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Quantization of gauge theories

(Lecture notes - a.a. 2021/22)

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1 Faddeev-Popov

The quantization of gauge theories requires additional considerations for constructing a well-defined QFT and its perturbative expansion. The problem is clear in the path integral approach, where a naive path integral quantization gives a diverging result. Let us reconsider the case of the abelian U(1) theory (free electromagnetism)

$$\begin{aligned} S[A] &= \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \\ Z &= \int DA\, e^{iS[A]} \sim \infty \end{aligned} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here the path integral diverges because one is summing over an infinite number of gauge equivalent configurations

$$A_\mu(x) \quad \to \quad A^g_\mu(x) = A_\mu(x) + ig(x)\partial_\mu g^{-1}(x) \;, \qquad g(x) = e^{i\alpha(x)} \in U(1) \tag{2}$$

which have the same value of the action, $S[A^g] = S[A]$. The field space decomposes into inequivalent gauge orbits, as in figure 1.

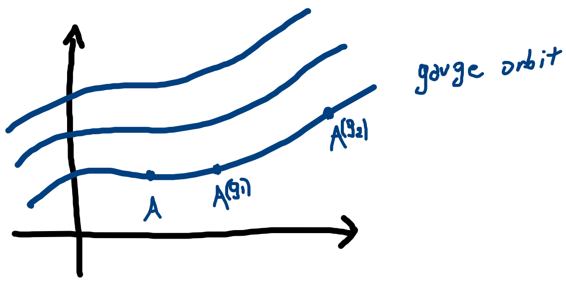

Figure 1: Gauge orbits

One would like to define the path integral in such a way of getting a finite and gauge invariant result

$$Z = \int \frac{DA}{\text{Vol(Gauge)}}\, e^{iS[A]} \sim \text{finite} \tag{3}$$

where 'Vol(Gauge)' formally indicates the infinite volume of the gauge group. This definition can be implemented concretely using a gauge fixing function à la Faddeev-Popov, where unphysical ghost fields are introduced to exponentiate a measure factor. Ideally, the gauge fixing function should pick just one representative from each gauge orbits, as sketched in figure 2.

The gauge-fixed action thus obtained will exhibit a global symmetry, called BRST symmetry, which is a remnant of the original gauge symmetry and which is instrumental in proving

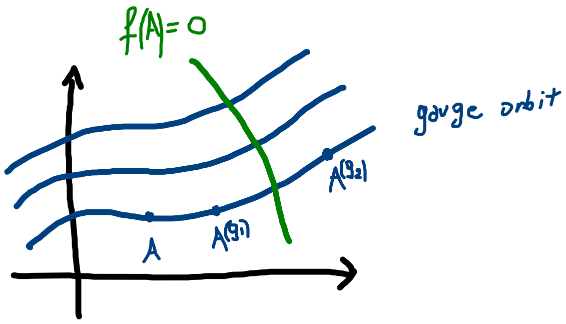

Figure 2: Gauge fixing by the condition $f(A) = 0$

many properties of the quantum gauge theories. In particular, it can be used to prove the independence of physical quantities from the gauge-fixing function. It is also the basis of the BRST lagrangian quantization method, which generalizes the Faddeev-Popov method to more general cases. An even more powerful method uses additional fields (the so-called antifields) and is known as the Batalin-Vilkovisky method.

Let us start describing the Faddeev-Popov method in the case of the free abelian gauge theory. The trick it to use a gauge-fixing condition, as depicted in figure 2, and insert the identity (written in a suitable way) in the path integral so to extract the volume of the gauge group. Consider the identity written as

$$1 = \int df \, \delta(f) = \int dy \frac{\partial f(y)}{\partial y} \delta(f(y)) \tag{4}$$

which generalizes to n-dimensions (with f that becomes a vector with n components)

$$1 = \int d^n f \, \delta^{(n)}(f) = \int d^n y \det \left(\frac{\partial f^i(y)}{\partial y^j} \right) \delta^{(n)}(f(y)) \, . \tag{5}$$

We generalize this further to functional integrals

$$1 = \int Dg \, \delta\Big(f(A^g(x))\Big) \mathrm{Det} \left(\frac{\delta f(A^g(x))}{\delta g(y)} \right) \tag{6}$$

where Dg is a suitable gauge invariant measure that also produces the volume of the gauge group, $\int Dg = \mathrm{Vol(Gauge)}$. 'Det' indicates a functional determinant. The delta function (actually, "delta functional") $\delta(f(x))$ means that the whole function $f(x)$ is set to vanish.

Then let us compute in a formal way the path integral in (3) inserting the identity in (6)

$$\begin{aligned} Z &= \int \frac{DA}{\mathrm{Vol(Gauge)}} e^{iS[A]} \\ &= \int \frac{DA}{\mathrm{Vol(Gauge)}} \int Dg \, \delta\Big(f(A^g(x))\Big) \mathrm{Det} \left(\frac{\delta f(A^g(x))}{\delta g(y)} \right) e^{iS[A]} \\ &= \int \frac{DA^g Dg}{\mathrm{Vol(Gauge)}} \, \delta\Big(f(A^g(x))\Big) \mathrm{Det} \left(\frac{\delta f(A^g(x))}{\delta g(y)} \right) e^{iS[A^g]} \\ &= \int \frac{Dg}{\mathrm{Vol(Gauge)}} \int DA \, \delta\Big(f(A(x))\Big) \mathrm{Det} \left(\frac{\delta f(A^g(x))}{\delta g(y)} \right) \Bigg|_{g=1} e^{iS[A]} \\ &= \int DA \, \delta\Big(f(A(x))\Big) \mathrm{Det} \left(\frac{\delta f(A^g(x))}{\delta g(y)} \right) \Bigg|_{g=1} e^{iS[A]} \end{aligned} \tag{7}$$

which is the gauge-fixed path integral we were looking for. In these manipulations we have first inserted the identity (6), and then used the fact the action and measure are both gauge invariant, namely $S[A^g] = S[A]$ and $DA^g = DA$. This is certainly true for the classical action, but it is an assumption for the measure (more concretely it is related to the regularization methods used to make sense of the diverging Feynman diagrams of the perturbative expansion[1]). Then, we have changed variables from A^g to A, so that nothing depends on $g(x)$ anymore, and the integration on Dg can be factorized out to cancel the infinite gauge volume Vol(Gauge).

The final formula is correct but can be written in more a useful form by:
(i) introducing ghosts, i.e. anticommuting fields $c(x)$ and $\bar{c}(x)$ to exponentiate the determinant, known as the Faddeev-Popov determinant,
(ii) modifying the gauge-fixing function to get rid of the delta functional from the path integral: instead of setting $f(A(x)) = 0$, one may equivalently set it to $f(A(x)) = h(x)$, where $h(x)$ is an arbitrary function, and then functionally average over the functions $h(x)$ with gaussian weight $e^{-\frac{i}{2\xi}\int h^2}$. The rationale behind this procedure is that different gauge-fixing functions must give the same gauge invariant result. In particular, nothing physical should depend on the value of the parameter ξ.

