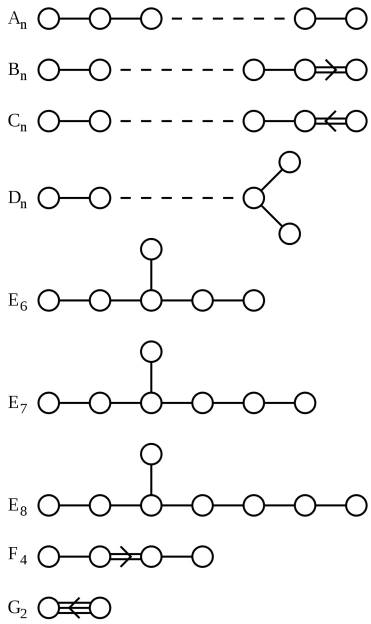

Figure 1: Dynkin diagrams