Path integrals in quantum mechanics

(Appunti per il corso di Fisica Teorica 1 – 2012/13)

Fiorenzo Bastianelli

Quantum mechanics can be formulated in two equivalent ways: (i) canonical quantization, also known as operatorial quantization, based on linear operators acting on the Hilbert space of physical states, (ii) path integrals, based on integration over a space of functions. The former was the first one to be developed, through the work of Heisenberg, Schrödinger, Dirac and others. The latter was introduced later on by Feynman, who extended previous suggestions by Dirac. Nowadays it is useful to know both formulations, as depending on the problem at hands, one may find technical advantages in using one with respect to the other. In worldline approaches one often uses the operatorial formulation to define the problem, and path integrals to calculate the answer.

The operatorial formulation of quantum mechanics is the one usually presented in introductory courses on quantum mechanics. Path integrals are introduced later on, when approaching the problem of quantizing gauge fields. Indeed with the advent of gauge theories, path integrals have become quite popular because the quantization of gauge fields is much more intuitive and transparent in such a context. In part I of this book we introduce path integrals for the quantization of point particles, as opposed to the quantization of field theories. The former contains a finite number of degrees of freedom, the latter deals with an infinite number of degrees of freedom, though formally they can be treated on the same footing. We assume only elementary notions of quantum mechanics in its operatorial form, and start developing path integrals from the beginning.

We begin this chapter by introducing path integrals for a non relativistic point particle. This case contains already the essence of path integrals. Then in the following chapter we provide a derivation of path integrals for fermionic systems, i.e. those which at the canonical level are quantized using anticommutators. Fermionic path integrals make use of Grassmann variables, anticommuting variables that allow the description of spin at the "classical level". Path integrals with bosonic and fermionic variables can be used to discuss supersymmetric systems, that often arise in the description of point particles with spin. In subsequent chapters we consider path integrals in the presence of background fields. For the case of bosonic path integrals, a scalar background potential $V(x)$ is treated without any particular effort already in this chapter, but in chapters 4 and 5 we discuss the regularization issues needed for coupling the particle first to a vector potential $A_i(x)$ and then to a tensor potential $g_{ij}(x)$, typically the metric of the space on which the particle propagates. Similar issues are discussed for fermionic path integrals as well.

1 Canonical quantization

Canonical quantization is constructed starting from the hamiltonian formulation of a classical system, and lifting its phase space coordinates, the generalized coordinates x^i and their conjugate momenta p_i, to linear operators $\hat{x}^i$ and $\hat{p}_i$. The latter act on a linear space endowed with a positive definite norm, the Hilbert space of physical states $\mathcal{H}$. The basic operators must satisfy commutation relations required to be equal $i\hbar$ times the value of the corresponding classical

Poisson brackets

$$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j \ , \quad [\hat{x}^i, \hat{x}^j] = 0 \ , \quad [\hat{p}_i, \hat{p}_j] = 0 \ . \tag{1}$$

As a consequence, all classical observables $A(x,p)$, which are functions on phase space, become linear operators $\hat{A}(\hat{x},\hat{p})$ acting on the Hilbert space $\mathcal{H}$. The most important example is given by the hamiltonian function $H(x,p)$, which upon quantization turns into the hamiltonian operator $\hat{H}(\hat{x},\hat{p})$. The latter generates the time evolution of any state $|\psi\rangle \in \mathcal{H}$ through the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \ . \tag{2}$$

The corresponding solution is a time dependent state $|\psi(t)\rangle$ that describes the evolution of the system. This set up is known as the Schrödinger picture of quantum mechanics. It is a formal quantization procedure that becomes operative once one finds an irreducible unitary representation of the algebra (1). It is a mathematical result, known as the Stone–von Neumann theorem, that in quantum mechanics all irreducible representations are unitarily equivalent, so that there is a unique procedure of quantizing a classical system up to equivalences[1]. Historically, this theorem made it clear that the Schrödinger formulation of quantum mechanics had to be equivalent to the one proposed by Heisenberg with its matrix mechanics (now known as the Heisenberg picture).

Using the coordinate representation, obtained by considering the eigenstates $|x\rangle$ of the position operator $\hat{x}$, that satisfy $\hat{x}|x\rangle = x|x\rangle$ with x a real number, and projecting the various states of the Hilbert space onto them to identify the standard wave functions, one finds the familiar way of realizing quantum mechanics as wave mechanics

$$\begin{aligned}
|\psi\rangle &\rightarrow \psi(x) && \Big(\psi(x) = \langle x|\psi\rangle\Big) \\
\hat{x} &\rightarrow x && \Big(\langle x|\hat{x}|x'\rangle = x\langle x|x'\rangle = x\delta(x-x')\Big) \\
\hat{p} &\rightarrow -i\hbar\frac{\partial}{\partial x} && \Big(\langle x|\hat{p}|x'\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|x'\rangle = -i\hbar\frac{\partial}{\partial x}\delta(x-x')\Big) \\
\hat{H} &\rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(q)
\end{aligned} \tag{3}$$

together with the standard form of the Schrödinger equation

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) \ . \tag{4}$$

We have reviewed the quantization procedure with a motion in one dimension, but extension to higher dimensions is straightforward.

Given an initial state $|\psi_i\rangle$ that describes the system at an initial time t_i, the solution of the Schrödinger equation in (2) for time independent hamiltonians is formally given by

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-t_i)}|\psi_i\rangle \tag{5}$$

which indeed satisfies the equation and the boundary condition $|\psi(t_i)\rangle = |\psi_i\rangle$. The amplitude to find the system at time t_f in state $|\psi_f\rangle$ is obtained by projecting the solution evaluated at

[1] And up to the problem of resolving the ordering ambiguities present when trying to relate a classical observable like the hamiltonian $H(x,p)$ to its quantum counterpart $\hat{H}(\hat{x},\hat{p})$.

time t_f onto the state $|\psi_f\rangle$

$$\langle\psi_f|\psi(t_f)\rangle = \langle\psi_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|\psi_i\rangle \ . \tag{6}$$

This amplitude is called "transition amplitude". In the following sections we shall find a path integral representation of such an amplitude.

2 Path integrals in phase space

We now proceed to derive a path integral that computes the transition amplitude starting from canonical quantization. For this purpose it is useful to insert twice the identity operator $\mathbb{1}$, expressed using the eigenstates of the position operator

$$\mathbb{1} = \int dx\, |x\rangle\langle x| \qquad\qquad \text{with } \langle x|x'\rangle = \delta(x-x')\,, \tag{7}$$

and rewrite (6) as

$$\begin{aligned}\langle\psi_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|\psi_i\rangle &= \langle\psi_f|\,\mathbb{1}\,e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}\,\mathbb{1}\,|\psi_i\rangle \\ &= \int dx_f \int dx_i\, \psi_f^*(x_f)\,\langle x_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|x_i\rangle\,\psi_i(x_i)\end{aligned} \tag{8}$$

where $\psi_i(x_i)$ and $\psi_f(x_f)$ are the wave functions for the initial and final states and $\psi_f^*(x) = \langle\psi_f|x\rangle = \langle x|\psi_f\rangle^*$. This shows that it is enough to consider the matrix element of the evolution operator between position eigenstates

$$A = \langle x_f|e^{-\frac{i}{\hbar}\hat{H}T}|x_i\rangle \tag{9}$$

where $T = (t_f - t_i)$ is the total propagation time.