Thus one finds

$$Z = \int DADcD\bar{c}Dh\, \delta\Big(f(A(x)) - h(x)\Big) \exp i \left(S[A] + \int d^4x d^4y\, \bar{c}(x) \left. \frac{\delta f(A^g(x))}{\delta g(y)} \right|_{g=1} c(y) - \frac{1}{2\xi} \int d^4x\, h^2(x) \right) \tag{8}$$

which is simplified by path integrating over $h(x)$ to eliminate the delta functional, and find in the exponent the gauge-fixed total action S_{tot}

$$\begin{aligned} Z &= \int DADcD\bar{c} \exp i \left(S[A] + \int d^4x d^4y\, \bar{c}(x) \left. \frac{\delta f(A^g(x))}{\delta g(y)} \right|_{g=1} c(y) - \frac{1}{2\xi} \int d^4x\, f^2(A(x)) \right) \\ &= \int DADcD\bar{c}\, e^{iS_{tot}[A,c,\bar{c}]} \,. \end{aligned} \tag{9}$$

To exemplify the above construction, let us choose as gauge-fixing function

$$f(A) = \partial^\mu A_\mu \tag{10}$$

which corresponds to the Lorenz gauge for the path integral in (7), and to a weighted Lorenz gauge for the path integral in (9) (also called R_ξ gauge). Under a gauge variation $\delta A_\mu = \partial_\mu \alpha$

$$\delta f(A) = \partial^\mu \delta A_\mu = \partial^\mu \partial_\mu \alpha \tag{11}$$

and one is led to interpret[2]

$$\left. \frac{\delta f(A^g(x))}{\delta g(y)} \right|_{g=1} \sim \frac{\delta f(A(x))}{\delta \alpha(y)} = \partial^\mu \partial_\mu \delta^4(x-y) \tag{12}$$

whose determinant (the Faddeev-Popov determinant) can be reproduced by path integrating over the ghost fields $c(x)$ and $\bar{c}(x)$. Thus, the gauge fixed action reads

$$S_{tot}[A, c, \bar{c}] = S[A] + \int d^4x \left(-\partial^\mu \bar{c} \partial_\mu c - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \right) \,. \tag{13}$$

[1]In chiral theories where the regularization might fail to maintain gauge invariance, one speaks of anomalies. The quantization procedure breaks down if the anomalies in the gauge symmetry are not canceled.

[2]The identification $\delta g \sim g^{-1} dg$ gives a group invariant measure, the so-called Haar measure.

The corresponding total lagrangian is

$$\mathcal{L}_{tot} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \partial^\mu \bar{c} \partial_\mu c - \frac{1}{2\xi} (\partial^\mu A_\mu)^2 \tag{14}$$

which (up to total derivatives that are usually dropped) may be written as

$$\mathcal{L}_{tot} = -\frac{1}{2} \partial^\mu A^\nu \partial_\mu A^\nu + \frac{1}{2} \left(1 - \frac{1}{\xi} \right) (\partial^\mu A_\mu)^2 - \partial^\mu \bar{c} \partial_\mu c \; . \tag{15}$$

In particular, the Feynman gauge is obtained for $\xi = 1$ and gives the simple action

$$\mathcal{L}_{tot} = -\frac{1}{2} \partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu \bar{c} \partial_\mu c \; . \tag{16}$$

The path integral is now well-defined, one can add sources, and compute propagators. In the Feynman gauge they read

$$\begin{aligned} \langle A_\mu(x) A_\nu(y) \rangle &= \int \frac{d^4p}{(2\pi)^4} \, e^{ip(x-y)} \, \frac{-i\eta_{\mu\nu}}{p^2 - i\epsilon} \\ \langle c(x) \bar{c}(y) \rangle &= \int \frac{d^4p}{(2\pi)^4} \, e^{ip(x-y)} \, \frac{-i}{p^2 - i\epsilon} \; . \end{aligned} \tag{17}$$

Exercize: Find the propagators in the R_ξ gauge.

At the stage one can also add fermions to obtain the complete QED action, and note that the ghosts can be integrated out and eliminated, as they contribute at most to an overall normalization factor that disappears in normalized correlation functions. This will not be the case for non-abelian gauge theories (and also for QED in curved space), but before addressing the non-abelian case, let us discover and study a rigid symmetry that is present in the gauge-fixed action and which has far reaching implications, the BRST symmetry.

2 BRST symmetry

The gauge-fixed action S_{tot} is not gauge invariant anymore, as the gauge symmetry has been fixed. However, the gauge symmetry survives in the form of BRST symmetry, a rigid symmetry that depends on a constant anticommuting parameter Λ. For the lagrangian in the Feynman gauge

$$\mathcal{L}_{tot} = -\frac{1}{2} \partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu \bar{c} \partial_\mu c \tag{18}$$

the BRST symmetry takes the form

$$\begin{aligned} \delta_B A_\mu(x) &= \Lambda \, \partial_\mu c(x) \\ \delta_B c(x) &= 0 \\ \delta_B \bar{c}(x) &= \Lambda \, \partial^\mu A_\mu(x) \end{aligned} \tag{19}$$

and gives $\delta_B \mathcal{L}_{tot} = 0$, up to total derivatives. Recall that Λ, $c(x)$ and $\bar{c}(x)$ anticommute (they are Grassmann valued quantities). A crucial property of this symmetry is that it is nilpotent, i.e. it is a Lie symmetry with a Grassmann odd parameter and satisfies

$$[\delta_B(\Lambda_1), \delta_B(\Lambda_2)] = 0 \; . \tag{20}$$

where one uses two different values of the parameter Λ. The nilpotency property is perhaps more evident if one defines the Slavnov operator s by factorizing the constant parameter Λ as

$$\delta_B(\Lambda) = \Lambda s\ . \tag{21}$$

The Slavnov operator s is a nilpotent graded variation, where graded means that it anticommutes with Grassmann odd fields, and where nilpotency means that is satisfies

$$s^2 = 0 \tag{22}$$

i.e., operating twice with it gives a vanishing result on any field. Explicitly, we get from (19)

$$\begin{aligned} sA_\mu(x) &= \partial_\mu c(x) \\ sc(x) &= 0 \\ s\bar{c}(x) &= \partial^\mu A_\mu(x) \end{aligned} \tag{23}$$

that indeed gives

$$s^2 A_\mu(x) = s^2 c(x) = 0 \qquad \text{and} \qquad s^2\bar{c}(x) = \partial^\mu\partial_\mu c(x) = 0 \tag{24}$$

where the ghost equations of motion have been used in the last equation. Because one needs to use the equations of motion, these particular BRST symmetry rules are said to be nilpotent on-shell.

As for any rigid symmetry, there is an associated conserved current J^μ_B, that can be found as usual by the Noether trick, i.e. extending $\Lambda \to \Lambda(x)$ in (19) and computing

$$\delta_B S_{tot} = \int d^4x\, (\partial_\mu\Lambda) J^\mu_B\ , \qquad J^\mu_B = -F^{\mu\nu}\partial_\nu c - \partial^\nu A_\nu \partial^\mu c \tag{25}$$

with associated BRST charge

$$Q_B = \int d^3x\, (\partial_\nu A^\nu \dot{c} - F^{0i}\partial_i c)\ . \tag{26}$$

There is a way of adding auxiliary fields to make the BRST symmetry nilpotent off-shell. To see how this happens, let us use a path integral representation of the delta functional in (6)

$$\delta(f(x)) = \int DB\ e^{-i\int d^4x\, B(x) f(x)}\ , \tag{27}$$

and use it into the path integral (8). Then, path-integrate over h by gaussian integration. The relevant part from (8) is

$$\begin{aligned} &\int Dh\, \delta\Big(f(A(x)) - h(x)\Big) e^{-i\int d^4x\, \frac{1}{2\xi}h^2(x)} = \int DhDB\ e^{-i\int d^4x\, \left(B(x)[f(A(x))-h(x)] + \frac{1}{2\xi}h^2(x)\right)} \\ &= \int DB\, e^{i\int d^4x\, \left(-B(x)f(A(x)) + \frac{\xi}{2}B^2(x)\right)}\ . \end{aligned} \tag{28}$$

From these manipulations one finds a new total lagrangian for the fields $(A_\mu, c, \bar{c}, B)$. As we have taken as gauge-fixing function $f(A) = \partial^\mu A_\mu$, the lagrangian takes the form

$$\mathcal{L}_{tot} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \partial^\mu\bar{c}\partial_\mu c - B\partial^\mu A_\mu + \frac{\xi}{2}B^2\ . \tag{29}$$

Of course, eliminating B by its algebraic equations of motion (equivalently, path-integrating over B) reproduces the original lagrangian in (35), so that this checks that the new lagrangian gives an equivalent description of the same theory. Now the BRST symmetry reads

$$\begin{aligned} \delta_B A_\mu &= \Lambda\,\partial_\mu c \\ \delta_B c &= 0 \\ \delta_B \bar{c} &= \Lambda B \\ \delta_B B &= 0 \end{aligned} \tag{30}$$

and is nilpotent off-shell. Recall that the fields A_μ and B are commuting (bosonic), while c and $\bar{c}$ are anticommuting (fermionic). The field $c(x)$ is called the *ghost*, and corresponds to the gauge parameter $\alpha(x)$, which in a sense is turned into a dynamical field, but with opposite Grassmann character ($\alpha(x) \to \Lambda c(x)$). The field $\bar{c}$ is also called *antighost*, while the field B is the *auxiliary field* (or *Nakanishi-Lautrup field*).

