QCD, background field method, and anomalies

(Lecture notes - a.a. 2020/21)

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1 Feynman rules for perturbative QCD

Having obtained the gauge-fixed action for the non-abelian gauge theories, one can write down the QCD gauge-fixed lagrangian by adding the fermions (the quark fields). Using an arbitrary R_ξ, gauge one obtains the following gauge-fixed lagrangian

$$\mathcal{L}_{tot} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - B^a\partial^\mu A^a_\mu + \frac{\xi}{2}(B^a)^2 - \partial^\mu\bar{c}^a D_\mu c^a - \sum_{f=1}^{6}\overline{\psi}_f(\gamma^\mu D_\mu + m_f)\psi_f \ . \tag{1}$$

Eliminating the auxiliary field B^a one finds

$$\boxed{\mathcal{L}_{tot} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 - \partial^\mu\bar{c}^a D_\mu c^a - \sum_{f=1}^{6}\overline{\psi}_f(\gamma^\mu D_\mu + m_f)\psi_f} \ . \tag{2}$$

This lagrangian can be split as $\mathcal{L}_{tot} = \mathcal{L}_2 + \mathcal{L}_{int}$, with $\mathcal{L}_2$ the quadratic part used to find the propagators, and $\mathcal{L}_{int}$ the interacting part used to get the vertices. Recalling that

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \tag{3}$$

one finds

$$\mathcal{L}_2 = -\frac{1}{2}(\partial_\mu A^a_\nu)^2 + (\frac{1}{2} - \frac{1}{2\xi})(\partial^\mu A^a_\mu)^2 - \partial^\mu\bar{c}^a\partial_\mu c^a - \sum_{f=1}^{6}\overline{\psi}_f(\gamma^\mu\partial_\mu + m_f)\psi_f \tag{4}$$

and

$$\begin{aligned}\mathcal{L}_{int} = & - gf^{abc}\partial^\mu A^{\nu a}A^b_\mu A^c_\nu - \frac{g^2}{4}f^{abc}f^{ade}A^b_\mu A^c_\nu A^{\mu d}A^{\nu e} \\ & - gf^{abc}\partial^\mu\bar{c}^a A^b_\mu c^c + igA^a_\mu\sum_{f=1}^{6}\overline{\psi}_f\gamma^\mu T^a\psi_f \ .\end{aligned} \tag{5}$$

Propagators

It is immediate to extract the propagators from (4). We recall that the perturbative propagators are the inverse of the kinetic terms: in an hypercondensed notation (for real fields ϕ^i and complex fields $\psi^i, \overline{\psi}_i$) we have the formulae

$$\begin{aligned} S[\phi] = -\frac{1}{2}\phi^i K_{ij}\phi^j \qquad &\to \quad \langle\phi^i\phi^j\rangle = -iG^{ij} \qquad (K_{ij}G^{jk} = \delta_i^k) \\ S[\psi,\bar\psi] = -\bar\psi_i K^i{}_j\psi^j \qquad &\to \quad \langle\psi^i\bar\psi_j\rangle = -iG^i{}_j \qquad (K^i{}_j G^j{}_k = \delta^i_k)\end{aligned} \tag{6}$$

valid for either commuting or anticommuting fields.

Thus, from $\mathcal{L}_2$ we get

$$
\begin{aligned}
\langle A^a_\mu(x) A^b_\nu(y)\rangle &= -i \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{\delta^{ab}}{p^2 - i\epsilon} \left(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right) \\
\langle c^a(x) \bar{c}^b(y)\rangle &= -i \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x-y)}\, \frac{\delta^{ab}}{p^2 - i\epsilon} \\
\langle \psi_f(x) \overline{\psi}_f(y)\rangle &= -i \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot(x-y)}\, \frac{-i\not{p} + m_f}{p^2 + m_f^2 - i\epsilon}
\end{aligned}
\tag{7}
$$

with no sum over f (the quark flavours), from which one extracts the propagators in momentum space

$$- i \frac{\delta^{ab}}{p^2 - i\epsilon}(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}$$

$$- i \frac{\delta^{ab}}{p^2 - i\epsilon} \tag{8}$$

$$- i \frac{-i\not{p} + m_f}{p^2 + m_f^2 - i\epsilon}$$

Vertices

As for the vertices, one can obtain the elementary Feynman rules by Fourier transforming the various monomials contained in $\mathcal{L}_{int}$.

Let us review our conventions by presenting the definition of the Fourier transform of the effective action for a field φ (the quantum action in Srednicki), and the *proper vertices* it gives rise to (the amputated 1PI Feynman diagrams). The effective action $\Gamma[\varphi]$ is expanded in a Taylor series as

$$
\begin{aligned}
\Gamma[\varphi] &= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^Dx_1 \ldots d^Dx_n\, \Gamma(x_1, ..., x_n)\varphi(x_1)...\varphi(x_n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^Dp_1}{(2\pi)^D} \cdots \frac{d^Dp_n}{(2\pi)^D}\, \tilde{\Gamma}(p_1, ..., p_n)\tilde{\varphi}(p_1)...\tilde{\varphi}(p_n) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^Dp_1}{(2\pi)^D} \cdots \frac{d^Dp_n}{(2\pi)^D}\, (2\pi)^D\delta(p_1 + .. + p_n)\Gamma(p_1, .., p_n)\tilde{\varphi}(p_1)...\tilde{\varphi}(p_n)
\end{aligned}
\tag{9}
$$

where we have defined the Fourier transform of the fields with *outgoing momenta* as

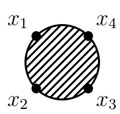
$$
\tilde{\varphi}(p) = \int d^Dx\, e^{-ipx}\varphi(x)\ , \qquad \varphi(x) = \int \frac{d^Dp}{(2\pi)^D}\, e^{ipx}\tilde{\varphi}(p) \tag{10}
$$

and the Fourier transform of the proper vertices with *ingoing momenta* as

$$
\begin{aligned}
\tilde{\Gamma}(p_1, .., p_n) &= \int d^Dx_1..d^Dx_n\, e^{ip_1x_1 + .. + ip_nx_n}\, \Gamma(x_1, .., x_n) \\
&= (2\pi)^D\delta(p_1 + .. + p_n)\Gamma(p_1, .., p_n)
\end{aligned}
\tag{11}
$$

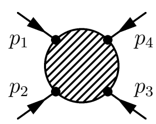
the second line following from momentum conservation (translational invariance of $\Gamma(x_1, .., x_n)$). Graphically, we denote these correlation functions by

$$\Gamma(x_1, .., x_4) =$$



;

$$\Gamma(p_1, .., p_4) = \tag{12}$$



where in the second graph conservation of momentum is understood, and the external lines indicate only the flow of momentum and do not include external propagators (the 1PI graphs are amputated).

