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QCD, background field method, and anomalies

(Lecture notes - a.a. 2024/25)

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1 Feynman rules for perturbative QCD

Having obtained the gauge-fixed action for the non-abelian gauge theories, one can write down the QCD gauge-fixed lagrangian by adding the fermions (the quark fields). Using an arbitrary R_ξ, gauge one obtains the following gauge-fixed lagrangian

$$\mathcal{L}_{tot} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - B^a\partial^\mu A^a_\mu + \frac{\xi}{2}(B^a)^2 - \partial^\mu \bar{c}^a D_\mu c^a - \sum_{f=1}^{6}\overline{\psi}_f(\gamma^\mu D_\mu + m_f)\psi_f \ . \tag{1}$$

Eliminating the auxiliary field B^a one finds

$$\boxed{\mathcal{L}_{tot} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - \partial^\mu \bar{c}^a D_\mu c^a - \sum_{f=1}^{6}\overline{\psi}_f(\gamma^\mu D_\mu + m_f)\psi_f} \ . \tag{2}$$

This is the starting point for developing the perturbative expansion. The lagrangian is split as $\mathcal{L}_{tot} = \mathcal{L}_2 + \mathcal{L}_{int}$, with $\mathcal{L}_2$ the quadratic part used to find the propagators, and $\mathcal{L}_{int}$ the interacting part used to get the vertices. Recalling that

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{3}$$

one finds

$$\mathcal{L}_2 = -\frac{1}{2}(\partial_\mu A^a_\nu)^2 + (\frac{1}{2} - \frac{1}{2\xi})(\partial^\mu A^a_\mu)^2 - \partial^\mu \bar{c}^a \partial_\mu c^a - \sum_{f=1}^{6}\overline{\psi}_f(\gamma^\mu \partial_\mu + m_f)\psi_f \tag{4}$$

and

$$\begin{aligned}\mathcal{L}_{int} =& - g f^{abc}\partial^\mu A^{\nu a} A^b_\mu A^c_\nu - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} \\ & - g f^{abc}\partial^\mu \bar{c}^a A^b_\mu c^c + i g A^a_\mu \sum_{f=1}^{6}\overline{\psi}_f \gamma^\mu T^a \psi_f \ .\end{aligned} \tag{5}$$

where the interaction vertices are all controlled by the same coupling constant g.

<u>Propagators</u>

It is immediate to extract the propagators from (4). We recall that the perturbative propagators are the inverse of the "kinetic" terms, i.e. the differential operators contained in the quadratic lagrangian. In an hypercondensed notation (for real fields ϕ^i and complex fields $\psi^i, \overline{\psi}_i$) we have

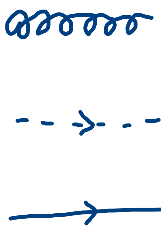
$$\begin{array}{lll} S[\phi] = -\frac{1}{2}\phi^i K_{ij}\phi^j & \to \quad \langle \phi^i \phi^j \rangle = -iG^{ij} & \quad (K_{ij}G^{jk} = \delta_i^k) \\ S[\psi,\bar\psi] = -\bar\psi_i K^i{}_j \psi^j & \to \quad \langle \psi^i \bar\psi_j \rangle = -iG^i{}_j & \quad (K^i{}_j G^j{}_k = \delta^i_k) \end{array} \tag{6}$$

valid for either commuting or anticommuting fields.

Thus, from $\mathcal{L}_2$ we get

$$
\begin{aligned}
\langle A^a_\mu(x) A^b_\nu(y)\rangle &= -i\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{\delta^{ab}}{p^2-i\epsilon}\left(\eta_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2}\right)\\
\langle c^a(x)\bar{c}^b(y)\rangle &= -i\int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{\delta^{ab}}{p^2-i\epsilon}\\
\langle \psi_f(x)\overline{\psi}_f(y)\rangle &= -i\int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot(x-y)}\, \frac{-i\not{p}+m_f}{p^2+m_f^2-i\epsilon}
\end{aligned}
\tag{7}
$$

with no sum over f (the quark flavours), from which one extracts the propagators in momentum space



$$
\begin{aligned}
&-i\frac{\delta^{ab}}{p^2-i\epsilon}(\eta_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2}\\
&-i\frac{\delta^{ab}}{p^2-i\epsilon}\\
&-i\frac{-i\not{p}+m_f}{p^2+m_f^2-i\epsilon}
\end{aligned}
\tag{8}
$$

<u>Vertices</u>

As for the vertices in momentum space, one must Fourier transform the various monomials contained in $\mathcal{L}_{int}$.

Let us review our conventions by presenting the Fourier transform of the effective action (the quantum action in Srednicki's book, see ch. 21 pag 127), and the relate *proper vertices* (the amputated 1PI Feynman diagrams). The effective action $\Gamma[\varphi]$ for a generic field φ is expanded in a Taylor series as

$$
\begin{aligned}
\Gamma[\varphi] &= \sum_{n=0}^{\infty}\frac{1}{n!}\int d^Dx_1\ldots d^Dx_n\,\Gamma(x_1,...,x_n)\varphi(x_1)...\varphi(x_n)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int \frac{d^Dp_1}{(2\pi)^D}\cdots\frac{d^Dp_n}{(2\pi)^D}\,\tilde{\Gamma}(p_1,...,p_n)\tilde{\varphi}(p_1)...\tilde{\varphi}(p_n)\\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int \frac{d^Dp_1}{(2\pi)^D}\cdots\frac{d^Dp_n}{(2\pi)^D}\,(2\pi)^D\delta(p_1+..+p_n)\Gamma(p_1,..,p_n)\tilde{\varphi}(p_1)...\tilde{\varphi}(p_n)
\end{aligned}
\tag{9}
$$

where we have defined the Fourier transform of the fields with *outgoing momenta* as

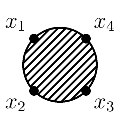
$$
\tilde{\varphi}(p) = \int d^Dx\, e^{-ipx}\varphi(x)\ , \qquad \varphi(x) = \int \frac{d^Dp}{(2\pi)^D}\, e^{ipx}\tilde{\varphi}(p) \tag{10}
$$

and the Fourier transform of the proper vertices with *ingoing momenta* as

$$
\begin{aligned}
\tilde{\Gamma}(p_1,..,p_n) &= \int d^Dx_1..d^Dx_n\, e^{ip_1x_1+..+ip_nx_n}\,\Gamma(x_1,..,x_n)\\
&= (2\pi)^D\delta(p_1+..+p_n)\Gamma(p_1,..,p_n)
\end{aligned}
\tag{11}
$$

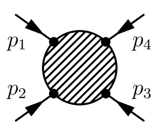
the second line following from momentum conservation (translational invariance of $\Gamma(x_1,..,x_n)$). Graphically, we may denote these correlation functions by

$$\Gamma(x_1,..,x_4) = $$



;

$$\Gamma(p_1,..,p_4) = $$



(12)

where in the second graph conservation of momentum is understood, and the external lines indicate only the flow of momentum and do not include external propagators (the 1PI graphs are amputated).

Denoting $\Gamma^n = \Gamma^n(p_1, .., p_n)$, one finds that Γ^0 is the zero point function, $\Gamma^1(p)$ is the one-point function, often set to vanish by adjusting the vev of the quantum field $\varphi(p)$, $\Gamma^2(-p,p)$ is the effective kinetic term appearing in

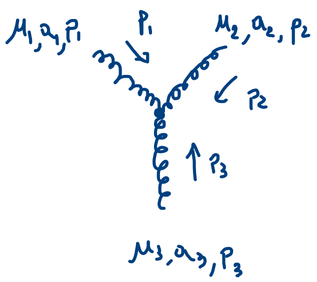
$$\begin{aligned}
\Gamma^2(\varphi) &= \frac{1}{2}\int \frac{d^D p_1}{(2\pi)^D}\frac{d^D p_2}{(2\pi)^D}\,(2\pi)^D\delta(p_1+p_2)\Gamma^2(p_1,p_2)\tilde\varphi(p_1)\tilde\varphi(p_2) \\
&= \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\,\tilde\varphi(-p)\Gamma^2(-p,p)\tilde\varphi(p) \\
&= \frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\,\tilde\varphi(-p)\left(-p^2-m^2+\Pi(p^2)\right)\tilde\varphi(p)
\end{aligned} \tag{13}$$

where $-p^2-m^2$ is the leading kinetic term whose inverse (times i) gives the perturbative propagator in momentum space $\frac{-i}{p^2+m^2}$, while $\Pi(p^2)$ is the self-energy contribution. The functions Γ^n for $n \geq 3$ give the proper vertices. Thus, it should be clear that the propagator and the vertices to be used in the Feynman rules are obtained by substituting the classical action $S[\varphi]$ suitably gauge-fixed to the effective action $\Gamma[\varphi]$.