3 Non-abelian gauge fields

Let us apply what we discussed so far to non-abelian gauge fields. Consider the Yang-Mills theory with lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \tag{31}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$, with infinitesimal gauge symmetry

$$\delta A^a_\mu(x) = D_\mu \alpha^a(x) = \partial_\mu \alpha^a(x) + g f^{abc} A^b_\mu(x) \alpha^c(x)\ . \tag{32}$$

One can choose as gauge-fixing functions (one for each gauge symmetry)

$$f^a(A) = \partial^\mu A^a_\mu \tag{33}$$

and recognize that the Faddeev-Popov determinant, just like in (12), arises from

$$\frac{\delta f^a(A(x))}{\delta \alpha^b(y)} = \partial^\mu D^{ab}_\mu \delta^4(x-y) \tag{34}$$

where the covariant derivative in the adjoint is written as $D^{ab}_\mu = \partial_\mu \delta^{ab} + g f^{acb} A^c_\mu(x)$. Now, using ghosts fields and considering the weighted Lorenz gauge (R_ξ gauge), one finds the gauge-fixed action

$$\mathcal{L}_{tot} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \partial^\mu \bar{c}^a D_\mu c^a - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2\ . \tag{35}$$

The ghosts do not decouple, as there is a covariant derivative appearing: there is a non-trivial interaction vertex sitting inside

$$\mathcal{L}_{gh} = -\partial^\mu \bar{c}^a D_\mu c^a = -\partial^\mu \bar{c}^a \partial_\mu c^a - g f^{abc} \partial^\mu \bar{c}^a A^b_\mu c^c\ . \tag{36}$$

Thus the ghosts enter the Feynman diagrams, and they will do their job to guarantee the consistency of the final result.

The gauge-fixed lagrangian has a nilpotent BRST symmetry. It is simpler to introduce first the auxiliary fields B^a and present the gauge-fixed action in the form

$$\mathcal{L}_{tot} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \partial^\mu \bar{c}^a D_\mu c^a - B^a \partial^\mu A^a_\mu + \frac{\xi}{2} (B^a)^2 \tag{37}$$

whose BRST symmetry takes the simple form

$$\begin{aligned} \delta_B A_\mu^a &= \Lambda\, D_\mu c^a \\ \delta_B c^a &= -\frac{g}{2} f^{abc} c^b c^c \Lambda \\ \delta_B \bar{c}^a &= \Lambda B^a \\ \delta_B B^a &= 0 \end{aligned} \tag{38}$$

and is nilpotent off-shell. To recognize the BRST variation of the ghosts c^a one could have imposed the nilponency of the BRST rule on A_μ^a

$$[\delta(\Lambda_1), \delta(\Lambda_2)] A_\mu^a = 0 \ . \tag{39}$$

Eliminating the auxiliary B^a by its equation of motion ($B^a = \frac{1}{\xi}\partial^\mu A_\mu^a$) modifies the BRST transformation rule of the antighost to

$$\delta_B \bar{c}^a = \Lambda \frac{1}{\xi} \partial^\mu A_\mu^a \tag{40}$$

making the BRST symmetry nilponent only on-shell.

Having found a gauge-fixed action, one can use it to generate the perturbative expansion and Feynman diagrams. In particular, one can deduce the Feynman rules corresponding to propagators and vertices (see ch. 72 of Srednicki, and a further set of lecture notes). But before doing that, we discuss the BRST quantization method and some of its properties.

4 BRST quantization

BRST quantization is an algebraic method that allows to find the complete gauge-fixed action entering the path integral, including the ghosts and gauge-fixing terms. It is recognized by extending the key features observed in the previous examples, and uses the BRST symmetry as guiding principle.

We employ the non-abelian gauge theory to present the main steps of the method. The BRST quantization consists in introducing the BRST symmetry first by starting from the gauge invariant lagrangian. Then, by adding a suitable set of non-minimal fields, one presents the gauge-fixed total lagrangian by requiring it to be BRST invariant. This provides a set-up where quite general gauges can be chosen.

Thus, one starts from the gauge field $A_\mu(x) = -iA_\mu^a(x) T^a$ with gauge invariant lagrangian

$$\mathcal{L} = \frac{1}{2g^2} \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) = \frac{2}{g^2} \mathrm{tr}\left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) \tag{41}$$

whose infinitesimal gauge symmetry is given by

$$\delta A_\mu(x) = D_\mu \alpha(x) \ . \tag{42}$$

The BRST symmetry on the gauge field is obtained by introducing the ghost $c(x) = -ic^a(x) T^a$ and replacing $\alpha(x) \to \Lambda c(x)$ in the infinitesimal gauge transformation, where Λ is the rigid anticommuting Lie parameter of the BRST symmetry. Evidently, the BRST symmetry on the gauge field is equivalent to a gauge transformation[3]

$$\delta A_\mu(x) = D_\mu \alpha(x) \qquad \to \qquad \delta_B A_\mu(x) = \Lambda D_\mu c(x) \tag{43}$$

[3]The covariant derivative of a field in the adjoint is $D_\mu \alpha = \partial_\mu \alpha + [A_\mu, \alpha]$, and similarly $D_\mu c = \partial_\mu c + [A_\mu, c]$.

and the classical action is manifestly BRST invariant. Having introduces the ghost c we have to find its BRST variation, which is obtained by requiring that the BRST transformation of the gauge field A_μ be nilpotent

$$[\delta_B(\Lambda_1), \delta_B(\Lambda_2)] A_\mu(x) = 0 \qquad \rightarrow \qquad \delta_B c(x) = -c(x)c(x)\Lambda \; . \tag{44}$$

Note that

$$c(x)c(x) = c^a(x)T^a c^b(x)T^b = c^a(x)c^b(x)T^a T^b = c^a(x)c^b(x)\frac{1}{2}[T^a, T^b] = c^a(x)c^b(x)\frac{i}{2}f^{abc}T^c \tag{45}$$

that implies

$$\delta_B c^a(x) = -\frac{1}{2} f^{abc} c^b(x) c^b(x) \Lambda \; . \tag{46}$$

It contains the structure constants of the gauge group and coincides[4] with the one in (38). Nilpotency of the BRST symmetry on the ghost is a consequence of the Jacobi identity satisfied by the structure constants of the gauge group.

At this stage the BRST symmetry is equivalent to the original gauge symmetry. The path integral with the ghosts has a chance of being finite

$$Z = \int DADc\, e^{iS[A]} = \text{finite} \tag{47}$$

but it is still useless in this from (comparing with (3) one may identify $\int Dc = \frac{1}{\text{Vol(Gauge)}}$) .

To make it more useful, one introduces additional fields, the non-minimal fields $(\bar{c}, B)$, with $\bar{c}$ fermionic and B bosonic[5], namely the antighost and the auxiliary field. They carry trivial BRST transformations

$$\begin{aligned} \delta_B \bar{c}(x) &= \Lambda B(x) \\ \delta_B B(x) &= 0 \end{aligned} \tag{48}$$

which are obviously nilpontent. They are used to introduce a gauge fermion Ψ of the form[6]

$$\Psi = \bar{c} f(A_\mu, B) \tag{49}$$

where $f(A_\mu, B)$ plays the role of the gauge-fixing function. The BRST variation of the gauge fermion (with the parameter Λ removed) is manifestly BRST invariant, as the BRST symmetry is nilpotent, and that the total lagrangian defined by

$$\mathcal{L}_{tot} = \mathcal{L} + s\Psi \tag{50}$$

is BRST invariant by construction

$$s\mathcal{L}_{tot} = s\mathcal{L} + s^2\Psi = 0 \tag{51}$$

(up to total derivatives, as usual).

Suitable choices of the function $f(A_\mu, B)$ deliver a gauge-fixed action that has a well-defined path integral. It has to be path integrated over $(A_\mu, c, \bar{c}, B)$ and can be used to generate the perturbative expansion in terms of Feynman diagrams.

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[4]The coupling constant is absorbed into A_μ and c.