For simplicity, we keep considering a one-dimensional motion for a non relativistic particle of mass m with quantum hamiltonian

$$\hat{H}(\hat{x},\hat{p}) = \frac{1}{2m}\hat{p}^2 + \hat{V}(\hat{x}) \ . \tag{10}$$

The derivation of the path integral now proceeds in the following fashion. One splits the transition amplitude as the product of N factors, and inserts the completeness relation (7) in between the factors $N-1$ times

$$\begin{aligned} A &= \langle x_f|e^{-\frac{i}{\hbar}\hat{H}T}|x_i\rangle = \langle x_f|\Big(e^{-\frac{iT}{\hbar N}\hat{H}}\Big)^N|x_i\rangle = \langle x_f|\underbrace{e^{-\frac{i\epsilon}{\hbar}\hat{H}}e^{-\frac{i\epsilon}{\hbar}\hat{H}}\cdots e^{-\frac{i\epsilon}{\hbar}\hat{H}}}_{N\text{ times}}|x_i\rangle \\ &= \langle x_f|e^{-\frac{i\epsilon}{\hbar}\hat{H}}\,\mathbb{1}\,e^{-\frac{i\epsilon}{\hbar}\hat{H}}\,\mathbb{1}\cdots\mathbb{1}\,e^{-\frac{i\epsilon}{\hbar}\hat{H}}|x_i\rangle = \int\Big(\prod_{k=1}^{N-1}dx_k\Big)\prod_{k=1}^{N}\langle x_k|e^{-\frac{i\epsilon}{\hbar}\hat{H}}|x_{k-1}\rangle \end{aligned} \tag{11}$$

where for convenience we have denoted $x_0 \equiv x_i$, $x_N \equiv x_f$, $\epsilon \equiv \frac{T}{N}$. We can now use N more times the resolution of the identity, but expressed in terms of the momentum egenstates

$$\mathbb{1} = \int \frac{dp}{2\pi\hbar}\,|p\rangle\langle p| \qquad\qquad \text{with } \langle p|p'\rangle = 2\pi\hbar\,\delta(p-p')\,, \tag{12}$$

to obtain

$$\begin{aligned} A &= \int \Big(\prod_{k=1}^{N-1} dx_k\Big) \prod_{k=1}^{N} \langle x_k | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | x_{k-1}\rangle = \int \Big(\prod_{k=1}^{N-1} dx_k\Big) \prod_{k=1}^{N} \langle x_k | \mathbb{1}\, e^{-\frac{i\epsilon}{\hbar}\hat{H}} | x_{k-1}\rangle \\ &= \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}\Big) \prod_{k=1}^{N} \langle x_k | p_k\rangle\langle p_k | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | x_{k-1}\rangle \ . \end{aligned} \tag{13}$$

This is an exact formula. There is one more integration over momenta than integrations over coordinates, as consequence of choosing coordinate eigenstates as initial and final states in the transition amplitude. Now one can manipulate this formula further by making approximations that are valid in the limit $N \to \infty$ ($\epsilon \to 0$). The crucial point is the evaluation of the following matrix element

$$\begin{aligned} \langle p | e^{-\frac{i\epsilon}{\hbar}\hat{H}(\hat{x},\hat{p})} | x\rangle &= \langle p | \Big(\mathbb{1} - \frac{i\epsilon}{\hbar}\hat{H}(\hat{x},\hat{p}) + \cdots\Big) | x\rangle \\ &= \langle p | x\rangle - \frac{i\epsilon}{\hbar}\langle p | \hat{H}(\hat{x},\hat{p}) | x\rangle + \cdots \\ &= \langle p | x\rangle \left(1 - \frac{i\epsilon}{\hbar} H(x,p) + \cdots\right) \\ &= \langle p | x\rangle \, e^{-\frac{i\epsilon}{\hbar}H(x,p)+\cdots} \ . \end{aligned} \tag{14}$$

These approximations are all valid in the limit of small ϵ. In addition, the substitution $\langle p|\hat{H}(\hat{x},\hat{p})|x\rangle = \langle p|x\rangle\, H(x,p)$ follows from the simple structure of the hamiltonian (10), that allows one to act with the momentum operator on the left and with the position operator on the right, and have them replaced with the corresponding eigenvalues. Notice that there is no need of commuting operators inside the hamiltonian, and the final result is that all operators are simply replaced by eigenvalues. This way the quantum hamiltonian $\hat{H}(\hat{x},\hat{p})$ gets replaced by the classical function $H(x,p) = \frac{p^2}{2m} + V(x)$. All these manipulations are justified for a wide class of physically interesting potentials $V(x)$. In this context there exists a rigorous proof that this is correct, which goes under the name of "Trotter formula". We shall not need to review it as the physically intuitive derivation given above is enough for our purposes.

Using now eq. (14) and remembering that the wave functions of the momentum eigenstates (the plane waves) are normalized as

$$\langle x | p\rangle = e^{\frac{i}{\hbar}px} \ , \quad \langle p | x\rangle = \langle x | p\rangle^* = e^{-\frac{i}{\hbar}px} \tag{15}$$

one obtains

$$\langle x_k | p_k\rangle\langle p_k | e^{-\frac{i\epsilon}{\hbar}\hat{H}} | x_{k-1}\rangle = e^{\frac{i}{\hbar}p_k(x_k - x_{k-1}) - \frac{i\epsilon}{\hbar}H(x_{k-1},p_k)} \tag{16}$$

up to terms that vanish for $\epsilon \to 0$. This expression can now be inserted in (13). At this stage the transition amplitude does not contain any more operators, bra and ket states, but just integrations of ordinary functions, though a big number of them (a number that tends to infinity)

$$\begin{aligned} A &= \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k\Big)\Big(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}\Big) e^{\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\left[p_k\frac{(x_k - x_{k-1})}{\epsilon} - H(x_{k-1},p_k)\right]} \\ &= \int DxDp \ e^{\frac{i}{\hbar}S[x,p]} \ . \end{aligned} \tag{17}$$

This is the path integral in phase space. One recognizes in the exponent a discretization of the classical phase space action

$$S[x,p] = \int_{t_i}^{t_f} dt \left(p\dot{x} - H(x,p) \right) \;\rightarrow\; \sum_{k=1}^{N} \epsilon \Big(p_k \frac{(x_k - x_{k-1})}{\epsilon} - H(x_{k-1}, p_k) \Big) \tag{18}$$

where $t_f - t_i = T = N\epsilon$ is the total propagation time. The last way of writing the amplitude in (17) is symbolic and suggestive: it indicates the formal sum over all paths in phase space weighted by the exponential of $\frac{i}{\hbar}$ times the classical action.