In particular, the Feynman rules for the vertices are extracted from iS_{tot} (as $e^{iS_{tot}}$ enters the path integral) after replacing $\gamma \to S$ in the above formulae. One obtaines the elementary vertices iS^n for a vertex with n fields in the Fourier transform formulae given above: the classical action gives the tree-level term of the effective action, while the full effective action contains also $\hbar$-corrections

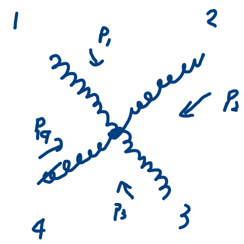
Thus, for QCD the trilinear gluon coupling in (5) give rises to the vertex $iS^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$

$$\begin{aligned}
iS^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) &= gf^{a_1a_2a_3}p_{1,\mu_2}\eta_{\mu_1\mu_3} + \text{permutations of external lines} \\
&= -gf^{a_1a_2a_3}\Big((p_1-p_2)_{\mu_3}\eta_{\mu_1\mu_2} + (p_2-p_3)_{\mu_1}\eta_{\mu_2\mu_3} + (p_3-p_1)_{\mu_2}\eta_{\mu_3\mu_1}\Big) \\
&=
\end{aligned} \tag{14}$$



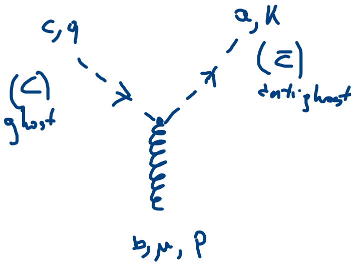
Similarly, one gets the 4-gluon vertex

$$\begin{aligned}
iS^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) &= -ig^2 f^{ba_1a_2}f^{ba_3a_4}\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} + \text{permutations of lines 2 3 4} \\
&= -ig^2[f^{ba_1a_2}f^{ba_3a_4}(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}) \\
&\quad + f^{ba_1a_3}f^{ba_4a_2}(\eta_{\mu_1\mu_4}\eta_{\mu_3\mu_2} - \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}) \\
&\quad + f^{ba_1a_4}f^{ba_2a_3}(\eta_{\mu_1\mu_2}\eta_{\mu_4\mu_3} - \eta_{\mu_1\mu_3}\eta_{\mu_4\mu_2})] \\
&=
\end{aligned} \tag{15}$$



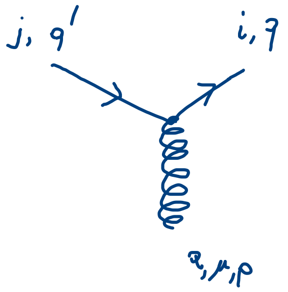
the ghost-antighost-gluon vertex

$$iS^{abc}_{\mu}(k,p,q) = gf^{abc}k_{\mu} \tag{16}$$



and the quark-quark-gluon vertex (for a fixed flavour)

$$i(S^{\mu a})^{i}{}_{j}(q,p,q') = -g\gamma^{\mu}(T^{a})^{i}{}_{j} \tag{17}$$



These rules are somewhat more complex then those for QED, nevertheless now one can start computing perturbatively various scattering processes, renomalize the theory loopwise by introducing counterterms, and compute the beta function.

2 Perturbative calculation and the beta function

To renomalize the theory one must introduce the counterterms that will cancel the infinities and match the renomalization conditions needed to define the theory in terms of a fixed number of observables.

Naively, looking at (4) and (5) one would expect at least nine Z factors (here we consider only one quark flavour) as there are 9 independent monomials in the fields. One writes them as follows

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}Z_3[(\partial_\mu A^a_\nu)^2 - (\partial^\mu A^a_\mu)^2] - \frac{1}{2\xi}Z_\xi(\partial^\mu A^a_\mu)^2 - Z_{2'}\partial^\mu\bar{c}^a\partial_\mu c^a - Z_2\overline{\psi}\gamma^\mu\partial_\mu\psi - Z_m m\overline{\psi}\psi \\ &- Z_{3g}gf^{abc}\partial^\mu A^{\nu a}A^b_\mu A^c_\nu - Z_{4g}\frac{g^2}{4}f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e} \\ &- Z_{1'}gf^{abc}\partial^\mu\bar{c}^a A^b_\mu c^c + iZ_1 gA^a_\mu\overline{\psi}\gamma^\mu T^a\psi\ .\end{aligned} \tag{18}$$

which must equal the bare lagrangian $\mathcal{L} = \mathcal{L}_0$, where

$$\begin{aligned}\mathcal{L}_0 = &-\frac{1}{2}[(\partial_\mu A^a_{0\nu})^2 - (\partial^\mu A^a_{0\mu})^2] - \frac{1}{2\xi_0}(\partial^\mu A^a_{0\mu})^2 - \partial^\mu\bar{c}^a_0\partial_\mu c^a + 0 - \overline{\psi}_0\gamma^\mu\partial_\mu\psi_0 - m_0\overline{\psi}_0\psi_0 \\ &- g_0 f^{abc}\partial^\mu A^{\nu a}_0 A^b_{0\mu}A^c_{0\nu} - \frac{\bar{g}_0^2}{4}f^{abc}f^{ade}A^b_{0\mu}A^c_{0\nu}A^{\mu d}_0 A^{\nu e}_0 \\ &- g'_0 f^{abc}\partial^\mu\bar{c}^a_0 A^b_{0\mu}c^c_0 + ig''_0 A^a_{0\mu}\overline{\psi}_0\gamma^\mu T^a\psi_0\ .\end{aligned} \tag{19}$$

However, gauge invariance (or more properly BRST invariance) can be used to derive Ward identities (that in this particular context are known as Slavnov-Taylor identites), which show that the Z factors are not all independent as one should have the same bare coupling constant $g_0 = g_0' = g_0'' = \bar{g}_0$. In particular, one finds that $Z_\xi = 1$ as in QED, and

$$g_0^2 = \frac{Z_1^2}{Z_2^2 Z_3} g^2 \tilde{\mu}^\epsilon = \frac{Z_{1'}^2}{Z_{2'}^2 Z_3} g^2 \tilde{\mu}^\epsilon = \frac{Z_{3g}^2}{Z_3^3} g^2 \tilde{\mu}^\epsilon = \frac{Z_{4g}}{Z_3^2} g^2 \tilde{\mu}^\epsilon \tag{20}$$

where $d = 4 - \epsilon$. As in the classical lagrangian, this means that there is only one independent coupling constant. Thus, to compute the β-function one could start calculating Z_1, Z_2 and Z_3 at one-loop, and use the first relation in (20).

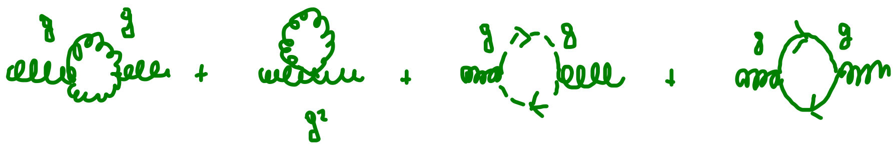
From the correction to the quark propagator one obtains Z_2 in the $\overline{\text{MS}}$ scheme

$$\rightarrow \qquad Z_2 = 1 - C(R) \frac{g^2}{8\pi^2} \frac{1}{\epsilon} + O(g^4) \; .$$

The study of the quark-quark-gluon vertex gives

$$\rightarrow \qquad Z_1 = 1 - [C(R) + T(A)] \frac{g^2}{8\pi^2} \frac{1}{\epsilon} + O(g^4) \; .$$

Finally, to renomalize the gluon wave function one must study the divergences in the gluon propagator



and a laborious calculation produces

$$Z_3 = 1 + \left[\frac{5}{3} T(A) - \frac{4}{3} n_F T(R) \right] \frac{g^2}{8\pi^2} \frac{1}{\epsilon} + O(g^4) \; .$$

Here, we have indicated by $T(R)$ and $C(R)$ the index and the quadratic Casimir in the representation R, respectively while A stands for the adjoint representation.

Now, one has all the elements to compute the one-loop beta function

$$\beta(g) = - \left[\frac{11}{3} T(A) - \frac{4}{3} n_F T(R) \right] \frac{g^3}{16\pi^2} + O(g^5) \tag{21}$$

which for QCD (where $T(A) = 3$ and $T(R) = \frac{1}{2}$) shows asymptotic freedom for a number of quark flavours $n_F \leq 16$.

If interested, these one-loop calculations can be followed in detail in chapter 73 of Srednicki.

3 Ward identities

Ward identities arising from the BRST symmetry can be used to deduce several properties of the quantum theory. One particular application is to show that scattering amplitudes containing gluons are transversal in the polarizations of the gluons. This means that if in a given total amplitude M we extract the physical polarization $\epsilon_\mu(p)$ of one of the gluons, say by setting $M = \epsilon_\mu(p) M^\mu(p)$, then after substituting this physical polarization $\epsilon_\mu(p)$ with a longitudinal one, $\epsilon_\mu(p) \to p_\mu$, one finds a vanishing result if all the other states are physical (i.e. with on-shell momenta and physical polarizations).