[5]Again with the expansion $\bar{c}(x) = -i\bar{c}^a(x)T^a$ and $B(x) = -iB^a(x)T^a$.

[6]For notational simplicity we neglect an overall $\frac{2}{g^2}\text{tr}(\cdots)$, that is reinstated at the end.

A useful choice for our model is

$$f(A_\mu, B) = \partial^\mu A_\mu - \frac{\xi}{2} B \qquad \rightarrow \qquad \Psi = \bar{c}\left(\partial^\mu A_\mu - \frac{\xi}{2} B\right) \tag{52}$$

which produces

$$s\Psi = B\left(\partial^\mu A_\mu - \frac{\xi}{2} B\right) - \bar{c}\,\partial^\mu D_\mu c \,. \tag{53}$$

Then, the correctly gauge-fixed total lagrangian is

$$\mathcal{L}_{tot} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \bar{c}\,\partial^\mu D_\mu c + B \partial^\mu A_\mu - \frac{\xi}{2} B^2 \,. \tag{54}$$

It compares successfully with (37), after reinstating the overall normalized trace $\frac{2}{g^2}\mathrm{tr}$.

Comments:

• One can introduce a *ghost number* by assigning ghost number 1 to the ghost c, and ghost number -1 to the antighost $\bar{c}$. Other fields have vanishing ghost number. The ghost number helps to classify the various quantities: the action has ghost number 0, the s operator changes ghost number by 1 (i.e. it has ghost number 1), and the gauge fermion Ψ has ghost number -1.

• The nilpontecy of the BRST symmetry brings in the concept of *cohomology*. It is used to identify the physical (i.e. gauge invariant) observables as *cohomology classes*: physical observables are BRST invariant functions, while two BRST invariant functions differing by the BRST variation of any function belong to the same class, and identify the same physical observable.

• S and $S_{tot} = S + s\int \Psi$ are equivalent actions and produce the same physical results.

• The BRST quantization is a very general approach. It can be applied to all cases where the gauge algebra has constant structure functions and closes off-shell (no equations of motions are needed to verify the closure of the gauge algebra). For example, it can be used to quantize gravity (i.e. the Einstein-Hilbert action), which however remains a non-renormalizable theory.

• More general theories, as for example supergravity, are often described by open gauge algebras[7]. Then, more general schemes are needed to get the correct gauge-fixed action. The Batalin-Vilkovisky method is such a scheme. It embeds the BRST method, and uses furthers concepts (antifields and antibrackets).

4.1 Application: gauge-fixing of perturbative quantum gravity

A further application of the BRST method is the gauge-fixing of the Einstein-Hilbert action for perturbative quantum gravity. In this theory the dynamical field is the metric tensor $g_{\mu\nu}(x)$ that defines the invariant length element in spacetime by

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \,. \tag{55}$$

The metric has a gauge symmetry associated with the *arbitrary changes of coordinates* (also known as *diffeomorphisms* or *reparametrizations*) according to which under a change of coordinates $x \to x'(x)$ it transforms according to the tensorial law

$$g_{\mu\nu}(x) \;\rightarrow\; g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \,. \tag{56}$$

[7] Equations of motions are needed to close the algebra of the gauge symmetries.

For an infinitesimal change of coordinates given by

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x) \tag{57}$$

where $\xi^\mu(x)$ is an arbitrary infinitesimal vector field, the metric transforms infinitesimally as

$$\begin{aligned}\delta g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x) &= \xi^\rho(x)\partial_\rho g_{\mu\nu}(x) + \partial_\mu \xi^\rho(x) g_{\rho\nu}(x) + \partial_\nu \xi^\rho(x) g_{\mu\rho}(x) \\ &= \nabla_\mu \xi_\nu(x) + \nabla_\nu \xi_\mu(x) \ .\end{aligned} \tag{58}$$

as directly deduced from the finite transformation rule above. There are 4 independent infinitesimal gauge functions contained the vector field ξ^μ. The gauge invariant action for the Einstein's theory of gravity is the Einstein-Hilbert action

$$S_{EH}[g_{\mu\nu}] = \frac{1}{2\kappa^2}\int d^4x \sqrt{g} R(g) \tag{59}$$

where $g = |\det g_{\mu\nu}|$, $R(g)$ is the Ricci scalar built from $g_{\mu\nu}$, and $\kappa^2 = 8\pi G$ with G the Newton constant. κ is taken as the coupling constant of the theory.

Our conventions follow from the definition of the covariant derivative ∇_μ, defined on vectors fields by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho \ , \qquad \nabla_\mu W_\nu = \partial_\mu W_\nu - \Gamma^\rho_{\mu\nu} W_\rho \tag{60}$$

where the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ (the components of the Levi-Civita connection) is

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \ . \tag{61}$$

Covariant derivatives do not commute, and are used to define the curvature tensors by

$$[\nabla_\mu, \nabla_\nu] V^\rho = R_{\mu\nu}{}^\rho{}_\sigma V^\sigma \ , \quad R_{\mu\nu} = R_{\rho\mu}{}^\rho{}_\nu \ , \quad R = R^\mu{}_\mu \tag{62}$$

known as Riemann tensor, Ricci tensor, and Ricci scalar, respectively.

Perturbatively, for small fluctuations around Minkowski space, one splits the metric as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \tag{63}$$

where $h_{\mu\nu}(x)$ is the field whose quanta are the "gravitons". The action expanded in terms of $h_{\mu\nu}(x)$, contains a quadratic part (it identifies the graviton propagator after gauge-fixing) plus an infinite number of vertices

$$S_{EH}[g_{\mu\nu}] = S_2[h_{\mu\nu}] + \sum_{n=3}^{\infty} S_n[h_{\mu\nu}] \tag{64}$$

and the action is fully nonlinear in $h_{\mu\nu}(x)$. The vertices have a quite complicated structure, but the quadratic part is easily derivable. It contains a kinetic operator that is not invertible because of the gauge symmetry (58), thus one cannot obtain the propagator for $h_{\mu\nu}$ and a gauge-fixing is required.

To start with the BRST quantization of (59), we first introduce the ghost c^μ by letting the gauge parameters $\xi^\mu(x) \to \Lambda c^\mu(x)$, and define the BRST rule on the metric $g_{\mu\nu}$ by

$$\delta_B g_{\mu\nu} = \Lambda(\nabla_\mu c_\nu + \nabla_\nu c_\mu) \ . \tag{65}$$

Requiring nilpotency fixes the BRST transformation rule of the ghosts

$$\delta_B c^\mu = \Lambda c^\nu \partial_\nu c^\mu \ . \tag{66}$$

This calculation is algebraically lenghty, but geometrical considerations simplify the task. To that purpose, let us notice that the gauge symmetry is generated by the Lie derivatives along ξ^μ (denoted by $\pounds_\xi$), which translates into the fact that (58) may be rewritten as

$$\delta g_{\mu\nu} = \pounds_\xi \, g_{\mu\nu} \tag{67}$$

(said differently, this defines the Lie derivative on the metric). More generally, any tensor transforms under the infinitesimal change of coordinates (57) by its Lie derivative along ξ^μ. In particular, a scalar field ϕ transforms as

$$\delta\phi = \pounds_\xi \, \phi = \xi^\mu \partial_\mu \phi \tag{68}$$

which would give a BRST transformation

$$\delta_B \phi = \Lambda c^\mu \partial_\mu \phi \ . \tag{69}$$

It is now simpler to study the nilpotency on the scalar ϕ to fix the BRST rule of the ghosts. This amounts to extract the structure constants of the algebra of the diffeomorphisms. Writing $\delta_B c^\mu = \Lambda s c^\mu$ in terms of the Slavnov variation s we impose

$$0 = [\delta_B(\Lambda_1), \delta_B(\Lambda_2)]\phi = 2\Lambda_2\Lambda_1(sc^\mu \partial_\mu \phi - c^\nu \partial_\nu c^\mu \partial_\mu \phi) \tag{70}$$

that fixes

$$sc^\mu = c^\nu \partial_\nu c^\mu \qquad \to \qquad \delta_B c^\mu = \Lambda c^\nu \partial_\nu c^\mu \ . \tag{71}$$