3 Path integrals in configuration space

The path integral in configurations space is now easily derived by integrating over the momenta. Indeed the dependence on momenta in the exponent of (17) is at most quadratic and can be eliminated by gaussian integration

$$\int_{-\infty}^{\infty} dp\, e^{-\frac{K}{2}p^2} = \sqrt{\frac{2\pi}{K}} \tag{19}$$

extended analytically to include complex values of K. Considering the form of the hamiltonian $H(x,p) = \frac{p^2}{2m} + V(x)$ and completing the squares (i.e. first rewriting $p_k \frac{(x_k - x_{k-1})}{\epsilon} - \frac{1}{2m} p_k^2 = -\frac{1}{2m}(p_k - m\frac{(x_k - x_{k-1})}{\epsilon})^2 + \frac{m}{2}\frac{(x_k - x_{k-1})^2}{\epsilon^2}$, and then changing integration variables $p_k \rightarrow \tilde{p}_k = p_k - m\frac{(x_k - x_{k-1})}{\epsilon}$, which leaves the integration measure invariant) one may perform the gaussian integrations over the momenta and obtain

$$\begin{aligned} A &= \lim_{N\to\infty} \int \Big(\prod_{k=1}^{N-1} dx_k \Big) \Big(\frac{m}{2\pi i\hbar\epsilon} \Big)^{\frac{N}{2}} e^{\frac{i\epsilon}{\hbar} \sum_{k=1}^{N} \left[\frac{m}{2} \frac{(x_k - x_{k-1})^2}{\epsilon^2} - V(x_{k-1}) \right]} \\ &= \int Dx\, e^{\frac{i}{\hbar} S[x]} \; . \end{aligned} \tag{20}$$

This is the path integral in configurations space. It contains in the exponent the classical configurations space action suitably discretized

$$S[x] \;=\; \int_{t_i}^{t_f} dt \left(\frac{m}{2}\dot{x}^2 - V(x) \right) \;\rightarrow\; \sum_{k=1}^{N} \epsilon \Big[\frac{m}{2} \Big(\frac{x_k - x_{k-1}}{\epsilon} \Big)^2 - V(x_{k-1}) \Big] \; . \tag{21}$$

Again, the last way of writing the expression in (20) is symbolic, and indicates the formal sum over paths in configurations space weighted by the exponential of $\frac{i}{\hbar}$ times the classical action. The space of paths is given by the space of functions $x(t)$ with boundary values $x(t_i) = x_i$ and $x(t_f) = x_f$. How to perform concretely the path integral over this functional space is defined precisely by the discretization, that approximates the function $x(t)$ by the $N+1$ values $x_0 \equiv x_i, x_1, x_2, \ldots, n_{N-1}, x_N \equiv x_f$.

3.1 Free particle

For a free particle ($V(x) = 0$) one may use repeatedly gaussian integrations and calculate from eq. (20) the exact transition amplitude

$$A(x_i, x_f; T) = \sqrt{\frac{m}{2\pi i\hbar T}}\, e^{\frac{i}{\hbar}\frac{m(x_f - x_i)^2}{2T}} \tag{22}$$

that indeed satisfies the free Schrödinger equation

$$i\hbar\frac{\partial}{\partial T}A(x_i,x_f;T) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_f^2}A(x_i,x_f;T) \tag{23}$$

with initial conditions

$$A(x_i,x_f;0) = \delta(x_f - x_i)\ . \tag{24}$$

The result is very suggestive: up to a prefactor it is given by the exponential of $\frac{i}{\hbar}$ times the classical action evaluated on the classical path, i.e. the path that satisfies the classical equations of motion. This is typical for the cases in which the semiclassical approximation is exact. One may interpret the prefactor as due to one-loop corrections to the classical (tree-level) result. The free particle case is also quite special: the correct result is obtained for any N, so that there is no need to take the limit $N \to \infty$. The case $N = 1$, which carries no integration at all in the x variable, is exact as well.

A useful but formal way of calculating gaussian path integrals is that of operating directly in the continuum limit. One does not need to consider the precise definition of the path integral measure, but uses only formal properties like its translational invariance. The calculation is formal in that one assumes properties of the path integral measure, that eventually have to be proved by an explicit regularization and construction. The calculation goes as follows. The classical action is $S[x] = \int_0^T dt\, \frac{m}{2}\dot{x}^2$, and the classical equations of motions are solved with the above boundary conditions by

$$x_{cl}(t) = x_i + (x_f - x_i)\frac{t}{T}\ . \tag{25}$$

Now one can represent a generic path $x(t)$ as a classical path $x_{cl}(t)$ plus quantum fluctuations $q(t)$

$$x(t) = x_{cl}(t) + q(t) \tag{26}$$

where the quantum fluctuations $q(t)$ must vanish at $t = 0$ and $t = T$ to preserve the boundary conditions. Then one computes the path integral as follows

$$\begin{aligned} A(x_i,x_f;T) &= \int Dx\, e^{\frac{i}{\hbar}S[x]} = \int D(x_{cl}+q)\, e^{\frac{i}{\hbar}S[x_{cl}+q]} \\ &= \int Dq\, e^{\frac{i}{\hbar}(S[x_{cl}]+S[q])} = e^{\frac{i}{\hbar}S[x_{cl}]}\int Dq\, e^{\frac{i}{\hbar}S[q]} \\ &= N e^{\frac{i}{\hbar}S[x_{cl}]} = N e^{\frac{i}{\hbar}\frac{m(x_f-x_i)^2}{2T}} \end{aligned} \tag{27}$$

where the translational invariance of the path integral measure ($Dx = D(x_{cl}+q) = Dq$) has been used. There is no linear term in q in the action because x_{cl} solves the classical equations of motion, so that for quadratic actions one has $S[x_{cl}+q] = S[x_{cl}] + S[q]$. The normalization coefficient $N = \int Dq\, e^{\frac{i}{\hbar}S[q]}$ is undetemined by this method, but can be fixed by requiring that the final result indeed satisfies the Schrödinger equation (thus finding $N = \sqrt{\frac{m}{2\pi i\hbar T}}$).