Let us sketch how to prove this result using the BRST symmetry. In the reduction LSZ formula, for each asymptotic gluon state there is the corresponding field $A_\mu(x)$ inside the correlation function, so that it creates the state form the vacuum. Moreover, the field $A_\mu(x)$:
i) is acted upon by the corresponding free wave operator $-\partial^2$ (in Feynman gauge),
ii) Fourier transformed with e^{ipx},
iii) contracted with the physical polarization $\epsilon_\mu(p)$.
Schematically, for an incoming state

$$a^\dagger_{in}(p) \ \to \ \epsilon^\mu(p) \int d^4x \ e^{ipx}(-\partial^2) A_\mu(x) \ . \tag{22}$$

Now the substitution $\epsilon^\mu(p) \to p^\mu$ would give

$$p^\mu \int d^4x \ e^{ipx}(-\partial^2) A_\mu(x) = -i \int d^4x \ (\partial^\mu e^{ipx})(-\partial^2) A_\mu(x) = i \int d^4x \ e^{ipx}(-\partial^2)\partial^\mu A_\mu(x) \tag{23}$$

which means that in the correlation functions the operator $\partial^\mu A_\mu(x)$ appears. However, we know that the operator $\partial^\mu A_\mu(x)$ is the BRST variation of the antighost field $\bar{c}(x)$, which in an operatorial language is generated by acting with the BRST charge on the antighost

$$\partial^\mu A_\mu(x) \sim \{Q_B, \bar{c}(x)\} \tag{24}$$

and thus it is a cohomologically trivial operator that produces a vanishing result once inserted inside correlation functions that include only physical operators and physical states (recall that in the LSZ formula one has a string of time-ordered physical operators, corresponding to the physical asymptotic states, sandwiched between the vacuum state, which is also physical).

4 Background field method

4.1 Background field method in a scalar theory

The effective action is the generator of one-particle irreducible graphs. A useful technique for computing the effective action is the background field method. We present it first for a simple scalar theory, and then extend it to gauge theories. The various generating functionals for a field ϕ with action $S[\phi]$ are

$$\begin{aligned} Z[J] &= e^{iW[J]} = \int D\phi \ e^{iS[\phi] + iJ_i\phi^i} \\ \Gamma[\varphi] &= \min_J \Big\{ W[J] - J_i \varphi^i \Big\} \ . \end{aligned} \tag{25}$$

In the *background field method* one first splits the variable ϕ as

$$\phi(x) = \varphi(x) + \tilde{\phi}(x) \tag{26}$$

where $\varphi(x)$ is taken as an arbitrary but fixed classical background, while $\tilde{\phi}(x)$ is the quantum field to be quantized (i.e. path-integrated over). The classical background $\varphi(x)$ is just an inert spectator in the quantization process and one defines the various quantum generating functionals accordingly

$$\begin{aligned} Z_B[\tilde{J};\varphi] &= e^{iW_B[\tilde{J};\varphi]} = \int D\tilde{\phi}\; e^{iS[\tilde{\phi}+\varphi]+i\tilde{J}_i\tilde{\phi}^i} \\ \Gamma_B[\tilde{\varphi};\varphi] &= \max_{\tilde{J}}\left\{W_B[\tilde{J};\varphi]-\tilde{J}_i\tilde{\varphi}^i\right\} . \end{aligned} \tag{27}$$

How these two different sets of generating functionals are related to each other? Changing the path integration variables $\tilde{\phi} \to \phi = \tilde{\phi} + \varphi$ in eq. (27) and considering that the measure is translational invariant, one finds

$$Z_B[\tilde{J};\varphi] = Z[\tilde{J}]\, e^{-i\tilde{J}_i\varphi^i} \quad \Rightarrow \quad W_B[\tilde{J};\varphi] = W[\tilde{J}] - \tilde{J}_i\varphi^i \quad \Rightarrow \quad \Gamma_B[\tilde{\varphi};\varphi] = \Gamma[\tilde{\varphi}+\varphi] \tag{28}$$

so that

$$\Gamma[\varphi] = \Gamma_B[0;\varphi] . \tag{29}$$

This equation says that the standard effective action $\Gamma[\varphi]$ can be computed as the sum of 1PI vacuum diagrams in the presence of the background field φ, see fig. 1 (see also exercise 21.3 of Srednicki).

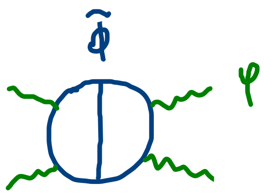


Figure 1: A one-particle irreducible graph contained in $\Gamma_B[0;\varphi]$ for the ϕ^3 theory.

4.2 Effective action, loop expansion, and 1PI diagrams

To clarify the meaning of the effective action, and reproduce the background field method from a different perspective, let us review once more the generating functionals in QFT, and check perturbatively that the effective action contains only one-particle irreducible graphs. We use the euclidean version of the QFT (this simplifies the correct insertions of factors of i, reintroduced by the inverse Wick rotation).

The standard functionals in euclidean QFT are defined by

$$\begin{aligned} Z[J] &= e^{\frac{1}{\hbar}W[J]} = \int D\phi\; e^{-\frac{1}{\hbar}S[\phi]+\frac{1}{\hbar}J_i\phi^i} \\ \Gamma[\varphi] &= J_i\varphi^i - W[J] \qquad \text{where} \qquad \varphi^i = \frac{\delta W[J]}{\delta J_i} \end{aligned} \tag{30}$$

with the effective action $\Gamma[\varphi]$ obtained by evaluating the right-hand side using the function $J_i = J_i(\varphi)$, that inverts the defining relation $\varphi^i(J) = \frac{\delta W[J]}{\delta J_i}$. Of course, this is the same as writing $\Gamma[\varphi] = \min_J\{J_i\varphi^i - W[J]\}$.

We keep $\hbar$, which we are going to use as loop counting parameter (at loop order L one gets a factor $\hbar^{L-1}$). One may invert the Legendre transform defining $\Gamma[\varphi]$ by

$$W[J] = J_i\varphi^i - \Gamma[\varphi] , \qquad J_i = \frac{\delta\Gamma[\varphi]}{\delta\varphi^i} . \tag{31}$$

With these relations at hand, one finds an equation for the effective action $\Gamma[\varphi]$

$$e^{-\frac{1}{\hbar}\left(\Gamma[\varphi]-\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\varphi^i\right)} = \int D\phi \; e^{-\frac{1}{\hbar}\left(S[\phi]-\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\phi^i\right)} \,, \tag{32}$$

and after a change of variables that implements the shift $\phi \to \phi + \varphi$ in the path integral one rewrites it as

$$\boxed{e^{-\frac{1}{\hbar}\Gamma[\varphi]} = \int D\phi \; e^{-\frac{1}{\hbar}S[\phi+\varphi]+\frac{1}{\hbar}\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\phi^i}} \,. \tag{33}$$

We use this equation to study the $\hbar$ expansion, i.e. the expansion in loops, which are counted by the parameter $\hbar$. We find the structure of the background field method explained previously.

It is convenient to use a compact notation and expand the classical action in a Taylor series

$$S[\phi+\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} S_n[\varphi]\phi^n \tag{34}$$

where $S_n[\varphi] \equiv \frac{\delta^n S[\varphi]}{\delta\varphi^n}$, and use a similar notation for $\frac{\delta\Gamma[\varphi]}{\delta\varphi} \equiv \Gamma_1[\varphi]$. Now, rescaling $\phi \to \sqrt{\hbar}\,\phi$ one finds

$$\exp\left(-\frac{1}{\hbar}\Gamma[\varphi] + \frac{1}{\hbar}S[\varphi]\right) = \int D\phi \exp\left(-\frac{1}{2}S_2[\varphi]\phi^2 - \sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S_n[\varphi]\phi^n + \frac{1}{\sqrt{\hbar}}(\Gamma_1[\varphi]-S_1[\varphi])\phi\right) \tag{35}$$

that depends only on $\bar{\Gamma}[\varphi] \equiv \Gamma[\varphi] - S[\varphi]$. Expanding $\bar{\Gamma}[\varphi]$ in powers of $\hbar$

$$\bar{\Gamma}[\varphi] = \sum_{n=1}^{\infty} \hbar^n \Gamma^{(n)}[\varphi] \tag{36}$$

(note the we start from $n = 1$, so the bar over $\Gamma^{(n)}$ is not needed anymore), one obtains the following master equation

$$\exp\Big(\underbrace{-\sum_{n=1}^{\infty}\hbar^{n-1}\Gamma^{(n)}[\varphi]}_{\text{1PI diagrams}}\Big) = \int D\phi\; \exp\Big(\underbrace{-\frac{1}{2}S_2[\varphi]\phi^2}_{\text{propagator}} \underbrace{-\sum_{n=3}^{\infty}\frac{\hbar^{\frac{n}{2}-1}}{n!}S_n[\varphi]\phi^n}_{\text{vertices}} + \underbrace{\sum_{n=1}^{\infty}\hbar^{n-\frac{1}{2}}\Gamma_1^{(n)}[\varphi]\phi}_{\text{extra vertices that remove diagrams that are not 1PI}}\Big) \,. \tag{37}$$

which we analyze by matching powers of $\hbar$ in the perturbative expansion.