This rule captures the structure constants carried by the commutator of the Lie derivatives

$$[\pounds_\xi, \pounds_\eta] = \pounds_{[\xi,\eta]} \tag{72}$$

which is the gauge algebra of the group of diffeomorphisms. Thus, we have found the BRST transformation that contain the information on the structure constants of the gauge group

$$\begin{aligned} \delta_B g_{\mu\nu} &= \Lambda(\nabla_\mu c_\nu + \nabla_\nu c_\mu) \\ \delta_B c^\mu &= \Lambda c^\nu \partial_\nu c^\mu \ . \end{aligned} \tag{73}$$

Now, to implement a gauge-fixing we add non-minimal fields $\bar{c}_\mu, B_\mu$ with trivial BRST transformations

$$\begin{aligned} \delta_B \bar{c}_\mu &= \Lambda B_\mu \\ \delta_B B_\mu &= 0 \ . \end{aligned} \tag{74}$$

A convenient gauge choice uses the functions

$$f^\mu = \partial_\nu(\sqrt{g} g^{\nu\mu}) \tag{75}$$

which fix the harmonic gauge (de Donder gauge) when set to zero. We use it in the gauge fermion

$$\Psi = \bar{c}_\mu(\partial_\nu(\sqrt{g} g^{\nu\mu}) + \alpha g^{\mu\nu} B_\nu) \tag{76}$$

which has the property of identifying a particularly simple graviton propagator for $\alpha = \frac{1}{2}$.

At this stage we have constructed the gauge-fixed action for pure gravity, given by

$$S_{tot} = S_{EH}[g] + s \int d^4x \, \Psi \tag{77}$$

which is manifestly BRST invariant. It is the starting point for the perturbative treatment of quantum gravity.

4.2 Graviton propagator in flat space

The gauge-fixed gravitational action (77) is algebraically quite complex, and perturbative calculation in quantum gravity are notoriously difficult.

Let us work out some details keeping only the quadratic approximation in $h_{\mu\nu}$, that leads to the perturbative propagator of the graviton (and ghosts). Expanding the Einstein-Hilbert action (59) to quadratic order in $h_{\mu\nu}$ one finds[8]

$$S_2[h] = \int d^4x \, \frac{1}{4}\Big\{ h^{\mu\nu}\partial^2 h_{\mu\nu} - \frac{1}{2}\, h\partial^2 h + 2\Big(\partial^\nu h_{\nu\mu} - \frac{1}{2}\partial_\mu h\Big)^2 \Big\} \tag{78}$$

with $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. Raising/lowering of indices is done with the flat metric $\eta_{\mu\nu}$. This quadratic action has a gauge symmetry under the linearized version of (58), that at lowest order reads

$$\delta h_{\mu\nu}(x) = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{79}$$

with $\xi_\mu = \eta_{\mu\nu}\xi^\nu$. The gauge fixing function (75) linearizes to the condition

$$f^\mu = -\Big(\partial_\nu h^{\nu\mu} - \frac{1}{2}\partial^\mu h\Big) \tag{80}$$

which inserted in the gauge fermion

$$\Psi = \bar{c}_\mu (f^\mu + \alpha B^\mu) \tag{81}$$

gives a BRST variation

$$s\Psi = B_\mu (f^\mu + \alpha B^\mu) + \bar{c}_\mu \partial^2 c^\mu \sim -\frac{1}{4\alpha} f^2 + \bar{c}_\mu \partial^2 c^\mu \tag{82}$$

where in the last expression the B_μ equations of motion have been used. Setting $\alpha = \frac{1}{2}$ gives the simplest form of the total gauge-fixed action

$$S_{2,tot} = S_2[h] + s \int d^4x \, \Psi = \int d^4x \, \Big\{ \frac{1}{4} h^{\mu\nu}\partial^2 h_{\mu\nu} - \frac{1}{8}\, h\partial^2 h + \bar{c}_\mu \partial^2 c^\mu \Big\} \,. \tag{83}$$

It is now a simple matter to get the propagators. The quadratic lagrangian for $h_{\mu\nu}$ is cast as

$$\mathcal{L}_h = \frac{1}{2} h_{\mu\nu} \tilde{P}^{\mu\nu,\alpha\beta} \partial^2 h_{\alpha\beta} \tag{84}$$

in terms of the tensor

$$\tilde{P}^{\mu\nu,\alpha\beta} = \frac{1}{4}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha}) - \frac{1}{4}\eta^{\mu\nu}\eta^{\alpha\beta} \tag{85}$$

which in arbitrary D dimensions has an inverse given by

$$P_{\mu\nu,\alpha\beta} = \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\alpha\beta} \,, \qquad \tilde{P}^{\mu\nu,\alpha\beta} P_{\alpha\beta,\rho\sigma} = \frac{1}{2}(\delta^\mu_\rho \delta^\nu_\sigma + \delta^\nu_\rho \delta^\mu_\sigma) \,. \tag{86}$$

In $D = 4$ one finds

$$\begin{aligned} \langle h_{\mu\nu}(x) h_{\alpha\beta}(y) \rangle &= \int \frac{d^4p}{i(2\pi)^4} \, e^{ip(x-y)} \, \frac{P_{\mu\nu,\alpha\beta}}{p^2 - i\epsilon} \\ \langle c_\mu(x) \bar{c}_\nu(y) \rangle &= \int \frac{d^4p}{i(2\pi)^4} \, e^{ip(x-y)} \, \frac{\eta_{\mu\nu}}{p^2 - i\epsilon} \end{aligned} \tag{87}$$

with $P_{\mu\nu,\alpha\beta} = \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}$. The derivation is valid in arbitrary dimensions, with obvious extentions and with $P_{\mu\nu,\alpha\beta}$ as in (86).

[8]Details of the calculation may be found in appendix A.

5 Cohomology

The power of the BRST symmetry is due to its nilpotency, that allows to define the concept of cohomology. The cohomology emerges as a vector space made up of elements that are equivalence classes. Different cohomology classes are identified with different physical observables, i.e. gauge invariant observables. For that purpose let us make an aside and review the concept of cohomology.

Let us consider a vector space V and a linear operator $\delta : V \longrightarrow V$ such that $\delta^2 = 0$. Such an operator is called *nilpotent*. One defines the kernel of δ, denoted by $\mathrm{Ker}(\delta)$, as all elements $\alpha \in V$ such that $\delta\alpha = 0$

$$\mathrm{Ker}(\delta) = \{\alpha \in V \mid \delta\alpha = 0\} \ . \tag{88}$$

Its elements are vectors that are said to be *closed*, and often called "cocycles". Then, one defines the image of δ, denoted by $\mathrm{Im}(\delta)$, as all elements $\beta \in V$ such that there exists an element $\gamma \in V$ for which $\beta = \delta\gamma$

$$\mathrm{Im}(\delta) = \{\beta \in V \mid \exists\gamma \in V \text{ for which } \beta = \delta\gamma\} \ . \tag{89}$$

Its elements are vectors that are said to be *exact*, and often called "coboundaries". Clearly, all exact elements are closed, $\mathrm{Im}(\delta) \subset \mathrm{Ker}(\delta)$, because of nilpotency. However, not all closed elements may be exact. The cohomology measures the amount of non-exactness of closed elements. It is defined as the set of equivalence classes $[\alpha]$ of closed elements that differ by exact elements

$$\alpha \sim \alpha' \qquad \text{if} \qquad \alpha' = \alpha + \delta\gamma \ . \tag{90}$$

The space of equivalent classes is denoted by

$$H(\delta) = \frac{\mathrm{Ker}(\delta)}{\mathrm{Im}(\delta)} \tag{91}$$

and is called the group of cohomology, or simply cohomology, see Figure 3.