3.2 Euclidean time and statistical mechanics

Continuing analytically the time parameter to purely imaginary values by $T \to -i\beta$ with real β, and setting $\hbar = 1$, the Schrödinger equation (23) turns into the heat equation

$$\frac{\partial}{\partial\beta}A = \frac{1}{2m}\frac{\partial^2}{\partial x_f^2}A\ . \tag{28}$$

Its fundamental solution, i.e. the solution with boundary condition $A \xrightarrow{\beta\to 0} \delta(x_f - x_i)$, given by

$$A = \sqrt{\frac{m}{2\pi\beta}}\, e^{-\frac{m(x_f-x_i)^2}{2\beta}} \ , \tag{29}$$

can be obtained from (22) by the same trick. This analytical continuation is called "Wick rotation". It can be performed directly on the path integral: analytically continuing the time variable as $t \to -i\tau$, the action with "minkowskian" time (i.e. with a real time t) turns into an "euclidean" action S_E defined by

$$iS[x] = i\int_0^T dt\, \frac{m}{2}\dot{x}^2 \quad \to \quad -S_E[x] = -\int_0^\beta d\tau\, \frac{m}{2}\dot{x}^2 \tag{30}$$

where in the euclidean action one defines $\dot{x} = \frac{dx}{d\tau}$, with τ usually called "euclidean time". The euclidean action is positive definite and the corresponding path integral

$$\int Dx\, e^{-S_E[x]} \tag{31}$$

for a free theory is truly gaussian, with an exponential damping rather than with increasingly rapid phase oscillations. It coincides with the functional integral introduced by Wiener in the 1920's to study brownian motion and the heat equation.

Such euclidean path integrals are quite useful in statistical mechanics, where β is related to the inverse temperature Θ by $\beta = \frac{1}{k\Theta}$ (k is the Boltzmann's constant). Indeed the trace of the evolution operator Z, that can be written equivalently using energy eigenstates labeled by n (if the spectrum is discrete) or position eigenstates labeled by q,

$$Z \equiv \mathrm{Tr}\, e^{-\frac{i}{\hbar}\hat{H}T} = \sum_n e^{-\frac{i}{\hbar}E_n T} = \int dq\, \langle q|e^{-\frac{i}{\hbar}\hat{H}T}|q\rangle \ , \tag{32}$$

can be Wick rotated with $T \to -i\beta$. Setting as usual $\hbar = 1$, one obtains the statistical partition function Z_E of the quantum system with hamiltonian $\hat{H}$

$$Z_E \equiv \mathrm{Tr}\, e^{-\beta\hat{H}} = \sum_n e^{-\beta E_n} = \int dq\, \langle q|e^{-\beta\hat{H}}|q\rangle \ . \tag{33}$$

It is now a simple task to obtain a representation of the statistical partition function in terms of path integrals: one performs a Wick rotation of the path integral action, sets the initial state (at euclidean time $\tau = 0$) equal to the final state (at euclidean time $\tau = \beta$), and sums over all possible states. The paths become closed paths, as $q(0) = q(\beta)$, and the partition function becomes

$$Z_E = \mathrm{Tr}\, e^{-\beta\hat{H}} = \int_{PBC} Dq\, e^{-S_E[q]} \tag{34}$$

where PBC stands for "periodic boundary conditions", indicating the sum over all paths that close on themselves in an euclidean time β.

Though introduced here for the free theory, the Wick rotation is supposed to be of more general value, relating quantum mechanics to statistical mechanics in the interacting case as well. Even if one is interested in the theory with a real time, often one works in the euclidean version of the theory, where factors of the imaginary unit i are absent, and path integral

convergence is more easily kept under control. Only at the very end one performs the inverse Wick rotation to read off the result for the minkowskian theory.

The Wick rotation procedure is better appreciated by considering the usual minowskian time as the real line of a complex plane: denoting the complex time by $t_\theta = te^{-i\theta}$, the usual real time corresponds to $\theta = 0$, while the euclidean time τ is obtained at $\theta = \frac{\pi}{2}$. The analytical continuation of all physical quantities is achieved by continually increasing θ form 0 to $\frac{\pi}{2}$, a clockwise rotation of the real axis into the imaginary one. The generalized partition function $Z_\theta = \mathrm{Tr}\, e^{-\frac{i}{\hbar}\hat{H}t_\theta}$ with a complex time $te^{-i\theta}$ with positive t has a damping factor for all $0 < \theta \leq \frac{\pi}{2}$ and for all hamiltonians that are bounded from below (up to an overall factor due to the value of the ground state energy, if that happens to be negative).

Similar considerations can be made for path integrals in minkowskian and euclidean times. Path integrals in euclidean times are mathematically better defined (one may develop a mathematically well defined measure theory on the space of functions), at least for quadratic actions and perturbations thereof. Path integral with a minkowskian time are more delicate, and physicists usually use the argument of rapid phase oscillations to make unwanted terms vanish. The Wick rotation suggests a way of defining the path integral in real time starting from the one with euclidean time. These points of mathematical rigor are not necessary for the applications described in this book, and the derivation of path integrals described previously is enough for our purposes.

3.3 Miscellaneous comments

We have seen that the quantization of a classical system with action $S[x]$ is achieved by the path integral $\int Dx\, e^{\frac{i}{\hbar}S[x]}$ that computes the transition amplitude.

At the classical level, the solutions of the equations of motion are those that extremize the action and make it a minimum ("principle of minimal action"). Thus $x_{cl}(t)$ solves the equations of motion only if the extremality condition $\delta S[x_{cl}] = 0$ is satisfied. This is generically all that one needs for classical problems. In the quantum theory one needs instead the value of the action for all configurations $x(t)$, as each configuration contributes to the total amplitude with the phase $e^{\frac{i}{\hbar}S[x(t)]}$. This indicates that a more extended use of the action is required for extracting the quantum properties of a physical system.

In the path integral formulation the classical limit is intuitive: macroscopic systems have large values of action in $\hbar$ units. Macroscopically small variations of paths can still make the phase variations $\frac{\delta S[x]}{\hbar}$ much bigger than π, so that amplitudes of nearby paths get canceled by destructive interference. This is true except for variations that make $\delta S[x] = 0$, which is precisely the condition that identifies the classical path. Nearby paths have amplitudes that sum coherently with the classical one, and the path integral is dominated by the classical trajectory.

The notation $\int Dx$ is symbolic and indicates the formal integration over the space of functions $x(t)$. To make it precise one has to regulate the functional space by making it finite dimensional (a procedure called "regularization"). Then one integrates over the regulated finite-dimensional space, and eventually takes the continuum limit by removing the regularization. If this procedure is done with enough care, the limit exists and gives the correct transition amplitude. In the previous derivation we have seen that the space of paths is regulated by approximating the functions $x(t)$ by their $N-1$ values computed at intermediate points, the x_k's with $k = 1, ..., N-1$. This makes the space of functions finite dimensional. The action is also discretized and evaluated using the approximated functions. At this stage the integration

over the regulated functional space is well defined. Eventually one takes the continuum limit ($N \to \infty$): if the integration measure is chosen appropriately, as in eq. (20), this limit exists and gives a viable definition of the path integral.