Approximation at 1-loop ($n = 1$)

From the master formula (37) we keep the $\hbar$ independent terms and get

$$e^{-\Gamma^{(1)}[\varphi]} = \int D\phi\; e^{-\frac{1}{2}S_2[\varphi]\phi^2+O(\hbar^{\frac{1}{2}})} = (\mathrm{Det}\, S_2[\varphi])^{-\frac{1}{2}} = e^{-\frac{1}{2}\ln\mathrm{Det}\, S_2[\varphi]} = e^{-\frac{1}{2}\mathrm{Tr}\ln S_2[\varphi]} \tag{38}$$

so that

$$\Gamma^{(1)}[\varphi] = \frac{1}{2}\ln\mathrm{Det}\, S_2[\varphi] = \frac{1}{2}\mathrm{Tr}\ln S_2[\varphi] \equiv \frac{1}{2}\;\bigcirc \tag{39}$$

where we have shown the Feynman diagram usually assigned to this type of term. Thus, at one-loop order, the effective action is given by

$$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2}\mathrm{tr}\ln S_2[\varphi] + O(\hbar^2) \,. \tag{40}$$

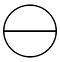
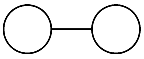
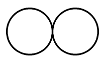
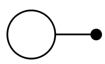
Approximation at 2 loops ($n=2$)

From the master formula one finds

$$\begin{aligned}
\mathrm{e}^{-\Gamma^{(1)}[\varphi]-\hbar\Gamma^{(2)}[\varphi]} &= \int D\phi\ \exp\Big(-\frac{1}{2}S_2[\varphi]\phi^2-\frac{\hbar^{\frac{1}{2}}}{3!}S_3[\varphi]\phi^3-\frac{\hbar}{4!}S_4[\varphi]\phi^4+\hbar^{\frac{1}{2}}\Gamma_1^{(1)}[\varphi]\phi+O(\hbar^{\frac{3}{2}})\Big)\\
&= (\mathrm{Det}\,S_2[\varphi])^{-\frac{1}{2}}\ \underbrace{\langle\exp(-S_{int})\rangle}_{\langle(1-S_{int}+\frac{1}{2}S_{int}^2+...)\rangle}\\
&= (\mathrm{Det}\,S_2[\varphi])^{-\frac{1}{2}}\exp(-\langle S_{int}\rangle_c+\frac{1}{2}\langle S_{int}^2\rangle_c+...)\\
&= (\mathrm{Det}\,S_2[\varphi])^{-\frac{1}{2}}\exp\Big[\frac{\hbar}{2}\frac{S_3[\varphi]}{3!}\langle\phi^3\phi^3\rangle_c\frac{S_3[\varphi]}{3!}-\hbar\frac{S_4[\varphi]}{4!}\langle\phi^4\rangle_c+\frac{\hbar}{2}\Gamma_1^{(1)}[\varphi]\langle\phi\phi\rangle_c\Gamma_1^{(1)}[\varphi]\\
&\quad-\hbar\frac{S_3[\varphi]}{3!}\langle\phi^3\phi\rangle_c\Gamma_1^{(1)}[\varphi]+O(\hbar^{\frac{3}{2}})\Big]
\end{aligned}\tag{41}$$

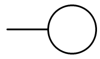
where $\langle..\rangle_c$ denote connected correlation functions. Using Wick contractions, one finds

$$\Gamma^{(2)}[\varphi]=-\frac{1}{2}\frac{S_3[\varphi]}{3!}\langle\phi^3\phi^3\rangle_c\frac{S_3[\varphi]}{3!}+\frac{S_4[\varphi]}{4!}\langle\phi^4\rangle_c-\frac{1}{2}\Gamma_1^{(1)}[\varphi]\langle\phi\phi\rangle_c\Gamma_1^{(1)}[\varphi]+\frac{S_3[\varphi]}{3!}\langle\phi^3\phi\rangle_c\Gamma_1^{(1)}[\varphi]$$

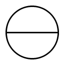
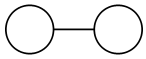
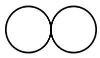
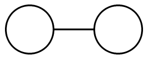
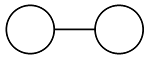
$= -\frac{1}{2}\Big[\frac{1}{3!}$  $+\frac{1}{4}$  $\Big]+\frac{1}{8}$  $-\frac{1}{2}$ $+\frac{1}{2}$  .

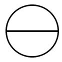
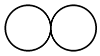
Observe now how the extra vertices that originate from $\Gamma_1^{(n)}[\varphi]\phi$ cancel graphs that are not 1PI. The extra vertex denoted by the dark blob coincides with

 $=\Gamma_1^{(1)}[\varphi]\phi=\frac{\delta\Gamma^{(1)}[\varphi]}{\delta\varphi}\phi=\frac{\delta}{\delta\varphi}\frac{1}{2}\mathrm{tr}\ln S_2[\varphi]\phi$

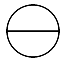
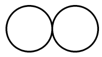
$=\frac{1}{2}\mathrm{tr}\Big[S_2^{-1}\frac{\delta S_2[\varphi]}{\delta\varphi}\Big]\phi=\frac{1}{2}$ "$\mathrm{tr}\,S_2^{-1}S_3$" $\phi=\frac{1}{2}$ 

so that

$\Gamma^{(2)}[\varphi]=-\frac{1}{12}$  $-\frac{1}{8}$  $+\frac{1}{8}$  $-\frac{1}{8}$  $+\frac{1}{4}$ 

$=-\frac{1}{12}$  $+\frac{1}{8}$  .

We have verified at this order that the effective action contains only 1PI graphs given by

$\Gamma^{(2)}[\varphi]=-\frac{1}{12}$  $+\frac{1}{8}$  . (42)

4.3 Background field method for gauge theories

A similar set-up can be used for gauge theories. Let's start with the Yang-Mills lagrangian for a non-abelian gauge field[1], to be denoted by W_μ

$$\mathcal{L}=\frac{1}{4}F_{\mu\nu}(W)F^{\mu\nu}(W)\tag{43}$$

[1]We expand all fields as $W_\mu(x)=-iW_\mu^a(x)T^a$, and use the generators T^a in the fundamental representation so that $\mathrm{tr}(T^aT^b)=\frac{1}{2}\delta^{ab}$. The lagrangian must then be traced with $\frac{2}{g^2}\mathrm{tr}(...)$, an operation which we leave understood for notational simplicity.

with gauge symmetry

$$\delta W_\mu = D_\mu(W)\alpha = \partial_\mu \alpha + [W_\mu, \alpha] \ . \tag{44}$$

Now one splits the gauge field W_μ as a *background part* A_μ (also known as the classical part) and a *quantum part* Q_μ to be path-integrated over (i.e. quantized)

$$W_\mu = A_\mu + Q_\mu \ . \tag{45}$$

We identify two different gauge symmetries:
i) a true gauge symmetry, where the background field is inert

$$\begin{aligned} \delta_1 A_\mu &= 0 \\ \delta_1 Q_\mu &= D_\mu(W)\alpha = \partial_\mu \alpha + [A_\mu + Q_\mu, \alpha] \end{aligned} \tag{46}$$

ii) a background gauge symmetry

$$\begin{aligned} \delta_2 A_\mu &= D_\mu(A)\alpha = \partial_\mu \alpha + [A_\mu, \alpha] \\ \delta_2 Q_\mu &= [Q_\mu, \alpha] \ . \end{aligned} \tag{47}$$

They both produce the same gauge transformation on W_μ, but only the first one must be gauge-fixed for quantization. The second one treats instead the background A_μ as the gauge field (the gauge connection in geometrical terms), while the quantum field Q_μ transforms as a tensor in the adjoint representation. This symmetry is called *background gauge symmetry.*

As said, the symmetry that must be gauge fixed is the first one, since it leaves the background field invariant, and thus it is a true dynamical symmetry. Upon gauge-fixing that gauge symmetry is lost, but it gives rise to the a rigid BRST symmetry. However, we can perform the gauge fixing in a useful way: we select a gauge fermion, and thus a gauge fixing function, in a way that it maintain the background gauge invariance. In this way the latter survives the quantization process, and remains as a useful symmetry of the effective action. In particular, it will constrain the counterterms needed to renormalize the theory.