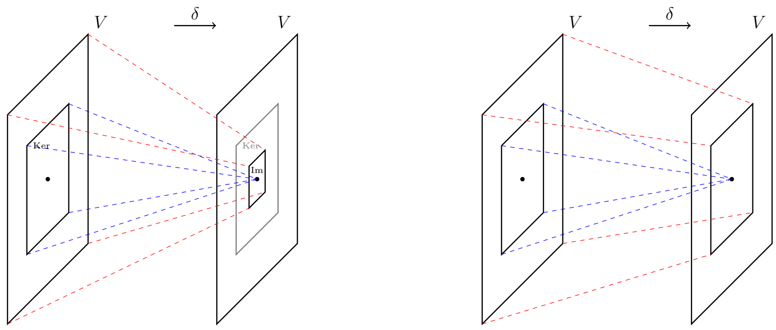

Figure 3: The cohomology measures the amount of non-exactness of closed elements. It is nontrivial when $\mathrm{Im}(\delta) \subset \mathrm{Ker}(\delta)$. The right-hand side depicts the case of vanishing cohomology, $\mathrm{Im}(\delta) = \mathrm{Ker}(\delta)$, where all closed elements are exact.

Returning to the BRST construction discussed previously, the Slavnov operator s is a nilpotent operator. It acts on functional of the fields, including ghosts and non-minimal fields. Physical observables must be BRST invariant quantities, i.e. annihilated by the operator s that defines the infinitesimal BRST transformations (the BRST symmetry takes the role of gauge invariance in the gauge fixed theory). Physical observables that differ by the BRST variation of some other quantity must be identified as equivalent: thus physical observables are defined by the cohomology classes of s. For example, the action is a nontrivial cohomological element at vanishing ghost number.

Similarly, in canonical quantization, the BRST charge Q becomes an operator $\hat{Q}$. It is the Noether charge associated to the BRST symmetry, that becomes an operator upon canonical quantization. It has the properties of being hermitian, with ghost number one, and nilpotent $\hat{Q}^2 = 0$. As all charges, like the hamiltonian, it has a dual role:
i) it is a conserved quantity due to a Lie symmetry of the system,
ii) it is the generator of that Lie symmetry on the system under study.

Physical states are defined by the cohomology of $\hat{Q}$ on the full BRST Hilbert space at vanishing ghost number. That is, physical states are given by vectors of the Hilbert space at vanishing ghost number that are BRST invariant, and thus satisfy $\hat{Q}|\psi_{ph}\rangle = 0$. In addition, physical states are equivalent if they differ by the BRST variation of another state: $|\psi_{ph}\rangle$ and $|\psi'_{ph}\rangle$ are equivalent if they are related by $|\psi'_{ph}\rangle = |\psi_{ph}\rangle + \hat{Q}|\chi\rangle$ for some $|\chi\rangle$. Similarly, *physical operators* are defined by cohomology classes: they are BRST invariant operators, meaning that they commute in a graded sense with the BRST charge $\hat{Q}$, i.e. $[\hat{Q}, \hat{A}_{ph}\} = 0$, and with an equivalence relation given by $\hat{A}_{ph} \sim \hat{A}'_{ph} = \hat{A}_{ph} + [\hat{Q}, \hat{B}\}$ for some operator $\hat{B}$.

One may check that matrix elements of physical operators between physical states do not depend on the representative chosen in the respective classes of equivalence, namely

$$\langle\psi_{ph}|\hat{A}_{ph}|\phi_{ph}\rangle = \langle\psi'_{ph}|\hat{A}'_{ph}|\phi'_{ph}\rangle \ . \tag{92}$$

For example, if $|\phi'_{ph}\rangle = |\phi_{ph}\rangle + \hat{Q}|\chi\rangle$, then

$$\langle\psi_{ph}|\hat{A}_{ph}|\phi'_{ph}\rangle = \langle\psi_{ph}|\hat{A}_{ph}|\phi_{ph}\rangle + \langle\psi_{ph}|\hat{A}_{ph}\hat{Q}|\chi\rangle \tag{93}$$

but the last term vanishes as $\langle\psi_{ph}|$ and $\hat{A}_{ph}$ are physical, so that

$$\langle\psi_{ph}|\hat{A}_{ph}\hat{Q}|\chi\rangle = \langle\psi_{ph}|[\hat{A}_{ph}, \hat{Q}]|\chi\rangle + \langle\psi_{ph}|\hat{Q}\hat{A}_{ph}|\chi\rangle = 0 \tag{94}$$

where we have taken $\hat{A}_{ph}$ to be bosonic for simplicity. Thus

$$\langle\psi_{ph}|\hat{A}_{ph}|\phi'_{ph}\rangle = \langle\psi_{ph}|\hat{A}_{ph}|\phi_{ph}\rangle \ . \tag{95}$$

Note that the full BRST Hilbert space cannot have a positive norm. For example, exact states like $\hat{Q}|\chi\rangle$ have vanishing norm, as

$$|\hat{Q}|\chi\rangle|^2 = \langle\chi|\hat{Q}\hat{Q}|\chi\rangle = 0 \tag{96}$$

where we have used the hermiticity of $\hat{Q}$ and its nilpotence. However, it is only important that the norm in the physical sector of the Hilbert space be positive definite.

For further details, please refer to chapter 74 of Srednicki.

6 Batalin-Vilkovisky and the antibracket

The method of Batalin-Vilkowsky generalizes the BRST scheme by adding to the action external sources that couple to the BRST variation of the fields. This set-up was originally considered by Zinn-Justin to derive useful Ward identities and study the renormalizability of gauge theories.

To introduce this method, let us consider the case of the $SU(N)$ Yang-Mills gauge-fixed theory. For each field $\phi^A(x)$ one introduces a source $\phi^*_A(x)$ of opposite Grassmann character, and adds to the lagrangian the term $\phi^*_A s\phi^A$, where $s\phi^A$ is the Slavnov variation of the field ϕ^A, and thus records its BRST transformation rules. The sources ϕ^*_A are called antifields, and are not to be path integrated over. They are just external sources for inserting the BRST variation of fields inside correlation function, and they may happens to be also nonlinear in the fields ("composite operators"), as in the case of the ghost c^a. It is actually convenient to multiply the source term with a suitable sign $\phi^*_A s\phi^A(-1)^{1+A}$, where A in the exponent is $+/-$ depending on the field ϕ^A being Grassmann even/odd, respectively. This to maintain conventions used so far for the ghost field. For the YM theory with gauge fixed action in (37) this construction reads

$$\begin{aligned} S[\phi^A,\phi^*_A] = \int d^4x \bigg(& -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \partial^\mu \bar{c}^a D_\mu c^a - B^a\partial^\mu A^a_\mu + \frac{\xi}{2}(B^a)^2 \\ & - A^{*\mu}_a D_\mu c^a - c^*_a \frac{g}{2} f^{abc}c^b c^c + \bar{c}^*_a B^a \bigg) \end{aligned} \tag{97}$$

and the BRST invariance is written as

$$\begin{aligned} & \int d^4x \frac{\delta_R S}{\delta\phi^A(x)}\delta_B\phi^A(x) = \int d^4x \frac{\delta_R S}{\delta\phi^A(x)}\Lambda s\phi^A(x) \\ & = \int d^4x \frac{\delta_R S}{\delta\phi^A(x)} s\phi^A(x)(-1)^{1+A}\Lambda = \int d^4x \frac{\delta_R S}{\delta\phi^A(x)}\frac{\delta_L S}{\delta\phi^*_A(x)}\Lambda = 0\ . \end{aligned} \tag{98}$$

This can be written more compactly defining the *antibracket*, that for any two functionals of $\phi^A(x)$ and $\phi^*_A(x)$ is defined by

$$(F,G) = \int d^4x \left(\frac{\delta_R F}{\delta\phi^A(x)}\frac{\delta_L G}{\delta\phi^*_A(x)} - \frac{\delta_R F}{\delta\phi^*_A(x)}\frac{\delta_L G}{\delta\phi^A(x)} \right)\ . \tag{99}$$

Dropping the anticommuting Λ, eq. (98) takes the form

$$(S,S) = 0\ . \tag{100}$$

A similar equation holds for the effective action

$$(\Gamma,\Gamma) = 0 \tag{101}$$

which constitute a set of Ward identities that are used to study the renomalizability of gauge theories through cohomological methods. We will not describe this topic in these lectures.