We have proceeded starting from canonical quantization and derived the above discretized form of the space of functions. This regularization is often called Time Slicing (TS). Viceversa, one can start directly with the path integral, regulate it suitably, and use it to construct the quantum theory. This can be viewed as an alternative approach to quantization. In the regularization procedure one must make several choices, and they may produce different transition amplitudes. For example, in a TS regularization one may discretize the potential term $V(x(t))$ in the action to $V(x_k)$ or $V(x_{k-1})$ or $V(\frac{1}{2}(x_k + x_{k-1})$. In the present case this makes no difference, and one obtains the same continuum limit, but for more complicated interactions, such as those arising from the coupling to gauge fields or in the presence of a nontrivial background metric, different discretizations may produce different final answers. These ambiguities are the path integral counterparts of the ordering ambiguities of canonical quantization. Given a regulated path integral, one may compute the transition amplitude and find the corresponding quantum hamiltonian by checking which Schrödinger equation it satisfies. Alternatively, one may compute some other observable to find the precise correspondence with canonical quantization.

Any regularization can be used to solve a physical problem, as different regularizations are related by "counterterms", that is extra potential terms that added to the action in one regularization scheme produce the result of another regularization scheme. Of course, a given regularization may be easier to deal with than others. It is often the case that one is interested in studying a system characterized by a particular quantum hamiltonian. To treat the problem with path integrals, once one has chosen a consistent regularization, he must add counterterms to make sure that the correct hamiltonian is reproduced by the regulated path integral. This is typical when dealing with motions on curved spaces, though in a milder form this happens also when considering the coupling to gauge fields. Counterterms are always local function. In the language of quantum field theory, quantum mechanics is a super-renormalizable QFT in one dimension, with coupling constants parametrized by the scalar, vector and tensor potentials. As we shall see later on, potential infinities that may appear at low loop order are actually absent if one considers the contributions coming from the nontrivial path integral measure. The remaining finite ambiguities are eliminated by the chosen regularization scheme.

A regularization scheme, alternative to TS, can be obtained by considering the Fourier expansion of the functions belonging to the space of paths (or alternatively an expansion in a complete set of orthonormal functions). The regularization is achieved by truncating the expansion at a given high mode M. Path integration is then defined as the integration over the finite number of mode coefficients. The continuum limit is obtained by removing the cut-off M, i.e. sending $M \to \infty$. Such a regularization is called Mode Regularization (MR).

Yet another regularization is based on extending the time dimension to higher dimensional spacetimes of dimension D, analytically continued to be a complex number. It is called Dimensional Regularization (DR), and though more abstract than the previous ones, it has its own virtues. All these regularizations will be exemplified when studying the motion in curved space in chapter **??**.

We have introduced path integrals by considering a single degree of freedom. Extension to a finite number of degrees of freedom is immedate, so that quantizing the motion of one or more particles in a finite dimensional space does not pose any conceptually new problem. For example, the motion of a nonrelativistic particle in $\mathbb{R}^3$ with cartesian coordinates $\vec{x}$, in the

presence of a scalar potential $V(\vec{x})$, is quantized by the following discretized path integral

$$\int Dx\, e^{\frac{i}{\hbar}S[x]} \;=\; \lim_{N\to\infty}\int \Big(\prod_{k=1}^{N-1} d^3x_k\Big)\Big(\frac{m}{2\pi i\hbar\epsilon}\Big)^{\frac{3N}{2}} e^{\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\left[\frac{m}{2}\frac{(\vec{x}_k-\vec{x}_{k-1})^2}{\epsilon^2}-V(\vec{x}_{k-1})\right]}\,. \tag{35}$$

Formally, one can also consider the case of an infinite number of degrees of freedom, as appropriate for a field theory. In this case convergence is not guaranteed, and the removal of the regularization may lead to an infinite result. In the class of theories called renormalizable, the infinites can be removed consistently by a renormalization procedure that redefines the field variables and the coupling constants, and allows to obtain, at least at the level of perturbation theory, finite results.

4 Correlation functions

Correlation functions are quantities used to describe physical observables in the quantum theory. They are useful to develop the perturbative expansion around the solvable gaussian path integral that corresponds to a free theory.

In our one dimensional example the normalized "n-point correlation functions" are defined by

$$\langle x(t_1)x(t_2)\dots x(t_n)\rangle = \frac{1}{Z}\int Dx\, x(t_1)x(t_2)\dots x(t_n)\, e^{\frac{i}{\hbar}S[x]} \tag{36}$$

where $Z=\int Dx\, e^{\frac{i}{\hbar}S[x]}$ provides the normalization to guarantee that $\langle 1\rangle = 1$. Thus, correlation functions are normalized averages of the product of n dynamical variables evaluated at different times and weighted by $e^{\frac{i}{\hbar}S}$. Of particular importance is the 2-point function $\langle x(t_1)x(t_2)\rangle$, often called the propagator. It is understood that correlation functions depend implicitly on the initial and final states. Very often, especially in quantum field theory, one chooses the initial and final states to be the vacuum state (the state with lowest energy), and also considers an infinite propagation time. We mostly consider amplitudes between positions eigenstates, but using (8) one can insert any desired state as boundary state.

In this book we are going to use path integrals, but it is useful to compare with the corresponding definition of correlation functions given in canonical quantization as well. We have employed the Schrödinger picture to evaluate the transition amplitude. In this picture operators are time independent and states acquire the time dependence by the Schrödinger equation. To state the equivalent definition of the n-point correlation function, given the times $t_1, t_2, ..., t_n$, one has to reorder them from the earliest to the latest one, i.e. use the permutation $T(1), T(2), ..., T(n)$ of the numbers $1, 2, ..., n$ such that $t_{T(1)} < t_{T(2)} < ... < t_{T(n)}$. Then one defines

$$\begin{aligned}\langle x(t_1)x(t_2)\cdots x(t_n)\rangle \;&=\; \frac{1}{Z}\langle x_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_{T(n)})}\hat{x}\, e^{-\frac{i}{\hbar}\hat{H}(t_{T(n)}-t_{T(n-1)})}\dots \\ &\dots\;\; e^{-\frac{i}{\hbar}\hat{H}(t_{T(3)}-t_{T(2)})}\hat{x}\, e^{-\frac{i}{\hbar}\hat{H}(t_{T(2)}-t_{T(1)})}\hat{x}\, e^{-\frac{i}{\hbar}\hat{H}(t_{T(1)}-t_i)}|x_i\rangle\end{aligned} \tag{37}$$

where $Z=\langle x_f|e^{-\frac{i}{\hbar}\hat{H}(t_f-t_i)}|x_i\rangle$ is the transition amplitude. The time ordering guarantees that in the path integral derivation, due to the time slicing procedure, each position operator is substituted by the eigenvalue of the eigenstate that is carried by the resolution of the identity

inserted next to the operator under consideration. This is always the case, as for very large N the time discretization is sufficiently fine to have a resolution of the identity next to the position of any operators $\hat{x}$.