Let us give some details. In the BRST quantization method, a gauge-fixing fermion that preserves the background gauge symmetry is given by

$$\Psi = \bar{c} f(Q_\mu, B; A_\mu) = \bar{c}\left(D^\mu(A) Q_\mu - \frac{\xi}{2} B \right) \tag{48}$$

which respects the background gauge symmetry: Q_μ transforms as a tensor in the adjoint representation and its background covariant derivatives does not destroy its tensorial character. Then, also B is taken to transform in the adjoint and the gauge fermion is a scalar under the background gauge symmetry. It produces

$$s\Psi = B\left(D^\mu(A) Q_\mu - \frac{\xi}{2} B \right) - \bar{c}\, D^\mu(A) D_\mu(A+Q) c \tag{49}$$

that must be added to the gauge invariant lagrangian. Upon elimination of the auxiliary B field, the gauge-fixed total lagrangian becomes

$$\boxed{\mathcal{L}_{B,tot} = \frac{1}{4} F_{\mu\nu}(A+Q) F^{\mu\nu}(A+Q) + D^\mu(A)\bar{c} D_\mu(A+Q) c + \frac{1}{2\xi}(D^\mu(A) Q_\mu)^2} \tag{50}$$

(the subscript B stands now for background), and is manifestly background gauge invariant. Fermions may also be added, with a gauge coupling to W_μ to maintain the background symmetry. Note that the field strength splits as

$$F_{\mu\nu}(W) = F_{\mu\nu}(A) + D_\mu(A) Q_\nu - D_\nu(A) Q_\mu + [Q_\mu, Q_\nu] \tag{51}$$

which shows the background gauge covariance (remember that Q is a tensor in the adjoint representation and $D(A)Q$ is also a tensor). Similarly, the ghost action is expanded as

$$\mathcal{L}_{gh} = D^\mu(A)\bar{c}D_\mu(A+Q)c = D^\mu(A)\bar{c}D_\mu(A)c + D^\mu(A)\bar{c}\,[Q_\mu, c] \tag{52}$$

with the first term that is background gauge invariant and fixes the propagator[2], while the second term is the vertex that also depends on A_μ and is background gauge invariant.

Now, one can quantize with path integrals and obtain[3]

$$\begin{aligned} Z_B[\tilde{J}; A] &= \mathrm{e}^{iW_B[\tilde{J};A]} = \int DQD\bar{c}Dc\ \mathrm{e}^{iS_{B,tot}[Q,\bar{c},c;A]+i\tilde{J}Q} \\ \Gamma_B[\tilde{Q}; A] &= \min_{\tilde{J}} \left\{ W_B[\tilde{J}; A] - \tilde{J}\tilde{Q} \right\} . \end{aligned} \tag{53}$$

The ghost are not physical external states, we omit adding sources for them (though one could do that as well). Then, $Z_B[\tilde{J}; A]$ and $W_B[\tilde{J}; A]$ are background gauge invariant under

$$\delta A_\mu = D_\mu(A)\alpha \ , \qquad \delta\tilde{J}^\mu = [\tilde{J}^\mu, \alpha] \ . \tag{54}$$

This is verified by making a change of variables in the path integral, with the new variables related to the old ones by a background symmetry transformation: $Q'_\mu = Q_\mu + [Q_\mu, \alpha]$ (similarly for the ghosts), and then considering that the path integral measure is invariant (unit jacobian i.e. no anomalies in this symmetry). Then, the effective action $\Gamma_B[\tilde{Q}; A]$ is also gauge invariant under

$$\delta A_\mu = D_\mu(A)\alpha \ , \qquad \delta\tilde{Q}_\mu = [\tilde{Q}_\mu, \alpha] \ . \tag{55}$$

In particular, $\Gamma_B[0; A]$ is gauge invariant: it contains 1PI graphs in terms of the field Q_μ that runs only inside the graphs (vacuum bubbles), and with external fields A_μ which do not propagate (as they are background fields), see figure 2.

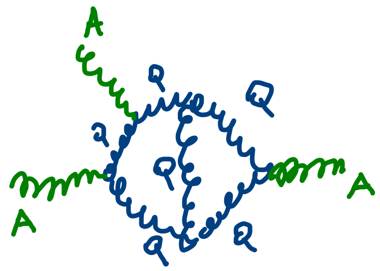


Figure 2: 1PI graph contained in $\Gamma_B[0; A]$.

The backgound gauge invariance is a powerful constraint on the form the effective action. Moreover, comparison with the result (29) of the scalar theory, suggests that it can be related to the standard effective action that captures 1PI diagrams, and therefore the complete physical information of the quantum theory.

To show that this is indeed the case, let us work out the details. We will find a relation with the effective action of the ordinary quantum field theory, but with an unusual gauge fixing

[2]It depends on the background field A_μ, and while it is difficult to compute it exactly, one can treat it in a perturbative expansion in A_μ.

[3]As a shorthand notation we denote $\tilde{J}Q = \int d^4x\, \tilde{J}^\mu(x)Q_\mu(x)$.

term that depends on the function A_μ. We start from the path integral in (53), that contains the action fixed by (50), and perform in the path integral a change of variables induced by the shift $Q_\mu \to Q_\mu - A_\mu$, which leave the measure invariant, but change the lagrangian (50) to the new lagrangian

$$\boxed{\mathcal{L}_{tot} = \frac{1}{4} F_{\mu\nu}(Q) F^{\mu\nu}(Q) + D^\mu(A)\bar{c} D_\mu(Q) c + \frac{1}{2\xi}(D^\mu(A) Q_\mu - \partial^\mu A_\mu)^2} \tag{56}$$

where we have used that $D^\mu(A) A_\mu = \partial^\mu A_\mu$. We see that this lagrangian is just as in ordinary gauge theory with the gauge field denoted by Q_μ, but with an unusual gauge-fixing function

$$f(Q_\mu; A_\mu) = D^\mu(A) Q_\mu - \partial^\mu A_\mu \ . \tag{57}$$

as recognized from $\frac{1}{2\xi} f^2$ that appears as the last term in (56). On top of ξ, the gauge-fixing term $\frac{1}{2\xi} f^2$ depends also on the arbitrary function $A_\mu(x)$. Being a gauge choice, we know that physical quantities cannot depend on the values of ξ and $A_\mu(x)$. We denote the corresponding path integral with gauge-fixed lagrangian (56) by

$$\begin{aligned} Z[J;A] &= \mathrm{e}^{iW[J;A]} = \int DQ D\bar{c} Dc \ \mathrm{e}^{iS_{tot}[Q,\bar{c},c;A] + iJQ} \\ \Gamma[Q;A] &= \min_J \Big\{ W[J;A] - JQ \Big\} \ . \end{aligned} \tag{58}$$

where the functional dependence on A reminds us that the gauge-fixing depends on it.

Now we are ready to describe the searched for relations. From (53), with the change of variables just mentioned, we find

$$Z_B[\tilde{J};A] = Z[\tilde{J};A] e^{-i\tilde{J}A} \tag{59}$$

$$W_B[\tilde{J};A] = W[\tilde{J};A] - \tilde{J}A \tag{60}$$

and finally

$$\Gamma_B[\tilde{Q};A] = \Gamma[\tilde{Q} + A;A] \tag{61}$$

which produces the searched for relation

$$\Gamma_B[0;A] = \Gamma[A;A] \tag{62}$$

that generalizes (29) to the gauge case. Note that functional differentiation of $\Gamma[A;A]$ with respect to the first variable yields the n-point 1PI diagrams, while differentiating with respect to the second A gives a vanishing result for physical quantities, as varying that A only corresponds to a change of the gauge-fixing term, a change which leaves physical quantities invariant. Relation (62) tells us that these 1PI diagrams can be computed in an equivalent way by using the background field method, and considering only the vacuum graphs of the quantum field $\tilde{Q}$. The latter depend on the background field A, as indicated in $\Gamma_B[0;A]$. It has the useful property of being gauge invariant under the background gauge symmetry.

β-function in the background field method

Recalling that

$$\Gamma_B[0;A] = \textit{classical action} + \hbar \ \textit{corrections} \tag{63}$$

with the $\hbar$ corrections including also the $Z-1$ counterterms, we find that the effective action $\Gamma_B[0;A]=\int d^4x\,\mathcal{L}_B(0;A)$ must have a lagrangian of the form (recall (50) with $Q=c=\bar{c}=0$)

$$\mathcal{L}_B(0;A)=\frac{1}{4}Z_{B,3}F_{\mu\nu}(A)F^{\mu\nu}(A)+\textit{other gauge invariant structures} \tag{64}$$

because it must necessarily be background gauge invariant. Inserting the Z factors as in eq. (18), with a subscript B for the background dependent lagrangian, we must have

$$\begin{aligned}\mathcal{L}_B &= -\frac{1}{2}Z_{B,3}[(\partial_\mu A_\nu^a)^2-(\partial^\mu A_\mu^a)^2]-Z_{B,3g}\,gf^{abc}\partial^\mu A^{\nu a}A_\mu^b A_\nu^c-Z_{B,4g}\frac{g^2}{4}f^{abc}f^{ade}A_\mu^b A_\nu^c A^{\mu d}A^{\nu e}\\ &= Z_{B,3}\left(-\frac{1}{2}(\partial_\mu A_\nu^a)^2+\frac{1}{2}(\partial^\mu A_\mu^a)^2-gf^{abc}\partial^\mu A^{\nu a}A_\mu^b A_\nu^c-\frac{g^2}{4}f^{abc}f^{ade}A_\mu^b A_\nu^c A^{\mu d}A^{\nu e}\right)\end{aligned} \tag{65}$$