Denoting $\Gamma^n = \Gamma^n(p_1, .., p_n)$, one finds that Γ^0 is the zero point function, $\Gamma^1(p)$ is the one-point function, often set to vanish by adjusting the vev of the quantum field $\varphi(p)$, $\Gamma^2(-p,p)$ is the effective kinetic term appearing in

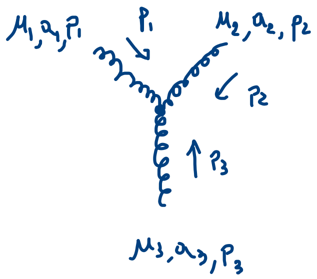
$$\begin{aligned}\Gamma^2(\varphi) &= -\frac{1}{2}\int \frac{d^D p_1}{(2\pi)^D}\frac{d^D p_2}{(2\pi)^D}\,(2\pi)^D\delta(p_1+p_2)\Gamma^2(p_1,p_2)\tilde{\varphi}(p_1)\tilde{\varphi}(p_2)\\ &= -\frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\,\tilde{\varphi}(-p)\Gamma^2(-p,p)\tilde{\varphi}(p)\\ &= -\frac{1}{2}\int \frac{d^D p}{(2\pi)^D}\,\tilde{\varphi}(-p)\left(p^2+m^2-\Pi(p^2)\right)\tilde{\varphi}(p)\end{aligned} \tag{13}$$

with $\Pi(p^2)$ the self-energy contribution: the remaining part in Γ^2 is the leading kinetic term, whose inverse (times $-i$) gives the perturbative propagator in momentum space $\frac{-i}{p^2+m^2}$. The functions Γ^n for $n \geq 3$ give the proper vertices.

From the above conventions it is clear that the propagator and the vertices are obtained by substituting the effective action $\Gamma[\varphi]$ the classical action $S[\varphi]$ suitably gauge fixed.

In particular, the trilinear gluon coupling in (5) give rises to the vertex $iV^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$

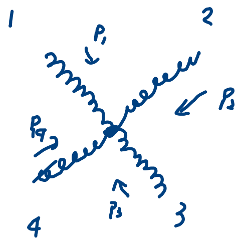
$$\begin{aligned}iV^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) &= gf^{a_1a_2a_3}p_{1,\mu_2}\eta_{\mu_1\mu_3} + \text{permutations of external lines}\\ &= -gf^{a_1a_2a_3}\Big((p_1-p_2)_{\mu_3}\eta_{\mu_1\mu_2} + (p_2-p_3)_{\mu_1}\eta_{\mu_2\mu_3} + (p_3-p_1)_{\mu_2}\eta_{\mu_3\mu_1}\Big)\end{aligned}$$

$=$  (14)

which is extracted from iS_{tot} (as $e^{iS_{tot}}$ enters the path integral) with $\Gamma \to S$ in the above formulae (i.e. the vertex iV^n corresponds to $i\Gamma^n$ for a vertex with n fields when the classical action S substitutes the effective action Γ in the Fourier transform formulae given above: the classical action gives the tree-level term of the effective action, which in addition contains also $\hbar$-corrections (i.e. the loop correction inlcuding the counterterms)).

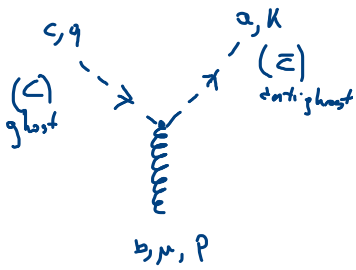
Similarly, we get the 4-gluon vertex

$$
\begin{aligned}
iV^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}(p_1,p_2,p_3,p_4) &= -ig^2 f^{ba_1a_2} f^{ba_3a_4}\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} + \text{permutations of lines 2 3 4} \\
&= -ig^2[f^{ba_1a_2} f^{ba_3a_4}(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} - \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}) \\
&\quad + f^{ba_1a_3} f^{ba_4a_2}(\eta_{\mu_1\mu_4}\eta_{\mu_3\mu_2} - \eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4}) \\
&\quad + f^{ba_1a_4} f^{ba_2a_3}(\eta_{\mu_1\mu_2}\eta_{\mu_4\mu_3} - \eta_{\mu_1\mu_3}\eta_{\mu_4\mu_2})] \\
&= \qquad (15)
\end{aligned}
$$



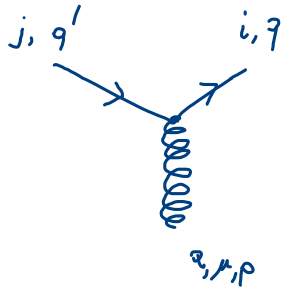
the ghost-antighost-gluon vertex

$$iV^{abc}_\mu(k,p,q) = gf^{abc}k_\mu \qquad (16)$$



and the quark-quark-gluon vertex (for a fixed flavour)

$$i(V^{\mu a})^i{}_j(q,p,q') = -g\gamma^\mu(T^a)^i{}_j \qquad (17)$$



These rules are somewhat more complex then those for QED, nevertheless now one can start computing perturbatively various scattering processes, renomalize the theory loopwise by introducing counterterms, and compute the beta function.

2 Perturbative calculation and the beta function

To renomalize the theory one must introduce the counterterms that will cancel the infinities and match the renomalization conditions needed to define the theory in terms of a fixed number of observables.

Naively, looking at (4) and (5) one would expect at least nine Z factors (here we consider only one quark flavour)

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}Z_3[(\partial_\mu A^a_\nu)^2 - (\partial^\mu A^a_\mu)^2] - \frac{1}{2\xi}Z_\xi(\partial^\mu A^a_\mu)^2 - Z_{2'}\partial^\mu \bar{c}^a \partial_\mu c^a - Z_2 \overline{\psi}\gamma^\mu\partial_\mu\psi - Z_m m\overline{\psi}\psi \\ &- Z_{3g} f^{abc}\partial^\mu A^{\nu a} A^b_\mu A^c_\nu - Z_{4g}\frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} \\ &- Z_{1'} g f^{abc}\partial^\mu \bar{c}^a A^b_\mu c^c + i Z_1 g A^a_\mu \overline{\psi}\gamma^\mu T^a \psi \; .\end{aligned} \tag{18}$$

however, gauge invariance (or more properly BRST invariance) can be used to derive Ward identities (that in this particular context are known as Slavnov-Taylor identites), which show that the Z factors are not all independent. In particular, one finds that $Z_\xi = 1$ as in QED, and

$$g_0^2 = \frac{Z_1^2}{Z_2^2 Z_3} g^2 \tilde{\mu}^\epsilon = \frac{Z_{1'}^2}{Z_{2'}^2 Z_3} g^2 \tilde{\mu}^\epsilon = \frac{Z_{3g}^2}{Z_3^3} g^2 \tilde{\mu}^\epsilon = \frac{Z_{4g}}{Z_3^2} g^2 \tilde{\mu}^\epsilon \tag{19}$$

where $d = 4 - \epsilon$. As in the classical lagrangian, this means that there is only one independent coupling constant g. Thus, to compute the β-function one could start calculating Z_1, Z_2 and Z_3 at one-loop, and use the first relation in (19).