Batalin-Vilkovisky (BV) generalized this construction to present a method for gauge-fixing any gauge theory. Using an hypercondensed notation, it goes as follows. They started with the original action $S_{cl}[\phi^i]$ with gauge symmetries $\delta\phi^i = R^i_\alpha \xi^\alpha$, and introduced the ghosts c^α associated with the gauge symmetries by letting the local parameter turn into the ghosts $\xi^\alpha = c^\alpha\Lambda$ (note the new definition for factorizing out Λ, which redefines some fields by a sign with respect to the previous treatment). Then, having defined a set of minimal fields ϕ^A, as given by the original fields plus the ghosts $\phi^A = (\phi^i, c^\alpha)$, BV required the action to be a proper solution of the master equation

$$(S,S) = 0 \tag{102}$$

with boundary conditions

$$\begin{aligned} & S|_{\phi^*_A=0} = S_{cl}[\phi^i] \\ & \frac{\partial_L S}{\partial\phi^*_i}|_{\phi^*_A=0} = R^i_\alpha c^\alpha \end{aligned} \tag{103}$$

where “proper” means that all gauge symmetries are taken care of in terms of BRST variations coupled to the antifields, i.e. they are all included in the second equation above. The action with the antifields that solves the master equation has a double role:
i) it gives the action (with sources),
ii) it acts as generator of BRST transformations through the antibrackets.
The BRST rules on the fields are then

$$\delta_B \phi^A = (\phi^A, S)\Lambda \ . \tag{104}$$

The master equation amounts to require the BRST invariance of the action S, with S acting also as generator of the BRST transformation through antibrackets.

For gauge algebras that close off-shell, and with constant structure functions, it amounts to find the BRST rule of the ghost. For YM the action satisfying this master equation reads

$$S[A, c, A^*, c^*] = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - A^{*a\mu} D_\mu c_a - c^*_a \frac{g}{2} f^{abc} c^b c^c \right) \ . \tag{105}$$

Technically, the solution must be “proper”, meaning that all the gauge symmetries must be included together with their own ghosts. In general, the proper solution contains all information about the gauge symmetries, their Jacobi identities, the generalization of the Jacobi identities appearing when the structure constants become functions and/or the case of gauge symmetries that close only on-shell.

At this stage this solution is not enough to perform a gauge-fixing, so one adds to the model non-minimal fields with trivial BRST transformation rules, and of course their respective antifields. For our case these new fields are $(\bar{c}^a, B^a)$ and $(\bar{c}^*_a, B^*_a)$. The extended action

$$S[\phi^A, \phi^*_A] = \int d^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - A^{*a\mu} D_\mu c_a - c^*_a \frac{g}{2} f^{abc} c^b c^c + \bar{c}^*_a B^a \right) \tag{106}$$

still solves the master equation. The non-minimal fields do not modify the cohomology, and the physical observables remain the same.

Now the gauge-fixing is achieved by selecting a gauge-fermion functional Ψ, and computing the total gauge fixed action by setting the antifields equal to the variation of the gauge fermion under the corresponding fields, i.e.

$$S_{tot}[\phi^A] = S\Big[\phi^A, \phi^*_A = \frac{\delta\Psi}{\delta\phi^A}\Big] \ . \tag{107}$$

Indeed, using as gauge fermion

$$\Psi = -\int d^4x \, \bar{c}^a \left(\partial^\mu A^a_\mu - \frac{\xi}{2} B^a \right) \tag{108}$$

one reobtains (37). Note that, for an anticommunting object like Ψ, left and right derivatives coincide, $\frac{\delta_L \Psi}{\delta\phi^A} = \frac{\delta_R \Psi}{\delta\phi^A}$.

The fact that the action with antifields satisfies the master equation (102) guarantees that physical observables do not depend on the choice of Ψ, even though many choices are of no practical use, as for example $\Psi = 0$. Formally, the path integral

$$Z = \int D\phi \, e^{iS[\phi^A, \phi^*_A = \delta\Psi/\delta\phi^A]} \tag{109}$$

can be shown to be independent of the gauge fermion Ψ.

A derivation of the master equation

Let us consider the path integral in (109) and check under which condition it is really independent on the gauge fermion $\Psi = \Psi(\phi)$, a function of the fields only. Of course, for $\Psi = 0$ the path integral is ill defined, but for suitable choices of Ψ this singular point might be resolved, making the path integral well-defined.

So we calculate its variation under a change $\delta\Psi$ of Ψ, and in hypercondensed notation we get [9]

$$\begin{aligned}
\delta Z &= \int D\phi \; \frac{\partial \delta\Psi}{\partial \phi^A}\Big(\frac{\partial_L}{\partial \phi^*_A} e^{iS[\phi,\,\phi*]}\Big)\Big|_{\phi^*=\partial\Psi/\partial\phi} \\
&= \int D\phi \; \frac{\partial_R \delta\Psi}{\partial \phi^A}\Big(\frac{\partial_L}{\partial \phi^*_A} e^{iS[\phi,\,\phi*]}\Big)\Big|_{\phi^*=\partial\Psi/\partial\phi} \\
&= -\int D\phi \; \delta\Psi \frac{\partial_R}{\partial \phi^A}\Big(\frac{\partial_L}{\partial \phi^*_A} e^{iS[\phi,\,\phi*]}\Big)\Big|_{\phi^*=\delta\Psi/\delta\phi} \\
&= -\int D\phi \; \delta\Psi \Big(\frac{\partial_R}{\partial \phi^A}\frac{\partial_L}{\partial \phi^*_A} e^{iS[\phi,\,\phi*]} + \frac{\partial_R}{\partial \phi^*_B}\frac{\partial_L}{\partial \phi^*_A} e^{iS[\phi,\,\phi*]}\frac{\partial_R}{\partial \phi^A}\frac{\partial_L \Psi}{\partial \phi^B}\Big)\Big|_{\phi^*=\delta\Psi/\delta\phi} \\
&= -\int D\phi \; \delta\Psi \frac{\partial_R}{\partial \phi^A}\frac{\partial_L}{\partial \phi^*_A} e^{iS[\phi,\,\phi*]}
\end{aligned} \tag{110}$$

which vanishes if

$$\frac{\partial_R}{\partial \phi^A}\frac{\partial_L}{\partial \phi^*_A} e^{iS[\phi,\,\phi*]} = 0 \; . \tag{111}$$

In the derivation we have formally integrated by parts in the path integral, and noted that the second term in the last-but-one line vanishes in the sum over indices.

The vanishing of (111) amounts to requiring the "quantum master equation"

$$(S, S) = 2i\Delta S \tag{112}$$

where

$$\Delta S \equiv \frac{\partial_R}{\partial \phi^A}\frac{\partial_L}{\partial \phi^*_A} S \tag{113}$$

This last term can often be set to vanish (if there are no anomalies in the BRST symmetry), and the condition reduces to the "classical master equation"

$$(S, S) = 0 \tag{114}$$

that we have already discussed.

Solution with an open gauge algebra

Consider an action $S[\phi]$ where all dynamical fields are collectively denoted by ϕ^i. We employ an hypercondensed notation where the index i stands for all possible indices which the fields may depend on, including the spacetime position. The equations of motion can be written as

$$S_{,i} \equiv \frac{\partial_R S}{\partial \phi^i} = 0 \tag{115}$$

[9] Note that left and right derivatives of on an anticommuting quantity Ψ coincide.

and we assume that there are gauge symmetries that we write as

$$\delta\phi^i = R^i_\alpha \xi^\alpha \tag{116}$$

where ξ^α are infinitesimal arbitrary parameters and the quantity R^i_α is in general field dependent. The fact that is a symmetry means that

$$\delta S = S_{,i} R^i_\alpha \xi^\alpha = 0 \tag{117}$$

and it is a gauge symmetry if the index α includes the time coordinate (or spacetime coordinates for relativistic theories) and the parameters ξ^α depending arbitrarily on it.