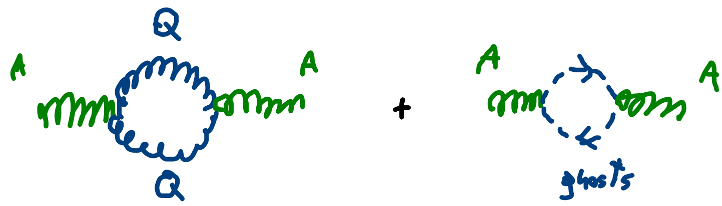
where in the second line we have imposed the background gauge covariance leading to (64), thus finding the identities

$$Z_{B,3g}=Z_{B,4g}=Z_{B,3}\ . \tag{66}$$

One consequence of these relations is that to obtain the β-function we have simply to compute $Z_{B,3}$, that renomalizes the gluon field, as one could employ the third relation in (20) that leads to

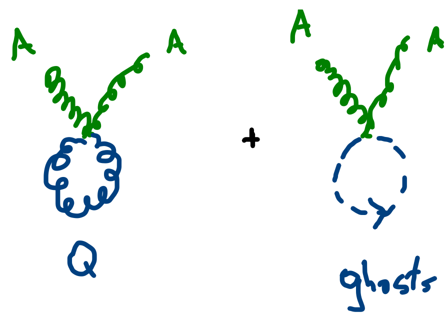
$$g_0^2=\frac{Z_{B,3g}^2}{Z_{B,3}^3}g^2\tilde{\mu}^\epsilon=\frac{1}{Z_{B,3}}g^2\tilde{\mu}^\epsilon\ . \tag{67}$$

The calculation of $Z_{B,3}$ means that we have to identify the diverging part of the following self-energy diagrams



(68)

which is much easier of the calculation described earlier. There are also other diagrams



which however vanish in dimensional regularization ("massless tadpole vanish in DR", as integrals of the type $\int d^4k\frac{1}{k^2}$ do not have any mass parameter that they might depend on, and are regulated to vanish).

We will not present here the detailed calculation (it be found on the textbooks such as Srednicki), and only describe some preliminary steps. For the calculation of the first graph in (68) one needs the AQQ vertices from (50). Setting $\xi = 1$, they are extracted from the gauge-fixed lagrangian

$$\begin{aligned} \mathcal{L}_{B,tot} &= \frac{1}{4}F_{\mu\nu}(A+Q)F^{\mu\nu}(A+Q) + \frac{1}{2}(D^\mu(A)Q_\mu)^2 + \text{ghosts} \\ &= \frac{1}{4}(F_{\mu\nu}(A) + D_\mu(A)Q_\nu - D_\nu(A)Q_\mu + [Q_\mu, Q_\nu])^2 + \frac{1}{2}(D^\mu(A)Q_\mu)^2 + ... \\ &= F_{\mu\nu}(A)Q_\mu Q_\nu + \frac{1}{2}(D_\mu(A)Q_\nu)^2 - \frac{1}{2}D_\mu(A)Q_\nu D^\nu(A)Q^\mu + \frac{1}{2}(D^\mu(A)Q_\mu)^2 + ... \\ &= 2F_{\mu\nu}(A)Q_\mu Q_\nu + \frac{1}{2}(D_\mu(A)Q_\nu)^2 + ... \end{aligned} \tag{69}$$

(recall that there is a trace understood, so that we can use the cyclicity of the trace to move terms around). Then, recalling the correct normalization (see previous footnote) plus considering the rescaling that brings the coupling constant into the vertices, we identify two independent structures for the AQQ vertex

$$\mathcal{L}_{int} = -gf^{abc}F^a_{\mu\nu}(A)Q^{\mu b}Q^{\nu c} - gf^{abc}A^{\mu a}Q^{\nu b}\partial_\mu Q^c_\nu \tag{70}$$

The calculation can be further simplified. Using twice the first vertex for the first diagram of (68) gives a contribution that can be computed more simply for a constant external $F^a_{\mu\nu}(A)$, and, considering that $f^{acd}f^{bcd} = T(A)\delta^{ab}$, gives a first contribution to $Z_{B,3}$. The second vertex is the same as that due to 4 real scalars in the adjoint, as described in chapter 78 of Srednicki, and using it on each vertex of the first diagram in (68) gives a second contribution. Finally, using both vertices once, gives a vanishing contribution.

Then, one should add the ghosts graphs, and at this stage one reproduces the beta function for pure YM theory. This is a much simpler task then the one performed in the ordinary QFT without the background field theory method. One can eventually add matter fields of spin 0 and 1/2 in arbitrary representations to find a more general expression for the beta function, which becomes

$$\beta(g) = -\left[\frac{11}{3}T(A) - \frac{4}{3}n_F T(R_F) - \frac{1}{3}n_S T(R_S)\right]\frac{g^3}{16\pi^2} + O(g^5) \tag{71}$$

where n_F and n_S are the number of Dirac fermions and complex scalars in representations T_F and T_S, respectively.

4.4 Anomalies

Let us now briefly touch on the topic of anomalies. Anomalies in QFT refer to the breaking of classical symmetries due to the quantization process, a breaking that cannot be cured unless one is willing to sacrifice other symmetries, or modifying the field content of the theory (in this regard, the Standard Model is consistent with the known field content of fermions, but would have been anomalous without the top quark, for example). The most important types of anomalies are *chiral anomalies* (anomalies in the conservation of Noether currents related to chiral symmetries) and *trace anomalies* (the trace of the stress tensor vanishes classically in conformal field theories, but may develop a trace at the quantum level). Also, *gravitational anomalies*, that can exist for theories coupled to gravity in dimensions 2, 6, 10 etc.., refers to the breaking of general coordinate invariance that may arise if chiral matter is present. They are of great importance in the construction of consistent string theories and supergravities.

The first anomaly to be discovered was the chiral anomaly in the axial $U(1)$ symmetry present in massless QED (thus the alternative name of *axial anomaly*, also known as the ABJ *anomaly* from Adler-Bell-Jackiw). For a massless Dirac fermion coupled to a $U(1)$ gauge theory, one has a vectorial gauge symmetry and a rigid axial symmetry. The gauge field needs not be quantized for discovering the anomaly, and it can be kept as a background. Regulating the theory so that the vector current is conserved leaves an axial current that is anomalous. Let us describe better the classical theory. The lagrangian of the model is given by a fermion field ψ coupled to a (background) abelian gauge field A_μ

$$\mathcal{L} = -\overline{\psi}\gamma^\mu(\partial_\mu - iA_\mu)\psi - m\overline{\psi}\psi \tag{72}$$

where a Dirac mass term will eventually be set to vanish. The gauge symmetry is given by

$$\begin{cases} \psi \to \psi' = e^{i\alpha}\psi \\ \overline{\psi} \to \overline{\psi}' = e^{-i\alpha}\overline{\psi} \\ A_\mu \to A'_\mu = A_\mu + \partial_\mu\alpha \end{cases} \tag{73}$$

that at the infinitesimal level $\alpha \ll 1$ reads

$$\begin{cases} \delta\psi = i\alpha\psi \\ \delta\overline{\psi} = -i\alpha\overline{\psi} \\ \delta A_\mu = \partial_\mu\alpha \ . \end{cases} \tag{74}$$

Note that A_μ that couples to the conserved vector current $J^\mu = i\overline{\psi}\gamma^\mu\psi$. The gauge symmetry is valid also for $m \neq 0$. It is a background gauge symmetry, as in (74) we transformed also the background. On the other hand, if one keeps the background fixed, there remains only a rigid $U(1)$ symmetry, that can be used to show that the vector current J^μ is conserved, $\partial_\mu J^\mu = 0$.