Equivalently, in the Heisenberg picture, one assigns the time evolution to the operators while states are time independent. Heisenberg's equations of motion (Heisenberg's "matrix mechanics") read as

$$i\hbar \frac{d\hat{x}_H}{dt} = [\hat{x}_H, \hat{H}] \tag{38}$$

where the subscript H refers to operators in the Heisenberg picture. They correspond to the quantum version of Hamilton's equations, with the Poisson bracket substituted by a commutator that takes the value of $i\hbar$ times the classical Poisson bracket. For a time independent hamiltonian the solution can formally be written as

$$\hat{x}_H(t) = e^{\frac{i}{\hbar}\hat{H}t} \hat{x}_H(0) e^{-\frac{i}{\hbar}\hat{H}t} \tag{39}$$

where the value of $\hat{x}_H(0)$ can be identified with the time independent Schrödinger operator $\hat{x}$. Eigenstates of $\hat{x}_H(t)$ can be written as $|x,t\rangle_H$

$$\hat{x}_H(t)|x,t\rangle_H = x|x,t\rangle_H \ . \tag{40}$$

The relation to the Schrödinger picture is simply given in terms of the unitary operator $e^{\frac{i}{\hbar}\hat{H}t}$. This operator relates the two pictures which are then unitarily equivalent, as guaranteed by the Stone–von Neumann theorem. Correlation functions in the Heisenberg picture are defined by

$$\langle x(t_1)x(t_2)...x(t_n)\rangle = \frac{1}{Z}{}_H\langle x_f, t_f | T\, \hat{x}(t_1)\hat{x}(t_2)\cdots\hat{x}(t_n)\, |x_i, t_i\rangle_H \tag{41}$$

where the symbol T indicates time ordering, i.e. the prescription of ordering the operators in such a way that they have an increasing value of time when going form right to left. The value of the transition amplitude that normalizes the expression is written in such a picture as $Z = {}_H\langle x_f, t_f | x_i, t_i\rangle_H$.

It is useful to collect all correlation functions into a single object $Z[J]$, called the generating functional of correlation functions. One uses an arbitrary function $J(t)$, called "source", and defines

$$\begin{aligned} Z[J] &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{\hbar}\right)^n \int dt_1 dt_2 \ldots dt_n \, \langle x(t_1)x(t_2)\cdots x(t_n)\rangle_U \, J(t_1)J(t_2)\cdots J(t_n) \\ &= \int Dx \, e^{\frac{i}{\hbar}(S[x] + \int dt\, Jx)} \end{aligned} \tag{42}$$

where the subscript "U" indicates un-normalized correlation functions, i.e. correlation functions obtained without dividing by Z. The expression in terms of the path integral is proven by functionally deriving n times with respect to the source $J(t)$, and then setting $J(t) = 0$.

Thus the correlation functions are obtained by functionally differentiation of the generating functional $Z[J]$ as follows

$$\langle x(t_1)x(t_2)\cdots x(t_n)\rangle = \frac{1}{Z}\left(\frac{\hbar}{i}\right)^n \left.\frac{\delta^n Z[J]}{\delta J(t_1)\delta J(t_2)\cdots\delta J(t_n)}\right|_{J=0} \tag{43}$$

where $Z = Z[0]$ is the normalizing factor.

To proceed swiftly, and give suitable gaussian formulae, it is useful to introduce an hypercondensed notation that allows to treat path integrals, including those for field theories, formally as ordinary integrals. The hypercondensed notation is defined by lumping together discrete and continuous indices into a single index, so that a variable ϕ^i can be used as a shorthand notation for the position $x(t)$ of the particle, identifying

$$\begin{aligned} x &\to \phi \\ t &\to i \end{aligned} \tag{44}$$

For a field, as for example the vector quadripotential $A_\mu(x^\nu)$, the hypercondensed notation is obtained by identifying

$$\begin{aligned} A &\to \phi \\ \mu, x^\nu \equiv \mu, x^0, x^1, x^2, x^3 &\to i \end{aligned} \tag{45}$$

where now the index i contains a discrete part (the discrete index $\mu = 0, 1, 2, 3$) and a continuous part (the space-time coordinates $x^\nu = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$). Indices may be lowered with a metric, which in many cases is simply given by the identity matrix but it allows for more general situations. Repeated indices are understood to be summed over (the Einstein summation convention). Thus, in the above cases the notation $\phi^i\phi_i$ stands for $\int dt\, x(t)x(t)$ and $\int d^4x\, A_\mu(x)A^\mu(x)$, respectively. Of course, one must pay attention to simple looking expressions as they include integrations or infinite sums and might not converge.

With such a notation we are ready to review the definition of correlation functions, introduce few more generating functionals, and present gaussian integration formulae. We are going to describe also the Wick theorem, that gives a simple way of computing correlation functions in a free (gaussian) theory in terms of the 2-point function.

The path integral in (21) can be written as

$$\int D\phi\, e^{\frac{i}{\hbar}S[\phi]} \tag{46}$$

and correlation functions by

$$\langle \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}\rangle = \frac{1}{Z}\int D\phi\ \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}\mathrm{e}^{\frac{i}{\hbar}S[\phi]}\ . \tag{47}$$

The generating functional takes the form

$$Z[J] = \int D\phi\ \mathrm{e}^{\frac{i}{\hbar}(S[\phi]+J_i\phi^i)} \tag{48}$$

and generates all correlation functions by differentiation (in hypercondensed notation functional derivatives read as standard derivatives, though we keep using the symbol δ of functional derivative)

$$\langle \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}\rangle = \frac{1}{Z[0]}\left(\frac{\hbar}{i}\right)^n \frac{\delta}{\delta J_{i_1}}\frac{\delta}{\delta J_{i_2}}\cdots\frac{\delta}{\delta J_{i_n}}Z[J]\bigg|_{J=0}\ . \tag{49}$$

We can now define the generating functional of connected correlation functions $W[J]$ by the relation

$$Z[J] = \mathrm{e}^{\frac{i}{\hbar}W[J]} \qquad \Rightarrow \qquad W[J] = \frac{\hbar}{i}\ln Z[J]\ . \tag{50}$$

One can prove that it generates connected correlation functions by

$$\langle \phi^{i_1} \phi^{i_2} \cdots \phi^{i_n} \rangle_c = \left(\frac{\hbar}{i} \right)^{n-1} \frac{\delta}{\delta J_{i_1}} \frac{\delta}{\delta J_{i_2}} \cdots \frac{\delta}{\delta J_{i_n}} W[J] \Big|_{J=0} . \tag{51}$$

It is also useful to define the effective action $\Gamma[\varphi]$ as the Legendre transform of the $W[J]$ functional

$$\Gamma[\varphi] = \min_J \Big\{ W[J] - J_i \varphi^i \Big\} \tag{52}$$

which can be treated as a classical action that includes all quantum corrections. It also generates the so called one-particle irreducible (1PI) correlation functions. The minimum in J is obtained at $\varphi^i = \frac{\delta W[J]}{\delta J_i}$, a relation that should be inverted to obtain $J_i = J_i(\varphi)$ and inserted into the right hand side of (52) to obtain the effective action as a functional of the variable φ^i.