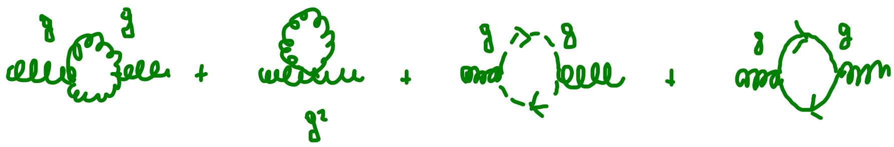
From the correction to the quark propagator one obtains Z_2 in the $\overline{\text{MS}}$ scheme

$$\rightarrow \qquad Z_2 = 1 - C(R)\frac{g^2}{8\pi^2}\frac{1}{\epsilon} + O(g^4) \; .$$

The study of the quark-quark-gluon vertex gives

$$\rightarrow \qquad Z_1 = 1 - [C(R) + T(A)]\frac{g^2}{8\pi^2}\frac{1}{\epsilon} + O(g^4) \; .$$

Finally, to renomalize the gluon wave function one must study the divergences in the gluon propagator



and a laborious calculation produces

$$Z_3 = 1 + \left[\frac{5}{3}T(A) - \frac{4}{3}n_F T(R)\right]\frac{g^2}{8\pi^2}\frac{1}{\epsilon} + O(g^4) \; .$$

Then, one has all the elements to compute the one-loop beta function

$$\beta(g) = -\left[\frac{11}{3}T(A) - \frac{4}{3}n_F T(R)\right]\frac{g^3}{16\pi^2} + O(g^5) \tag{20}$$

which for QCD (where $T(A) = 3$ and $T(R) = \frac{1}{2}$) shows asymptotic freedom for a number of quark flavours $n_F \leq 16$.

If interested, these one-loop calculations can be followed in detail in chapter 73 of Srednicki.

3 Ward identities

Ward identites arising from the BRST symmetry can be used to deduce several properties of the theory. One particular application is to show that scattering amplitudes are transversal in the polarization of the gluons. This means that if in a given total amplitude M we extract the physical polarization $\epsilon_\mu(p)$ of one gluon, say by setting $M = \epsilon_\mu(p)M^\mu$, then after substituting this physical polarization $\epsilon_\mu(p)$ with a longitudinal one, $\epsilon_\mu(p) \to p_\mu$, one finds a vanishing result if all the other states are physical (i.e. with on-shell momenta and physical polarizations).

Let us sketch how to prove this result using the BRST symmetry. In the LSZ formula for each asymptotic state corresponding to a gluon there appears in the correlation function the field $A_\mu(x)$, which: *i)* is acted upon by the corresponding free wave operator $-\partial^2$, *ii)* Fourier transformed with e^{ipx}, and *iii)* contracted with the physical polarization $\epsilon_\mu(p)$. Schematically,

$$a^\dagger(p)_{in} \;\to\; \epsilon^\mu(p) \int d^4x \; e^{ipx}(-\partial^2)A_\mu(x) \tag{21}$$

Now the substitution $\epsilon^\mu(p) \to p^\mu$ would give instead

$$p^\mu \int d^4x \; e^{ipx}(-\partial^2)A_\mu(x) = -i \int d^4x \; (\partial^\mu e^{ipx})(-\partial^2)A_\mu(x) = i \int d^4x \; e^{ipx}(-\partial^2)\partial^\mu A_\mu(x) \tag{22}$$

which means that in the correlation functions there appears the operator $\partial^\mu A_\mu(x)$. However, we know that the operator $\partial^\mu A_\mu(x)$ is the BRST variation of the antighost $\bar{c}$

$$\partial^\mu A_\mu(x) \sim \{Q_B, \bar{c}\} \tag{23}$$

and thus it is a cohomologically trivial operator that gives a vanishing result once inserted in correlation functions that include only physical operators and physical states (recall that in the LSZ formula one has a string of time-ordered physical operators, corresponding to the physical asymptotic states, sandwiched between the vacuum state, which is also physical).

4 Background field method

4.1 Background field method in a scalar theory

A useful technique for computing the effective action is the background field method. We present it first for a simple scalar theory, and then we extend it to gauge theories. The generating functionals for a field ϕ with action $S[\phi]$ are

$$\begin{aligned} Z[J] &= e^{iW[J]} = \int D\phi \; e^{iS[\phi]+iJ_i\phi^i} \\ \Gamma[\varphi] &= \min_J \left\{ W[J] - J_i\varphi^i \right\} . \end{aligned} \tag{24}$$

In the *background field method* one first split the variable ϕ as

$$\phi(x) = \varphi(x) + \tilde{\phi}(x) \tag{25}$$

where $\varphi(x)$ is taken as an arbitrary but fixed classical background, while $\tilde{\phi}(x)$ as the quantum field to be quantized (i.e. path integrated over). The classical background is just an inert spectator in the quantization process. Then one computes

$$\begin{aligned} Z_B[\tilde{J};\varphi] &= \mathrm{e}^{iW_B[\tilde{J};\varphi]} = \int D\tilde{\phi} \; \mathrm{e}^{iS[\tilde{\phi}+\varphi]+i\tilde{J}_i\tilde{\phi}^i} \\ \Gamma_B[\tilde{\varphi};\varphi] &= \max_J \left\{ W_B[\tilde{J};\varphi] - \tilde{J}_i\tilde{\varphi}^i \right\} . \end{aligned} \tag{26}$$

Changing path integration variables $\tilde{\phi} \to \phi = \tilde{\phi} + \varphi$ in eq. (26) one finds

$$Z_B[\tilde{J};\varphi] = Z[\tilde{J}]\, e^{-i\tilde{J}_i\varphi^i} \quad \Rightarrow \quad W_B[\tilde{J};\varphi] = W[\tilde{J}] - \tilde{J}_i\varphi^i \quad \Rightarrow \quad \Gamma_B[\tilde{\varphi};\varphi] = \Gamma[\tilde{\varphi}+\varphi] \tag{27}$$

so that

$$\Gamma[\varphi] = \Gamma_B[0;\varphi]\ . \tag{28}$$

Therefore, $\Gamma[\varphi]$ can be computed as the sum of 1PI vacuum diagrams in the presence of the background field φ see fig. 1 (see also exercise 21.3 of Srednicki).