In addition, there is an axial symmetry for $m = 0$ given by

$$\begin{cases} \psi \to \psi' = e^{i\beta\gamma^5}\psi \\ \overline{\psi} \to \overline{\psi}' = \overline{\psi}e^{i\beta\gamma^5} \\ A_\mu \to A'_\mu = A_\mu \end{cases} \tag{75}$$

that at the infinitesimal level $\beta \ll 1$ becomes

$$\begin{cases} \delta\psi = i\beta\gamma^5\psi \\ \delta\overline{\psi} = i\beta\overline{\psi}\gamma^5 \\ \delta A_\mu = 0 \end{cases} \tag{76}$$

where β is an infinitesimal constant phase. Indeed, the usual Noether procedure (using an arbitrary function $\beta(x)$ in (76)) gives a variation of the action

$$\delta S = \delta \int d^4x\, \mathcal{L} = \int d^4x \Big[-(\partial_\mu\beta)\underbrace{i\overline{\psi}\gamma^\mu\gamma^5\psi}_{J_5^\mu} - 2mi\beta\overline{\psi}\gamma^5\psi \Big] \ . \tag{77}$$

Then, using the equations of motion ($\delta S = 0$) and the arbitrariness of $\beta(x)$, one finds

$$\partial_\mu J_5^\mu = 2mi\overline{\psi}\gamma^5\psi \ . \tag{78}$$

For $m = 0$, the axial current J_5^μ is conserved

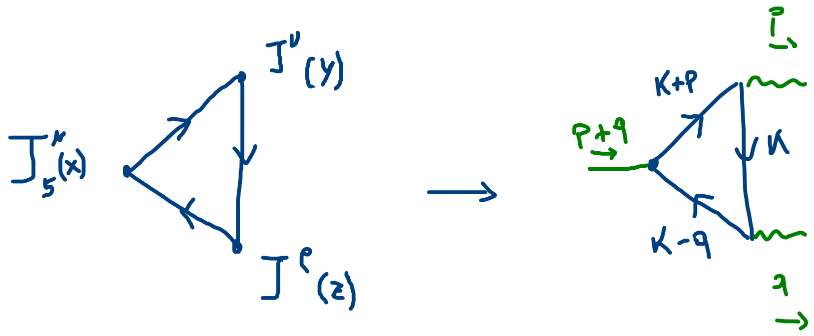
$$\partial_\mu J_5^\mu = 0 \tag{79}$$

and the axial transformation is a symmetry of the massless theory.

At the quantum level, the vector and axial currents become operators. One could then study the Ward identities, and in particular the one related to the expectation value of

$$\langle J_5^\mu(x) J^\nu(y) J^\rho(z) \rangle \tag{80}$$

that in momentum space gives rise to a triangle diagram of the form



(81)

The regularization of the diagram (e.g. using DR suitably extended to chiral theories, which however is a sublte issue) allows to preserve the conservation of the vector current, thus maintaining the gauge symmetry, but it leaves an anomalous divergence in the axial current

$$\partial_\mu J_5^\mu = \frac{1}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \; . \tag{82}$$

This implies that the chiral symmetry is broken by the quantization process (i.e. by the need of regulating and eventually renormalizing loop diagrams to get finite results), and is said to be anomalous.

There are several ways to compute the anomaly associated to the triangle diagram. One method is the direct calculation of the Feynman diagram in eq. (81). A second one is known as the Fujikawa's method and uses directly the path integral.

Let us give some details on the latter. In the Fujikawa's path integral method, one recognizes the anomaly as arising from the non-invariance under a symmetry variation of the measure $D\phi$ of the path integral (we use the euclidean version in our exemplification)

$$Z = \int D\phi \; e^{-S[\phi]} \; . \tag{83}$$

We review it for the case of a global symmetry and consider an infinitesimal symmetry transformation of the form $\delta_\alpha \phi^i = \alpha f^i(\phi, \partial_\mu \phi)$, with infinitesimal constant parameter α, that leaves the action invariant, i.e. $\delta_\alpha S[\phi] = 0$. Promoting the parameter α to be an arbitrary function $\alpha(x)$, one identifies as usual the Noether current J_μ associated to the symmetry by calculating the variation of the action

$$\delta_{\alpha(x)} S[\phi] = \int d^4x \; J^\mu \partial_\mu \alpha(x) \; . \tag{84}$$

Terms proportional to an undifferentiated α cannot be present, as for constant parameter one must recover the symmetry of the action. Now, using the equations of motion we know that $\delta S[\phi] = 0$ for arbitrary variations (least action principle), and in particular $\delta_{\alpha(x)} S[\phi] = 0$, so that, after performing an integration by parts in (84), one deduces that the Noether current J^μ is classically conserved

$$\partial_\mu J^\mu = 0 \; . \tag{85}$$

The quantum theory is defined by the path integral, which is left invariant by a dummy change of integration variables

$$\int D\phi \; e^{-S[\phi]} = \int D\phi' \; e^{-S[\phi']} \; . \tag{86}$$

Let us apply this property to an infinitesimal change of variables of the form

$$\phi^i \to \phi'^i = \phi^i + \delta_{\alpha(x)}\phi^i \tag{87}$$

where $\delta_{\alpha(x)}\phi^i$ is given by the infinitesimal symmetry transformation with the parameter α replaced by an arbitrary function $\alpha(x)$. In relating the path integral written in terms of ϕ'^i to the one written in terms of ϕ^i (in a condensed notation we include the space-time dependence into the index i), one uses

$$S[\phi'] = S[\phi] + \delta_{\alpha(x)} S[\phi] \tag{88}$$

and consider that the path integral jacobian $\mathcal{J}$ takes the form

$$\mathcal{J} = \text{Det}\, \frac{\partial \phi'^i}{\partial \phi^j} = 1 + \text{Tr}\, \frac{\partial \delta_{\alpha(x)}\phi^i}{\partial \phi^j} \equiv 1 + \text{Tr}\, J \; . \tag{89}$$

Thus, one finds from (86) the expectation value

$$\langle \text{Tr}\, J - \delta_{\alpha(x)} S[\phi] \rangle = 0 \tag{90}$$

that with an integration by parts is rewritten as

$$\int d^4x \, \alpha(x) \partial_\mu \langle J^\mu \rangle = -\text{Tr}\, J \; . \tag{91}$$

This shows that the Noether current is not conserved at the quantum level if the path integral measure carries a nontrivial jacobian under (87)

$$\partial_\mu \langle J^\mu \rangle \neq 0 \; . \tag{92}$$

We have indicated by $\langle \cdots \rangle$ the quantum expectation values defined by normalized averages within the path integral. We have assumed that the jacobian is independent of the quantum fields, and we have pulled it out of the expectation value.

To proceed further, one must define carefully the formal expressions appearing in the above formal reasoning. Ideally, one would like to fully specify the path integration measure, so that the evaluation of the jacobian would be a well-defined task. In practice, one is able to compute gaussian path integrals only, resorting to perturbative methods for more complicated cases. Nevertheless, one can still obtain the one-loop anomalies by regulating the trace in (91), as shown by Fujikawa. Employing a positive definite operator $\mathcal{R}$, the candidate anomaly is regulated as

$$\text{Tr}\, J \quad \to \quad \lim_{M\to\infty} \text{Tr}\, J \, e^{-\frac{\mathcal{R}}{M^2}} \; . \tag{93}$$

This functional trace is written in a more explicit notation (for a single scalar field) as

$$\text{Tr}\, J = \int d^4x \int d^4y \, J(x,y) \delta^4(x-y) \; , \qquad J(x,y) = \frac{\delta(\delta_{\alpha(x)}\phi(x))}{\delta\phi(y)} \tag{94}$$

and regulated by the differential operator $\mathcal{R}(x)$ acting on the x coordinates as

$$\lim_{M\to\infty} \text{Tr}\, J \, e^{-\frac{\mathcal{R}}{M^2}} = \lim_{M\to\infty} \int d^4x \int d^4y \, J(x,y) \, e^{-\frac{\mathcal{R}(x)}{M^2}} \, \delta^4(x-y) \; . \tag{95}$$

The choice of the correct regulator to be used may be subtle, though it must be related to the differential operator fixed by the action one is starting from. There exists an algorithm that derives the regulator $\mathcal{R}$ precisely from a Pauli-Villars (PV) regularization of the QFT, making it unambiguous.

Let us apply this set up to the calculation of the chiral anomaly.

Chiral anomaly

Let us consider the euclidean theory of a massless Dirac fermion coupled to an external abelian gauge field. It is obtained from the lagrangian (72) and reads

$$S[\psi, \overline{\psi}; A] = \int d^D x\, \overline{\psi}\gamma^\mu(\partial_\mu - iA_\mu)\psi \,. \tag{96}$$

The fields ψ and $\overline{\psi}$ are to be considered as independent fields in euclidean. The gauge symmetry in (74) and the chiral symmetry in (76) are classical symmetries.

The path integral defines the one-loop effective action $\Gamma[A]$, the effective action induced by the fermion loop, which is related to the functional determinant of the Dirac operator

$$Z[A] = e^{-\Gamma[A]} = \int D\psi D\overline{\psi}\, e^{-S[\psi,\overline{\psi};A]} = \mathrm{Det}(\not{D}(A)) \tag{97}$$

where $\not{D}(A) = \gamma^\mu D_\mu(A) = \gamma^\mu(\partial_\mu - iA_\mu)$. We expect the effective action to be gauge invariant, but the chiral symmetry will be anomalous.