The last two functionals find their main applications in quantum field theory, but we will have the chance of employing the generating functional of connected correlation functions in later applications of the worldline approach.

4.1 Digression over gaussian integrals

Gaussian integrals in one or more variables are easily computed. For a real variable ϕ they are given by

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} \, \mathrm{e}^{-\frac{1}{2} K \phi^2} &= \frac{1}{\sqrt{K}} \\ \int_{-\infty}^{\infty} \frac{d\phi}{\sqrt{2\pi}} \, \mathrm{e}^{-\frac{1}{2} K \phi^2 + J\phi} &= \frac{1}{\sqrt{K}} \, \mathrm{e}^{\frac{1}{2} \frac{1}{K} J^2} \end{aligned} \tag{53}$$

with K a real positive number. The first one is the standard gaussian integral, whose square is easily computed in polar coordinates. The second one is obtained by square completion, i.e. writing $-\frac{1}{2} K \phi^2 + J\phi = -\frac{1}{2} K (\phi - \frac{J}{K})^2 + \frac{1}{2} \frac{1}{K} J^2$, then shifting the measure from ϕ to $\phi - \frac{J}{K}$ to obtain the desired result.

They are straightforwardly extended to n real variables

$$\begin{aligned} \int \frac{d^n \phi}{(2\pi)^{\frac{n}{2}}} \, \mathrm{e}^{-\frac{1}{2} \phi^i K_{ij} \phi^j} &= (\det K_{ij})^{-\frac{1}{2}} \\ \int \frac{d^n \phi}{(2\pi)^{\frac{n}{2}}} \, \mathrm{e}^{-\frac{1}{2} \phi^i K_{ij} \phi^j + J_i \phi^i} &= (\det K_{ij})^{-\frac{1}{2}} \, \mathrm{e}^{\frac{1}{2} J_i G^{ij} J_j} \end{aligned} \tag{54}$$

where K_{ij} is a real positive definite matrix K_{ij} (all eigenvalues must be strictly positive), and G^{ij} its inverse (so that $K_{ij} G^{jk} = \delta_i^k$). The first integral is immediate if K_{ij} is diagonal, and valid in full generality by noting that K_{ij} is diagonalizable by an orthogonal transformation which leaves the measure invariant. The last integral is obtained again by square completion.

These gaussian integrals are suitable for euclidean path integrals, as in the hypercondensed notation path integrals look very much like ordinary integrals. Of course the definition of determinants for infinite dimensional matrices is delicate and requires a regularization procedure.

By analytical extension one obtains gaussian integrals suitable for quantum mechanics

$$\int \frac{d^n\phi}{(-2\pi i)^{\frac{n}{2}}}\ \mathrm{e}^{-\frac{i}{2}\phi^i K_{ij}\phi^j} = (\det K_{ij})^{-\frac{1}{2}}$$
$$\int \frac{d^n\phi}{(-2\pi i)^{\frac{n}{2}}}\ \mathrm{e}^{-\frac{i}{2}\phi^i K_{ij}\phi^j + iJ_i\phi^i} = (\det K_{ij})^{-\frac{1}{2}}\ \mathrm{e}^{\frac{i}{2}J_iG^{ij}J_j} \tag{55}$$

where again G^{ij} is the inverse of K_{ij}. Convergence to the given values is guaranteed if K_{ij} has a small negative imaginary part (e.g. $K = K_0 - i\epsilon$ with K_0 real and $\epsilon > 0$) that assure a gaussian damping for $|\phi| \to \infty$ (in quantum field theory this corresponds to the causal $i\epsilon$ Feynman prescription). In an hypercondensed notation, with the identification of $\int D\phi \equiv \frac{d^n\phi}{(-2\pi i)^{\frac{n}{2}}}$, these formulae give the formal solution of path integrals and corresponding generating functional for free theories without gauge invariances, in either quantum mechanics or quantum field theory. Gauge invariances correspond to the vanishing of $\det K_{ij}$, and one must apply a gauge fixing procedure to obtain a finite answer.

4.2 Free theory

It is useful to study the free theory as it provides additional intuition. A free theory is described by a quadratic action

$$S[\phi] = -\frac{1}{2}\phi^i K_{ij}\phi^j \tag{56}$$

which gives the linear equations of motion $K_{ij}\phi^j = 0$. We assume K_{ij} invertible so that there are no gauge symmetries. ??? Denoting $\mathcal{D}\phi \equiv \frac{d^n\phi}{(-2\pi i)^{\frac{n}{2}}}$, setting $\hbar = 1$, and using eq. (55), one can calculate the path integral with sources

$$Z[J] = \int \mathcal{D}\phi\ \mathrm{e}^{i(S[\phi]+J_i\phi^i)} = (\det K_{ij})^{-\frac{1}{2}}\ \mathrm{e}^{\frac{i}{2}J_iG^{ij}J_j}\ . \tag{57}$$

Then, recalling eq. (49) one immediately obtains the following correlation functions

$$\begin{aligned} &\langle 1\rangle = 1 \\ &\langle \phi^i\rangle = 0 \\ &\langle \phi^i\phi^j\rangle = -iG^{ij}\ . \end{aligned} \tag{58}$$

The first one is a consequence of the normalization, the second one reflects the symmetry $\phi^i \to -\phi^i$, and the third one is the propagator, which we recognize to be proportional to the inverse of the kinetic matrix K_{ij}.

Continuing with the calculation of higher point functions, we see that all correlation functions with an odd number of points vanish, again a signal of the symmetry $\phi^i \to -\phi^i$. Those with an even number n factorize instead into a sum of $(n-1)!!$ terms given by the product of the 2-point functions which connect any two points in all possible ways, a fact known also as the "Wick theorem". For example, the 4-point correlation function is given by

$$\langle\phi^1\phi^2\phi^3\phi^4\rangle = \langle\phi^1\phi^2\rangle\langle\phi^3\phi^4\rangle + \langle\phi^1\phi^3\rangle\langle\phi^2\phi^4\rangle + \langle\phi^1\phi^4\rangle\langle\phi^2\phi^3\rangle \tag{59}$$

that indeed contains the sum of 3!! terms. This correlation function is not connected, as it disconnects into the sum of products of correlation functions of lower order.

The generating functional of connected correlation functions $W[J]$ is easily obtained from eq. (50)

$$W[J] = \frac{1}{2} J_i G^{ij} J_j - \Lambda \tag{60}$$

where $\Lambda = -\frac{i}{2} \ln \det(K_{ij}) = -\frac{i}{2} \operatorname{tr} \ln(K_{ij})$ is a constant, whose precise value is often not needed. One easily verifies that all of its correlation functions (0- and 2-points) are connected.