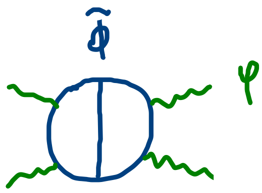


Figure 1: A graph contained in $\Gamma_B[0;\varphi]$ in the ϕ^3 theory.

4.2 Effective action, loop expansion, and 1PI diagrams

To clarify the meaning of the effective action, and reproduce the previous derivation from an alternative perspective, let us review the generating functionals in QFT. We now use the euclidean version of the QFT (this simplifies the correct insertions of the imaginary unit i, which are then reintroduced by the inverse Wick rotation).

The standard functionals used in euclidean QFT are defined as

$$\begin{aligned} Z[J] &= \mathrm{e}^{\frac{1}{\hbar}W[J]} = \int D\phi\ \mathrm{e}^{-\frac{1}{\hbar}S[\phi]+\frac{1}{\hbar}J_i\phi^i} \\ \Gamma[\varphi] &= J_i\varphi^i - W[J]\ , \qquad \varphi^i = \frac{\delta W[J]}{\delta J_i} \end{aligned} \tag{29}$$

with the effective action $\Gamma[\varphi]$ obtained by evaluating the right-hand side using $J_i = J_i(\varphi)$ that inverts the relation $\varphi^i(J) = \frac{\delta W[J]}{\delta J_i}$. We keep $\hbar$ that is going to be used as the loop counting parameter (a loop L gets a factor of $\hbar^{L-1}$). Also, one inverts this Legendre transform as

$$W[J] = J_i\varphi^i - \Gamma[\varphi]\ , \qquad J_i = \frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\ . \tag{30}$$

Using these relations one obtains an equation for the effective action $\Gamma[\varphi]$

$$\mathrm{e}^{-\frac{1}{\hbar}\left(\Gamma[\varphi]-\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\varphi^i\right)} = \int D\phi\ \mathrm{e}^{-\frac{1}{\hbar}\left(S[\phi]-\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\phi^i\right)} \tag{31}$$

that with a change of variables $\phi \to \phi + \varphi$ in the path integral becomes

$$\boxed{\mathrm{e}^{-\frac{1}{\hbar}\Gamma[\varphi]} = \int D\phi\ \mathrm{e}^{-\frac{1}{\hbar}\left(S[\phi+\varphi]-\frac{\delta\Gamma[\varphi]}{\delta\varphi^i}\phi^i\right)}}\ . \tag{32}$$

This is the basic equation that can be used to study the $\hbar$ expansion, i.e. the expansion in loops (the loops are counted by the parameter $\hbar$).

It is convenient to use a compact notation and expand the classical action in a Taylor series

$$S[\phi+\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} S_n[\varphi]\phi^n \tag{33}$$

where $S_n[\varphi] = \frac{\delta^n S[\varphi]}{\delta\varphi^n}$. Using a similar notation also for

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi} \equiv \Gamma_1[\varphi] \tag{34}$$

and rescaling $\phi \to \sqrt{\hbar}\,\phi$ one finds

$$\exp\left(-\frac{1}{\hbar}\Gamma[\varphi] + \frac{1}{\hbar}S[\varphi]\right) = \int D\phi \exp\left(-\frac{1}{2}S_2[\varphi]\phi^2 - \sum_{n=3}^{\infty} \frac{\hbar^{\frac{n}{2}-1}}{n!} S_n[\varphi]\phi^n + \frac{1}{\sqrt{\hbar}}(\Gamma_1[\varphi] - S_1[\varphi])\phi\right) \tag{35}$$

that depends only on $\bar{\Gamma}[\varphi] \equiv \Gamma[\varphi] - S[\varphi]$. Expanding in powers of $\hbar$

$$\bar{\Gamma}[\varphi] = \sum_{n=1}^{\infty} \hbar^n \bar{\Gamma}^{(n)}[\varphi] \tag{36}$$

one obtains the following master equation

$$\exp\left(-\sum_{n=1}^{\infty} \hbar^{n-1}\bar{\Gamma}^{(n)}[\varphi]\right) = \int D\phi \exp\left(-\frac{1}{2}S_2[\varphi]\phi^2 - \sum_{n=3}^{\infty} \frac{\hbar^{\frac{n}{2}-1}}{n!} S_n[\varphi]\phi^n + \sum_{n=1}^{\infty} \hbar^{n-\frac{1}{2}}\bar{\Gamma}_1^{(n)}[\varphi]\phi\right) .$$

This equation will be used to study the loop expansion of the effective action, obtained by matching the powers of $\hbar$ needed at each perturbative order. As we shall see

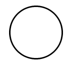
$$\exp\Big(\underbrace{-\sum_{n=1}^{\infty} \hbar^{n-1}\bar{\Gamma}^{(n)}[\varphi]}_{\text{1PI diagrams}}\Big) = \int D\phi\ \exp\Big(\underbrace{-\frac{1}{2}S_2[\varphi]\phi^2}_{\text{propagator}} \underbrace{- \sum_{n=3}^{\infty} \frac{\hbar^{\frac{n}{2}-1}}{n!} S_n[\varphi]\phi^n}_{\text{vertices}} + \underbrace{\sum_{n=1}^{\infty} \hbar^{n-\frac{1}{2}}\bar{\Gamma}_1^{(n)}[\varphi]\phi}_{\substack{\text{extra vertices that remove}\\ \text{diagrams that are not 1PI}}}\Big) .$$

Approximation at 1-loop ($n=1$)

From the master formula we get

$$e^{-\bar{\Gamma}^{(1)}[\varphi]} = \int D\phi\ e^{-\frac{1}{2}S_2[\varphi]\phi^2 + O(\hbar^{\frac{1}{2}})} = (\det S_2[\varphi])^{-\frac{1}{2}} = e^{-\frac{1}{2}\mathrm{tr}\ln \mathrm{S}_2[\varphi]} \tag{37}$$

so that

$$\Gamma^{(1)}[\varphi] = \frac{1}{2}\mathrm{tr}\ln \mathrm{S}_2[\varphi] \equiv \frac{1}{2}$$  (38)

where we have used the Feynman diagram usually adopted for this type of term. Thus, at one-loop the effective action is given by

$$\Gamma[\varphi] = S[\varphi] + \frac{\hbar}{2}\mathrm{tr}\ln S_2[\varphi] + O(\hbar^2) . \tag{39}$$