Let us now comment study the algebraic structure of the symmetries. The product of two transformations of the type (116) will still leave the action invariant, and it is natural to consider the commutator algebra (the Lie algebra) We shall assume, without loss of generality, that the set of symmetries is complete, i.e. all independent symmetries are taken into account by the general formula in (116). We will also assume that these symmetries are linearly independent (if this is not the case the algebra is called *reducible*). The commutator algebra will then be of the general form

$$[\delta(\xi_1), \delta(\xi_2)]\phi^i = (R^i_{\alpha,j} R^j_\beta - (-)^{\xi^\alpha \xi^\beta} R^i_{\beta,j} R^j_\alpha)\xi^\beta_1 \xi^\alpha_2 = R^i_\gamma f^\gamma_{\alpha\beta} \xi^\beta_1 \xi^\alpha_2 + S_{,j} E^{ji}_{\alpha\beta} \xi^\beta_1 \xi^\alpha_2 \,. \tag{118}$$

The last step follows since, by assumption, the original set of symmetries is complete and the most general thing that can happen is that the algebra closes modulo on-shell trivial symmetries (these on-shell trivial symmetries, which take the general form $\delta\phi^i = S_{,j} E^{ji}$ with E^{ji} graded antisymmetric but otherwise arbitrary, are present in any field theory; they vanish on shell and do not imply the absence of degrees of freedom from the theory). The structure functions $f^\gamma_{\alpha\beta}$ and the coefficients $E^{ij}_{\alpha\beta}$, graded antisymmetric in α, β and i, j, characterize the classical symmetry algebra. If the coefficients $E^{ij}_{\alpha\beta}$ are non-zero, one speaks of an *open* algebra. By using ghosts fields c^α, fields of opposite statistics than the symmetry parameters ξ^α, the relation (118) can be cast in the following way

$$(-)^{\xi^\alpha}(2R^i_{\alpha,j} R^j_\beta - R^i_\gamma f^\gamma_{\alpha\beta} - S_{,j} E^{ji}_{\alpha\beta}) c^\beta c^\alpha = 0 \tag{119}$$

Additional relations and higher order structure functions are obtained by considering Jacobi identities. They will be taken care of by the the antibracket formalism of Batalin-Vilkovisky. For the fields $\phi^A = (\phi^i, c^\alpha)$ one introduces the antifields $\phi^*_A = (\phi^*_i, c^*_\alpha)$ and one ask to satisfy the classical master equation

$$(S, S) = 0 \tag{120}$$

with boundary condition

$$S|_{\phi^*_A=0} = S_{cl}[\phi^i] \,, \qquad \left.\frac{\partial_L S}{\partial \phi^*_i}\right|_{\phi^*_A=0} = R^i_\alpha c^\alpha \tag{121}$$

where we denoted by S_{cl} the original action.

To see how the gauge algebra is cast into $S[\phi^A, \phi^*_A]$, let's consider the simple case in which higher order structure functions do not arise in the Jacobi identities. One expand S into powers of antifields

$$S = S_0 + S_1 + S_2 + \dots \tag{122}$$

and the master equation (120) splits into several pieces

$$(S_0, S_0) = 0 \tag{123}$$
$$(S_0, S_1) = 0 \tag{124}$$
$$(S_1, S_1) + 2(S_0, S_2) = 0 \tag{125}$$
$$(S_0, S_3) + (S_1, S_2) = 0 \tag{126}$$
$$\cdots\cdots\cdots\cdots \tag{127}$$

Employing the boundary conditions (121), one sees that eq.(123) is trivially satisfied since S_0 does not depend on antifields, eq.(124) is the statement about the invariance of S_{cl}, eq.(125) describe the algebraic closure given in (119), and eq.(126) is automatically satisfied because we restricted ourselves to the situation in which the higher order structure functions vanish. In this case, the general solution of the master equation reads

$$S = S_{cl} + \phi_i^* R_\alpha^i c^\alpha + \frac{1}{2} c_\gamma^* (-)^{\xi^\alpha} f_{\alpha\beta}^\gamma c^\beta c^\alpha - \frac{1}{4} \phi_i^* \phi_j^* (-)^{\xi^\alpha + \phi^i} E_{\alpha\beta}^{ji} c^\beta c^\alpha \; . \tag{128}$$

A Quadratic approximation of Einstein-Hilbert action

Some of the details are as follows: the metric is expanded as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{129}$$

where now we absorb the coupling constant κ in $h_{\mu\nu}$. Then, at linear order

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x) \ , \quad g = |\det g_{\mu\nu}| = 1 + h \ , \quad \sqrt{g} = 1 + \frac{1}{2}h \tag{130}$$

with indices raised/lowered with the flat metric $\eta_{\mu\nu}$. The Christoffel symbols linearize as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) \ , \tag{131}$$

the Riemann tensor as

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + ... = \frac{1}{2}\partial_\sigma(\partial_\mu h_\nu{}^\rho - \partial_\nu h_\mu{}^\rho) - \frac{1}{2}\partial^\rho(\partial_\mu h_{\nu\sigma} - \partial_\nu h_{\mu\sigma}) \ , \tag{132}$$

and the Ricci tensor as

$$R_{\nu\sigma} = R_{\mu\nu}{}^{\mu}{}_{\sigma} = \frac{1}{2}(\partial_\nu \partial^\mu h_{\sigma\mu} + \partial_\sigma \partial^\mu h_{\nu\mu} - \partial_\nu \partial_\sigma h - \partial^2 h_{\nu\sigma}) \ . \tag{133}$$

To get the quadratic approximation one needs to keep at least a linear order in the variation of the $\sqrt{g}g^{\mu\nu}$ part of the Einstein-Hilbert action, as at the quadratic level the Ricci tensor will not contribute. This is seen recalling that in a first order formalism, the action depends on the metric and Christoffel symbols independently

$$S_{EH}[g,\Gamma] = \frac{1}{2\kappa^2}\int d^4x \sqrt{g} g^{\mu\nu} R_{\mu\nu}(\Gamma) \tag{134}$$

The equation of motions of $g_{\mu\nu}$ give

$$R_{\mu\nu}(\Gamma) - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}R_{\alpha\beta}(\Gamma) = 0 \tag{135}$$

while the equations of motion from varying $\Gamma^{\rho}_{\mu\nu}$ give algebraic equations whose solutions are precisely the ones defining the usual Christoffel symbols in terms of the metric. This solution can be substituted back into the action and in (135). The latter giving the Einstein equation in its second order form

$$R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g) = 0 \tag{136}$$

The latter could be obtained as well from the action in the second order form, varying only the $\sqrt{g}g^{\mu\nu}$ part, the remaining $g_{\mu\nu}$ dependence does not need to be varied as that variation automatically vanishes (schematically $\frac{\delta R_{\mu\nu}}{\delta g} = \frac{\delta R_{\mu\nu}}{\delta \Gamma}\frac{\delta \Gamma}{\delta g}$, but $\frac{\delta R_{\mu\nu}}{\delta \Gamma}$ vanish so does $\frac{\delta R_{\mu\nu}}{\delta g}$). Thus, at the linear order the variation of the Einstein-Hilbert action in the second order formulation is written as

$$\delta S_{EH}[g] = \frac{1}{2\kappa^2}\int d^4x \sqrt{g}\, \delta g^{\mu\nu}\Big(R_{\mu\nu}(g) - \frac{1}{2}g_{\mu\nu}R(g)\Big) \ . \tag{137}$$

For a second variation, needed to identify the quadratic approximation, only the linear variations of R and $R_{\mu\nu}$ are needed, as all other term would vanish when setting $g_{\mu\nu}(x) = \eta_{\mu\nu}$.

Therefore, at the quadratic order we keep only the terms that will contribute, i.e.

$$S_2 = \frac{1}{2\kappa^2}\int d^D x \left(1+\frac{1}{2}h\right)(\eta^{\nu\sigma} - h^{\nu\sigma})\,\frac{1}{2}(\partial_\nu\partial^\mu h_{\sigma\mu} + \partial_\sigma\partial^\mu h_{\nu\mu} - \partial_\nu\partial_\sigma h - \partial^2 h_{\nu\sigma}) \tag{138}$$

leading to (after some integration by parts to collect similar terms)

$$S_2 = \frac{1}{4\kappa^2}\int d^D x\,(h^{\mu\nu}\partial^2 h_{\mu\nu} - h\partial^2 h + 2h\partial^\mu\partial^\nu h_{\mu\nu} + 2(\partial^\mu h_{\mu\nu})^2). \tag{139}$$

Finally, redefining $h_{\mu\nu} \to \kappa h_{\mu\nu}$, using some further integration by parts and grouping terms, one obtains (78).