Let us apply Fujikawa's argument directly to the chiral symmetry. One may write

$$Z[A] = \int D\psi D\overline{\psi}\, e^{-S[\psi,\overline{\psi};A]} = \int D\psi' D\overline{\psi}'\, e^{-S[\psi',\overline{\psi}';A]} \tag{98}$$

and take the primed variables to be related to the original ones by a change of variables given by the chiral transformation with parameter β extended to be an arbitrary function $\beta(x)$, namely

$$\psi'(x) = \psi(x) + i\beta(x)\gamma^5\psi(x) \,, \qquad \overline{\psi}'(x) = \overline{\psi}(x) + i\beta(x)\overline{\psi}(x)\gamma^5 \,. \tag{99}$$

Then, considering the change of the classical action,

$$S[\psi', \overline{\psi}'; A] = S[\psi, \overline{\psi}; A] + \int d^D x\, (\partial_\mu\beta(x))J_5^\mu(x) \qquad \text{with} \qquad J_5^\mu = i\overline{\psi}\gamma^\mu\gamma^5\psi \tag{100}$$

and the Jacobian that brings the measure back to the original variables

$$D\psi' = D\psi\, \mathrm{Det}\left(\frac{\partial\psi'(x)}{\partial\psi(y)}\right)^{-1} = D\psi\,(1 - i\mathrm{Tr}[\beta(x)\gamma^5]) \tag{101}$$

with a similar formula for $\overline{\psi}$, one finds from (98) (after an integration by parts)

$$\int d^D x\, \beta(x)\partial_\mu\langle J_5^\mu(x)\rangle = 2i\mathrm{Tr}[\beta(x)\gamma^5] \tag{102}$$

where the factor 2 is for both ψ and $\overline{\psi}$. The normalized expectation value is computed with the path integral in the background of the gauge field A_μ, while the functional trace is extracted from the path integral as it is independent of the fermionic fields ψ and $\overline{\psi}$.

As discussed, the Fujikawa's Jacobian must be regulated. We take as regulator the square of the Dirac operator

$$\mathcal{R} = -\not{D}^2(A) = -D^\mu D_\mu(A) + \frac{i}{2} F_{\mu\nu}\gamma^\mu\gamma^\nu \ . \tag{103}$$

In euclidean, it gives rise to an elliptic differential operator of the second order which cuts off the high-frequency modes of the fermionic fields and regulates the infinite dimensional trace. Thus, the right-hand side of (102) is regulated by

$$\text{Tr}[\beta(x)\gamma^5] \qquad \to \qquad \lim_{T\to 0} \text{Tr}[\beta(x)\gamma^5\, e^{-T\mathcal{R}}] = \lim_{T\to 0} \text{Tr}[\beta(x)\gamma^5\, e^{T\not{D}^2(A)}] \tag{104}$$

with $T = \frac{1}{M^2}$, recall (95). To explicitly compute this anomaly, it is useful to recall some results on the heat kernel. So, let us take an aside and describe the heat kernel.

<u>Heat kernel</u>

Let us consider a flat D-dimensional spacetime and an operator H of the form

$$H = -D^2 + V \tag{105}$$

where V is a matrix-valued potential and $D^2 = D^\mu D_\mu$, with $D_\mu = \partial_\mu + W_\mu$ the gauge covariant derivative satisfying

$$[D_\mu, D_\nu] = \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu] = \mathcal{F}_{\mu\nu} \ . \tag{106}$$

The heat kernel is the operator

$$K(T) = e^{-TH} \tag{107}$$

that satisfies the heat equation (the Schrödinger equation in euclidean time)

$$\frac{\partial K(T)}{\partial T} = -HK(T) \tag{108}$$

with boundary condition $K(0) = \mathbb{1}$.

The trace of the heat kernel has a small time expansion given by

$$\begin{aligned}
\text{Tr}\left[Je^{-TH}\right] &= \int d^Dx\, \text{tr}\Big[J(x) \underbrace{\langle x|e^{-TH}|x\rangle}_{(4\pi T)^{-\frac{D}{2}}\sum_{n=0}^\infty a_n(x,H)T^n} \Big] \\
&= \int \frac{d^Dx}{(4\pi T)^{\frac{D}{2}}} \sum_{n=0}^{\infty} \text{tr}[J(x)a_n(x,H)]T^n \\
&= \int \frac{d^Dx}{(4\pi T)^{\frac{D}{2}}}\, \text{tr}\left[J(x)(a_0(x,H) + a_1(x,H)T + a_2(x,H)T^2 + ...)\right]
\end{aligned} \tag{109}$$

where the symbol "tr" is a trace on the remaining discrete matrix indices, $J(x)$ is an arbitrary matrix-valued function, and $a_n(x,H)$ are the heat kernel, or Seeley-DeWitt, coefficients. They are matrix-valued, and the first ones are well-known and given by

$$\begin{aligned}
a_0(x,H) &= \mathbb{1} \\
a_1(x,H) &= -V \\
a_2(x,H) &= \frac{1}{2}V^2 - \frac{1}{6}D^2V + \frac{1}{12}\mathcal{F}_{\mu\nu}^2
\end{aligned} \tag{110}$$

where $D_\mu V = \partial_\mu V + [W_\mu, V]$, and so on.

Chiral anomalies in $D = 2, 4$

Getting back to the evaluation of the anomalies, we see that the regulator $\mathcal{R}$ in (103) corresponds to a Hamiltonian H with

$$W_\mu \equiv -iA_\mu, \qquad V \equiv \frac{i}{2} F_{\mu\nu}\gamma^\mu\gamma^\nu \tag{111}$$

Thus, in $D = 2$, we see that the chiral anomaly from (102) is obtained by the limit $T \to 0$ that produces

$$\partial_\mu \big\langle J_5^\mu(x) \big\rangle = \frac{2i}{4\pi} \mathrm{tr}(\gamma^5 a_1(x)) = -\frac{i}{2\pi}\mathrm{tr}(\gamma^5 V) = \frac{1}{4\pi}\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu)F_{\mu\nu} = \frac{i}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} \tag{112}$$

where we used $\gamma^5 = -i\gamma^1\gamma^2$ and $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu) = 2i\epsilon^{\mu\nu}$.

Similarly, in $D = 4$ we get

$$\begin{aligned} \partial_\mu \big\langle J_5^\mu(x) \big\rangle &= \frac{2i}{(4\pi)^2}\mathrm{tr}(\gamma_5\, a_2(x)) = \frac{2i}{(4\pi)^2}\mathrm{tr}\Big(\gamma_5 \frac{1}{2}V^2 + \cdots\Big) \\ &= -\frac{i}{(4\pi)^2}\frac{1}{4}\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta)F_{\mu\nu}F_{\alpha\beta} \\ &= -\frac{i}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \end{aligned} \tag{113}$$

where dots indicate terms that vanish under the trace, $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$, and $\mathrm{tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = 4\epsilon^{\mu\nu\alpha\beta}$. This leads to (82) after Wick rotating back to Minkowski space.

Now, let us verify that the gauge symmetry does not have anomalies, i.e. that the effective action in (97) is gauge invariant $\Gamma[A'] = \Gamma[A]$. Considering an infinitesimal gauge transformation (given in (73) and (74)), we write

$$\begin{aligned} e^{-\Gamma[A']} &= \int D\psi D\overline{\psi}\, e^{-S[\psi,\overline{\psi};A']} = \int D\psi' D\overline{\psi}'\, e^{-S[\psi',\overline{\psi}';A']} \\ &= \int D\psi(1 - \mathrm{Tr}(i\alpha))D\overline{\psi}(1 + \mathrm{Tr}(i\alpha))\, e^{-S[\psi,\overline{\psi};A]} \\ &= \int D\psi D\overline{\psi}\, e^{-S[\psi,\overline{\psi};A]}(1 - \mathrm{Tr}(i\alpha) + \mathrm{Tr}(i\alpha)) = \int D\psi D\overline{\psi}\, e^{-S[\psi,\overline{\psi};A]} \\ &= e^{-\Gamma[A]} \end{aligned} \tag{114}$$

which proves the gauge invariance. This gauge invariance

$$\delta\Gamma[A] = \int d^4x\, \delta A_\mu(x)\frac{\delta\Gamma[A]}{\delta A_\mu(x)} = \int d^4x\, (\partial_\mu\alpha(x))\frac{\delta\Gamma[A]}{\delta A_\mu(x)} = -\int d^4x\, \alpha(x)\partial_\mu\frac{\delta\Gamma[A]}{\delta A_\mu(x)} = 0 \tag{115}$$

translates into the conservation of $\langle J^\mu(x)\rangle = \frac{\delta\Gamma[A]}{\delta A_\mu(x)}$, the expectation value of the gauge current $J^\mu = i\overline{\psi}\gamma^\mu\psi$, namely

$$\partial_\mu\langle J^\mu(x)\rangle = 0\ . \tag{116}$$

Final comments

The gauge symmetry may become anomalous if the fermions are chiral. For example, if one takes the fermion in (96) to be Weyl fermions, rather than Dirac fermion, the gauge symmetry itself becomes anomalous. In such a case, the definition of the path integral in terms of a determinant of the Dirac operator is more subtle, as the Dirac operator maps left-handed fermions to right-handed ones. This situation happens for some chiral coupling of the Standard Model, however possible anomalies in the gauge symmetries are absent thanks to a cancellation arising from summing the contribution to the anomaly from the various species of the fermion running in the loop (the cancellation happens family by family).