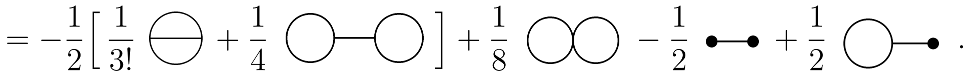
Approximation at 2 loops ($n=2$)

From the master formula one finds

$$
\begin{aligned}
\mathrm{e}^{-\bar\Gamma^{(1)}[\varphi]-\hbar\Gamma^{(2)}[\varphi]} &= \int D\phi\ \exp\Big(-\frac{1}{2}S_2[\varphi]\phi^2-\frac{\hbar^{\frac12}}{3!}S_3[\varphi]\phi^3-\frac{\hbar}{4!}S_4[\varphi]\phi^4+\hbar^{\frac12}\bar\Gamma^{(1)}_1[\varphi]\phi+O(\hbar^{\frac32})\Big)\\
&= (\det S_2[\varphi])^{-\frac12}\exp\Big[\frac{\hbar}{2}\frac{S_3[\varphi]}{3!}\langle\phi^3\phi^3\rangle_c\frac{S_3[\varphi]}{3!}-\hbar\frac{S_4[\varphi]}{4!}\langle\phi^4\rangle_c+\frac{\hbar}{2}\bar\Gamma^{(1)}_1[\varphi]\langle\phi\phi\rangle_c\bar\Gamma^{(1)}_1[\varphi]\\
&\quad-\hbar\frac{S_3[\varphi]}{3!}\langle\phi^3\phi\rangle_c\bar\Gamma^{(1)}_1[\varphi]+O(\hbar^{\frac32})\Big]
\end{aligned}
\tag{40}
$$

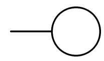
where $\langle ..\rangle_c$ denote connected correlation functions (however at this order there are no disconnected pieces). Then using Wick contractions one finds

$$
\Gamma^{(2)}[\varphi] = -\frac12\frac{S_3[\varphi]}{3!}\langle\phi^3\phi^3\rangle_c\frac{S_3[\varphi]}{3!}+\frac{S_4[\varphi]}{4!}\langle\phi^4\rangle_c-\frac12\bar\Gamma^{(1)}_1[\varphi]\langle\phi\phi\rangle_c\bar\Gamma^{(1)}_1[\varphi]+\frac{S_3[\varphi]}{3!}\langle\phi^3\phi\rangle_c\bar\Gamma^{(1)}_1[\varphi]
$$

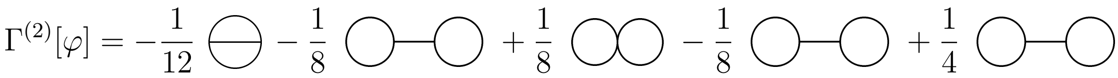


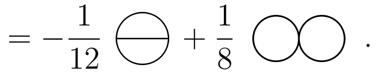
Observe now how the extra vertices that originate from $\bar\Gamma^{(n)}_1[\varphi]\phi$ cancel graphs that are not 1PI. The extra vertex denoted by the dark blob coincides with

$$
\begin{aligned}
&= \bar\Gamma^{(1)}_1[\varphi]\phi = \frac{\delta\bar\Gamma^{(1)}[\varphi]}{\delta\varphi}\phi = \frac{\delta}{\delta\varphi}\frac12\mathrm{tr}\ln S_2[\varphi]\phi\\
&= \frac12\mathrm{tr}\Big[S_2^{-1}\frac{\delta S_2[\varphi]}{\delta\varphi}\Big]\phi = \frac12\text{“}\,\mathrm{tr}\,S_2^{-1}S_3\text{”}\,\phi = \frac12
\end{aligned}
$$

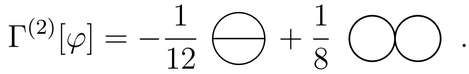


so that





We have verified again that the effective action contains only 1PI graphs



(41)

4.3 Background field method for gauge theories

A similar set-up can be used for gauge theories. Let's start with the Yang-Mills lagrangian for a non-abelian gauge field[1], to be denoted by $\mathcal{A}_\mu$

$$
\mathcal{L} = \frac14 F_{\mu\nu}(\mathcal{A})F^{\mu\nu}(\mathcal{A}) \tag{42}
$$

and split the gauge field $\mathcal{A}_\mu$ as a *background part* A_μ (also known as the classical part) and a *quantum part* Q_μ to be path integrated over (i.e. quantized)

$$
\mathcal{A}_\mu = A_\mu + Q_\mu\ . \tag{43}
$$

[1]We expand all fields as $\mathcal{A}_\mu(x) = -i\mathcal{A}^a_\mu(x)T^a$, and use the generators T^a in the fundamental representation so that $\mathrm{tr}(T^aT^b)=\frac12\delta^{ab}$. The following lagrangians must then be traced with $\frac{2}{g^2}\mathrm{tr}(...)$, an operation which we leave understood for notational simplicity.

We can identify two different gauge symmetries: a true gauge symmetry where the background field is inert

$$\begin{aligned}\delta_1 A_\mu &= 0\\ \delta_1 Q_\mu &= \partial_\mu \alpha + [A_\mu + Q_\mu, \alpha] = D_\mu(\mathcal{A})\alpha\end{aligned} \tag{44}$$

and a background gauge symmetry

$$\begin{aligned}\delta_2 A_\mu &= \partial_\mu \alpha + [A_\mu, \alpha] = D_\mu(A)\alpha\\ \delta_2 Q_\mu &= [Q_\mu, \alpha]\ .\end{aligned} \tag{45}$$

They both produce the same gauge transformation on $\mathcal{A}_\mu$, namely $\delta\mathcal{A}_\mu = D_\mu(\mathcal{A})\alpha$, but only the first one must be gauge-fixed for quantization. The second one treats instead the background A_μ as the gauge field (the gauge connection in geometrical terms), while the quantum field Q_μ transforms as a tensor in the adjoint representation. This symmetry is called *background gauge symmetry*.

As said, the symmetry that must be gauge fixed is the first one, since it leaves the background field invariant, and thus it is a true dynamical symmetry. Upon gauge-fixing that symmetry is lost, but it gives rise to the a rigid BRST symmetry. However, the gauge-fixing can be done in such a way as to preserve the second symmetry, the background gauge invariance. In this way the latter survives the quantization process, and remains as a useful symmetry of the effective action. In particular, it constrains the counterterms needed to renormalize the theory.

Let us expose some details. Using the BRST quantization method, a gauge-fixing function that preserves the background gauge symmetry is given by

$$f(Q_\mu, B; A_\mu) = D^\mu(A)Q_\mu - \frac{\xi}{2}B \qquad \to \qquad \Psi = \bar{c}\left(D^\mu(A)Q_\mu - \frac{\xi}{2}B\right) \tag{46}$$

which respects the background gauge symmetry (Q_μ transforms as a tensor in the adjoint representation, its background covariant derivatives does not destroy its tensorial character, and finally B is taken to transform in the adjoint and it expands as indicated in the previous footnote in terms of the generators: the gauge fermion is thus a scalar under the background gauge symmetry). This produces

$$s\Psi = B\left(D^\mu(A)Q_\mu - \frac{\xi}{2}B\right) - \bar{c}\,D^\mu(A)D_\mu(A+Q)c\ . \tag{47}$$

The gauge-fixed total lagrangian, upon elimination of the auxiliary B field, becomes

$$\mathcal{L}_{B,tot} = \frac{1}{4}F_{\mu\nu}(A+Q)F^{\mu\nu}(A+Q) + D^\mu(A)\bar{c}D_\mu(A+Q)c + \frac{1}{2\xi}(D^\mu(A)Q_\mu)^2 \tag{48}$$

(the subscript B stands for background). Evidently, it is background gauge invariant. Fermions may also be added, with a gauge coupling to $\mathcal{A}_\mu$ to maintain the background symmetry. Note that the field strength splits as

$$F_{\mu\nu}(\mathcal{A}) = F_{\mu\nu}(A) + D_\mu(A)Q_\nu - D_\nu(A)Q_\mu + [Q_\mu, Q_\nu] \tag{49}$$

which shows the background gauge covariance (Q is a tensor in the adjoint representation and DQ is also a tensor). Similarly, the ghost action is expanded as

$$\mathcal{L}_{gh} = D^\mu(A)\bar{c}D_\mu(A+Q)c = D^\mu(A)\bar{c}D_\mu(A)c + D^\mu(A)\bar{c}\,[Q_\mu, c] \tag{50}$$

with the first term that is background gauge invariant and fixes the propagator (it depends on the background field A_μ, one can formally construct it, but often is considered in a perturbative expansion in A_μ), and the second term is the vertex that also depends on A_μ, and again is background gauge invariant.

Now, one can quantize with path integrals and obtain[2]

$$\begin{aligned} Z_B[\tilde{J};A] &= \mathrm{e}^{iW_B[\tilde{J};A]} = \int DQD\bar{c}Dc \; \mathrm{e}^{iS_{B,tot}[Q,\bar{c},c;A]+i\tilde{J}Q} \\ \Gamma_B[\tilde{Q};A] &= \min_{\tilde{J}} \left\{ W_B[\tilde{J};A] - \tilde{J}\tilde{Q} \right\} . \end{aligned} \tag{51}$$

As the ghost are not physical external states, we omit adding sources for them, though one could do that as well. Then, $Z_B[\tilde{J};A]$ and $W_B[\tilde{J};A]$ are background gauge invariant under

$$\delta A_\mu = D_\mu(A)\alpha \; , \qquad \delta \tilde{J}^\mu = [\tilde{J}^\mu, \alpha] \tag{52}$$

which is seen by changing the path integration variables with the new variables related to the old ones by a background symmetry transformation: $Q'_\mu = Q_\mu + [Q_\mu, \alpha]$ (and similarly for the ghosts), and then considering that the path integral measure is invariant (unit jacobian i.e. no anomalies in this symmetry). Similarly, the effective action $\Gamma_B[\tilde{Q};A]$ is gauge invariant under

$$\delta A_\mu = D_\mu(A)\alpha \; , \qquad \delta \tilde{Q}_\mu = [\tilde{Q}_\mu, \alpha] \; . \tag{53}$$

In particular, $\Gamma_B[0;A]$ is gauge invariant. It contains 1PI graphs in terms of the field Q_μ that runs only inside the graphs (vacuum bubbles), and with external fields A_μ which do not propagate (as they are background fields), see figure 2.

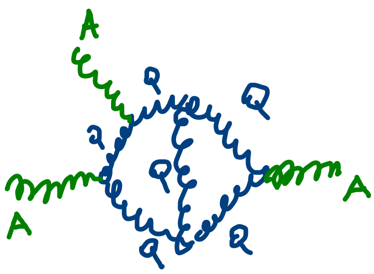


Figure 2: 1PI graph contained in $\Gamma_B[0;A]$.

Moreover, comparison with (28) suggests that it could be used as the effective action that captures the 1PI diagrams, and thus the complete physical information of the quantum theory.

To show that this is indeed the case, let us work out the details. We will find a relation with the effective action of the ordinary quantum field theory, but with an unusual gauge fixing term that depends on the function A_μ. We start from the path integral in (51), that contains the action fixed by (48), and perform in the path integral a change of variables induced by the shift $Q_\mu \to Q_\mu - A_\mu$, which leave the measure invariant, but change the lagrangian (48) to the new lagrangian

$$\mathcal{L}_{tot} = \frac{1}{4}F_{\mu\nu}(Q)F^{\mu\nu}(Q) + D^\mu(A)\bar{c}D_\mu(Q)c + \frac{1}{2\xi}(D^\mu(A)Q_\mu - \partial^\mu A_\mu)^2 \tag{54}$$

[2]As a shorthand notation we denote $\tilde{J}Q = \int d^4x \, \tilde{J}^\mu(x) Q_\mu(x)$.

where we have used that $D^\mu(A)A_\mu = \partial^\mu A_\mu$. We see that this is just an ordinary gauge theory, with the gauge field denoted by Q_μ, but with an unusual gauge fixing term constructed using a gauge fixing-function f as recognized from $\frac{1}{2\xi}f^2$ that appears in the last term of (54), i.e.

$$f(Q_\mu; A_\mu) = D^\mu(A)Q_\mu - \partial^\mu A_\mu \ . \tag{55}$$

On top of ξ, the gauge-fixing term $\frac{1}{2\xi}f^2$ depends also on the arbitrary function $A_\mu(x)$. Being a gauge choice, we know that physical quantities cannot depend on the values of ξ and of $A_\mu(x)$. We denote the corresponding path integral with lagrangian (54) by

$$\begin{aligned} Z[J;A] &= \mathrm{e}^{iW[J;A]} = \int DQD\bar{c}Dc \ \mathrm{e}^{iS_{tot}[Q,\bar{c},c;A]+iJQ} \\ \Gamma[Q;A] &= \min_J \Big\{ W[J;A] - JQ \Big\} \ . \end{aligned} \tag{56}$$

where the functional dependence on A reminds us that the gauge-fixing depends on it.

Now we are ready to find some useful relations. From (51), with the change of variables just mentioned, we find

$$Z_B[\tilde{J};A] = Z[\tilde{J};A]e^{-i\tilde{J}A} \tag{57}$$

$$W_B[\tilde{J};A] = W[\tilde{J};A] - \tilde{J}A \tag{58}$$

and finally

$$\Gamma_B[\tilde{Q};A] = \Gamma[\tilde{Q}+A;A] \tag{59}$$

which produces the searched for relation

$$\Gamma_B[0;A] = \Gamma[A;A] \tag{60}$$

that generalizes (28) to the gauge case. Note that functional differentiation of $\Gamma[A;A]$ with respect to the first variable yields the n-point 1PI diagrams, while differentiating with respect to the second A gives a vanishing result on physical quantities, as varying that A corresponds to just a change of the gauge-fixing term, a change which leaves physical quantities invariant. Relation (60) tells us that these 1PI diagrams can be computed in an equivalent way by using the background field method and considering only the vacuum graphs of the quantum field $\tilde{Q}$. The latter depend on the background field A, as indicated in $\Gamma_B[0;A]$. It has the useful property of being gauge invariant under the background gauge symmetry.

β-function in the background field method

Recalling that

$$\Gamma_B[0;A] = \textit{classical action} + \hbar \ \textit{corrections} \tag{61}$$

with the $\hbar$ corrections including also the $Z-1$ counterterms, we find that the effective action $\Gamma_B[0;A] = \int d^4x \, \mathcal{L}_B(0;A)$ must have a lagrangian of the form (recall (48) with $Q = c = \bar{c} = 0$)

$$\mathcal{L}_B(0;A) = \frac{1}{4} Z_{B,3} F_{\mu\nu}(A)F^{\mu\nu}(A) + \textit{other gauge invariant structures} \tag{62}$$

because it must be background gauge invariant. Thus, inserting the Z factors as in eq. (18), with a subscript B for the background dependent lagrangian, we must have the identities

$$Z_{B,3g}\sqrt{Z_{B,3}} = 1 \ , \qquad Z_{B,4g}Z_{B,3} = 1 \ . \tag{63}$$

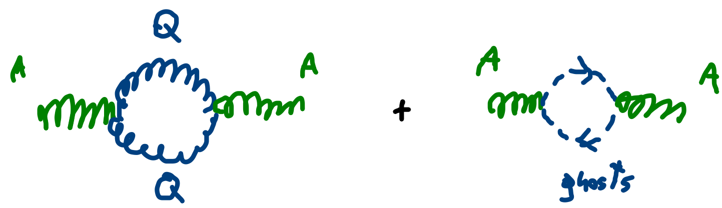
Indeed, using the components A^a_μ, we must consider the counterterms

$$
\begin{aligned}
\mathcal{L}_B &= -\frac{1}{2}Z_{B,3}[(\partial_\mu A^a_\nu)^2 - (\partial^\mu A^a_\mu)^2] - Z_{B,3g} f^{abc}\partial^\mu A^{\nu a} A^b_\mu A^c_\nu - Z_{B,4g}\frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e} \\
&= Z_{B,3}\left(-\frac{1}{2}(\partial_\mu A^a_\nu)^2 + \frac{1}{2}(\partial^\mu A^a_\mu)^2 - f^{abc}\partial^\mu A^{\nu a} A^b_\mu A^c_\nu - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e}\right)
\end{aligned}
\tag{64}
$$

where in the second line we have imposed the background gauge covariance leading to (62). Thus, relations (63) follow. One consequence of these relations is that to obtain the β-function we have simply to compute $Z_{B,3}$, that renomalizes the gluon field, as one could employ the third relation in (19) that leads to

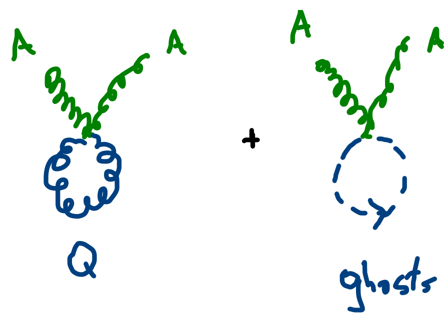
$$
g_0^2 = \frac{Z_{B,3g}^2}{Z_{B,3}^3} g^2 \tilde{\mu}^\epsilon = \frac{1}{Z_{B,3}^4} g^2 \tilde{\mu}^\epsilon \ . \tag{65}
$$

The calculation of $Z_{B,3}$ means that we have to identify the diverging part of the following self-energy diagrams



(66)

which is much easier of the calculation described earlier. There are also other diagrams



which however vanish in dimensional regularization ("massless tadpole vanish in DR", as integrals of the type $\int d^4k \frac{1}{k^2}$ do not have any mass parameter they might depend on, and are regulated to vanish).

We will not present here the detailed calculation (it be found on the textbooks such as Srednicki), and only describe some preliminary steps. For the calculation of the first graph in (77) one needs the AQQ vertices from (48). Setting $\xi = 1$, they are extracted from the gauge-fixed lagrangian

$$
\begin{aligned}
\mathcal{L}_{B,tot} &= \frac{1}{4}F_{\mu\nu}(A+Q)F^{\mu\nu}(A+Q) + \frac{1}{2}(D^\mu(A)Q_\mu)^2 + \text{ghosts} \\
&= \frac{1}{4}(F_{\mu\nu}(A) + D_\mu(A)Q_\nu - D_\nu(A)Q_\mu + [Q_\mu, Q_\nu])^2 + \frac{1}{2}(D^\mu(A)Q_\mu)^2 + \dots \\
&= F_{\mu\nu}(A)Q_\mu Q_\nu + \frac{1}{2}(D_\mu(A)Q_\nu)^2 - \frac{1}{2}D_\mu(A)Q_\nu D^\nu(A)Q^\mu + \frac{1}{2}(D^\mu(A)Q_\mu)^2 + \dots \\
&= 2F_{\mu\nu}(A)Q_\mu Q_\nu + \frac{1}{2}(D_\mu(A)Q_\nu)^2 + \dots
\end{aligned}
\tag{67}
$$

(recall that there is a trace understood, so that we can use the cyclicity of the trace to move terms around). Then, recalling the correct normalization (see previous footnote) plus considering the rescaling that brings the coupling constant into the vertices, we identify two independent structures for the AQQ vertex

$$\mathcal{L}_{int} = -gf^{abc}F^a_{\mu\nu}(A)Q^{\mu b}Q^{\nu c} - gf^{abc}A^{\mu a}Q^{\nu b}\partial_\mu Q^c_\nu \tag{68}$$

The calculation can be further simplified. Using twice the first vertex for the first diagram of (77) gives a contribution that can be computed more simply for a constant external $F^a_{\mu\nu}(A)$, and, considering that $f^{acd}f^{bcd} = T(A)\delta^{ab}$, gives a first contribution to $Z_{B,3}$. The second vertex is the same as that due to 4 real scalars in the adjoint, as described in chapter 78 of Srednicki, and using it on each vertex of the first diagram in (77) gives a second contribution. Finally, using both vertices once, gives a vanishing contribution.

Then, one should add the ghosts graphs, so that at this stage one reproduces the beta function for pure YM theory. This is a much simpler task then the one performed in the ordinary QFT without the background field theory method. One can also add matter fields of spin 0 and 1/2 in arbitrary representations to find a more general expression for the beta function, which becomes

$$\beta(g) = -\left[\frac{11}{3}T(A) - \frac{4}{3}n_F T(R_F) - \frac{1}{3}n_S T(R_S)\right]\frac{g^3}{16\pi^2} + O(g^5) \tag{69}$$

where n_F and n_S are the number of Dirac fermions and complex scalars in representations T_F and T_S, respectively.

4.4 Anomalies

Let us now briefly touch on the topic of anomalies, which for lack of space and time we cannot treat in any detail.

Anomalies in QFT refer to the breaking of classical symmetries due to the quantization process, a breaking that cannot be cured unless one is willing to sacrifice other symmetries, or modifying the field content of the theory (in this regard, the Standard Model is consistent with the known field content of fermions, but would have been anomalous without the top quark). The most important types of anomalies are *chiral anomalies* (anomalies in the conservation of Noether currents related to chiral symmetries) and *trace anomalies* (the trace of the stress tensor vanishes classically in conformal field theories, but may develop a trace at the quantum level). Also, *gravitational anomalies*, that can exist for theories coupled to gravity in dimensions 2, 6, 10 etc.., refers to the breaking of general coordinate invariance that can arise if chiral matter is present. They are of great importance in the construction of consistent string theories and supergravities.

The first anomaly to be discovered was the chiral anomaly in the axial $U(1)$ symmetry present in massless QED (thus the alternative name of *axial anomaly*, also known as the ABJ *anomaly* from Adler-Bell-Jackiw). For a massless Dirac fermion coupled to a $U(1)$ gauge theory, one has a vectorial gauge symmetry and a rigid axial symmetry. The gauge field needs not be quantized for discovering the anomaly, and it can be kept as a background. Regulating the theory so that the vector current is conserved leaves an axial current that is anomalous. Let us describe better the classical theory. The lagrangian of the model is given by a fermion field ψ coupled to a (background) abelian gauge field A_μ

$$\mathcal{L} = -\overline{\psi}\gamma^\mu(\partial_\mu - iA_\mu)\psi - m\overline{\psi}\psi \tag{70}$$

where a Dirac mass term will eventually be set to vanish. The gauge symmetry is given by

$$\begin{cases} \delta\psi = i\alpha\psi \\ \delta\overline{\psi} = -i\alpha\overline{\psi} \\ \delta A_\mu = \partial_\mu\alpha \end{cases} \tag{71}$$

with A_μ that couples to the conserved vector current $J^\mu = i\overline{\psi}\gamma^\mu\psi$. The gauge symmetry is valid also for $m \neq 0$. It is a background gauge symmetry, as in (71) we transformed also the background. On the other hand, if one keeps the background fixed, there remains only a rigid $U(1)$ symmetry, that can be used to show that the vector current J^μ is conserved, $\partial_\mu J^\mu = 0$.

In addition, there is an axial symmetry for $m = 0$. The corresponding infinitesimal rigid chiral transformations are given by

$$\begin{cases} \delta\psi = i\beta\gamma^5\psi \\ \delta\overline{\psi} = i\beta\overline{\psi}\gamma^5 \\ \delta A_a = 0 \end{cases} \tag{72}$$

where β is an infinitesimal constant phase. Indeed, the usual Noether procedure (using an arbitrary function $\beta(x)$ in (72)) gives a variation of the action

$$\delta S = \delta \int d^4x\, \mathcal{L} = \int d^4x \Big[-(\partial_\mu\beta)\, \underbrace{i\overline{\psi}\gamma^\mu\gamma^5\psi}_{J_5^\mu} - 2mi\beta\overline{\psi}\gamma^5\psi \Big] \,. \tag{73}$$

Then, using the equations of motion ($\delta S = 0$) and the arbitrariness of $\beta(x)$, one finds

$$\partial_\mu J_5^\mu = -2mi\overline{\psi}\gamma^5\psi \,. \tag{74}$$

For $m = 0$, the axial current J_5^μ is conserved

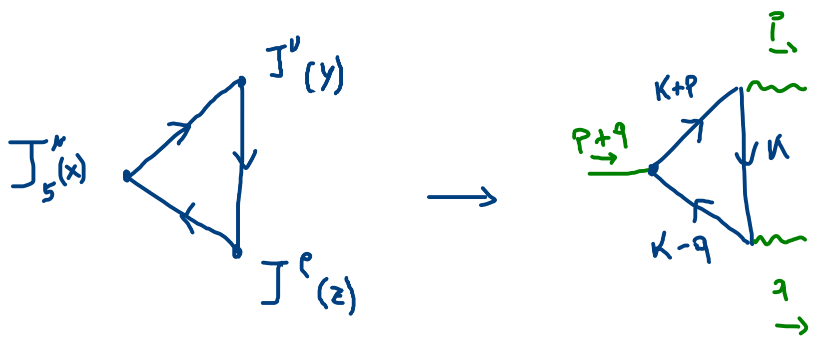
$$\partial_\mu J_5^\mu = 0 \tag{75}$$

and the axial transformation is a symmetry of the massless theory.

At the quantum level, the vector and axial currents become operators. One could then study the Ward identities, and in particular the one related to the expectation value of

$$\langle J_5^\mu(x) J^\nu(y) J^\rho(z) \rangle \tag{76}$$

that in momentum space gives rise to a triangle diagram of the form



(77)

The regularization of the diagram (e.g. using DR suitably extended to chiral theories, which however is a sublte issue) allows to preserve the conservation of the vector current, thus maintaining the gauge symmetry, but it leaves an anomalous divergence in the axial current

$$\partial_\mu J_5^\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \,. \tag{78}$$

This implies that the chiral symmetry is broken by the quantization process (i.e. by the need of regulating and eventually renormalizing loop diagrams to get finite results), and is said to be anomalous.