Let us also calculate the effective action. The minimum in J of eq. (52) is achieved for

$$\frac{\delta W}{\delta J_i} = \varphi^i \quad \Longrightarrow \quad \varphi^i = G^{ij} J_j \quad \Longrightarrow \quad J_i = K_{ij} \varphi^j \tag{61}$$

so that

$$\Gamma[\varphi] = -\frac{1}{2} \varphi^i K_{ij} \varphi^j - \Lambda \, . \tag{62}$$

Thus we see that for a free theory the effective action $\Gamma[\varphi]$ coincides with the original action $S[\varphi]$ up to the additive constant Λ, that can be interpreted as a vacuum energy of quantum origin. The latter can be disregarded if the gravitational interactions are neglected. In general, the effective action can be considered as a classical action that contains all the effects of quantization in its couplings (and thus should not be quantized again).

4.3 Harmonic oscillator

Let us work out explicitely the case of a harmonic oscillator with unit mass

$$S[x] = \int_{-\infty}^{\infty} dt \Big(\frac{1}{2} \dot{x}^2 - \frac{\omega^2}{2} x^2 \Big) \, , \qquad Z[J] = \int Dx \, e^{\frac{i}{\hbar}(S[x] + \int dt \, Jx)} \tag{63}$$

already solved formally in the previous section. We can repeat briefly the deduction without using the hypercondensed notation. We consider an infinite propagation time and a transition amplitude between the ground state, which classically is achieved for $x = 0$. The action in the exponent can be manipulated with an integration by parts without producing boundary terms. Indeed imposing that $x(t)$ be in its classical vacuum at initial and final times gives a vanishing boundary term (another justification will be given later on when treating the euclidean version of the problem)

$$\begin{aligned} S[x] &= -\int dt \, \frac{1}{2} x(t) \Big(\frac{d^2}{dt^2} + \omega^2 \Big) x(t) \\ &= -\int dt dt' \, \frac{1}{2} x(t) \Big(\frac{d^2}{dt^2} + \omega^2 \Big) \delta(t - t') x(t') \\ &\equiv -\int dt dt' \, \frac{1}{2} x(t) K(t, t') x(t') \end{aligned} \tag{64}$$

where a Dirac delta function $\delta(t - t')$ has been introduced to expose the "kinetic matrix" $K(t, t') = (\frac{d^2}{dt^2} + \omega^2)\delta(t - t')$, i.e. the differential operator of the harmonic oscillator. The inverse of this matrix (the Green function of the differential operator), is easily written in Fourier transform

$$G(t, t') = -\int \frac{dp}{2\pi} \, \frac{e^{-ip(t-t')}}{p^2 - \omega^2} \tag{65}$$

that indeed satisfies the defining equation

$$\int dt''\, K(t,t'')G(t'',t') = \Big(\frac{d^2}{dt^2}+\omega^2\Big)G(t,t') = \delta(t-t')\ . \tag{66}$$

Now one can complete the square in (63) and find

$$\begin{aligned} Z[J] &= \int Dx\, e^{\frac{i}{\hbar}(S[x]+\int dt\, Jx)} \\ &= \int Dx\, \exp\frac{i}{\hbar}\Big[-\int dtdt'\,\Big(\frac{1}{2}x(t)K(t,t')x(t') - J(t)\delta(t-t')x(t') \\ &\qquad \pm\frac{1}{2}J(t)G(t,t')J(t')\Big)\Big] \\ &= \exp\Big(\frac{i}{2\hbar}\int dtdt'\, J(t)G(t,t')J(t')\Big)\underbrace{\int D\tilde{x}\,\exp\Big(-\frac{i}{\hbar}\int dtdt'\,\frac{1}{2}\tilde{x}(t)K(t,t')\tilde{x}(t')\Big)}_{\det{}^{-1/2}[\frac{1}{\hbar}K(t,t')]\equiv N} \\ &= N\exp\Big(\frac{i}{2\hbar}\int dtdt'\, J(t)G(t,t')J(t')\Big)\ . \end{aligned} \tag{67}$$

Adding the Feynman $i\epsilon$ prescription for specifying how to integrates around the poles $p=\pm\omega$ ($\omega^2\to\omega^2-i\epsilon$ with $\epsilon\to 0^+$) one may compute

$$G(t,t') = -\int\frac{dp}{2\pi}\,\frac{e^{-ip(t-t')}}{p^2-\omega^2+i\epsilon} = \frac{i}{2\omega}e^{-i\omega|t-t'|}\ . \tag{68}$$

The 2-point function (i.e. the Feynman propagator D_F) is then

$$\begin{aligned} \langle x(t)x(t')\rangle &= \frac{\int Dx\, x(t)x(t')e^{\frac{i}{\hbar}S[x]}}{\int Dx\, e^{\frac{i}{\hbar}S[x]}} = \frac{1}{Z[0]}\left(\frac{\hbar}{i}\right)^2\frac{\delta^2 Z[J]}{\delta J(t)\delta J(t')}\bigg|_{J=0} = -i\hbar G(t,t') \\ &= \frac{\hbar}{2\omega}e^{-i\omega|t-t'|}\equiv D_F(t,t')\ . \end{aligned} \tag{69}$$

4.3.1 Harmonic oscillator in euclidean time

The statistical partition function in the limit of vanishing temperature ($\Theta\to 0$), or equivalently for an infinite euclidean propagation time ($\beta\to\infty$), takes a simple form

$$Z_E = \mathrm{Tr}\, e^{-\beta\hat{H}} = \sum_n e^{-\beta E_n} \quad\overset{\beta\to\infty}{\longrightarrow}\quad e^{-\beta E_0} + \text{subleading terms.} \tag{70}$$

This is true even in the presence of a source J if one assumes that the source is nonvanishing only in a finite interval of time: the remaining infinite time is sufficient to project the operator $e^{-\beta\hat{H}}$ onto the ground state. This allows us to rewrite the generating functional $Z[J]$ in the euclidean case in a simpler way, justifying in yet another way the dropping of boundary terms in the integration by parts. The statistical partition function is obtained by using periodic boundary conditions, so that for large β one gets the projection onto the ground state

$$\begin{aligned} Z_E[J] &= \int_{PBC} Dx\, e^{-S_E[x]+\int d\tau\, Jx} = \lim_{\beta\to\infty}\Big\langle 0,\frac{\beta}{2}\Big|0,-\frac{\beta}{2}\Big\rangle_J \\ S_E[x] &= \int_{-\infty}^{\infty} d\tau\Big(\frac{1}{2}\dot{x}^2+\frac{\omega^2}{2}x^2\Big) \end{aligned} \tag{71}$$

where we have indicated the transition amplitude between vacuum states using the Heisenberg picture with the subscript J that indicates the presence of a source. We can now repeat the previous calculation in the present context. For closed paths one may integrate by parts without encountering boundary terms, and the path integral is strictly gaussian

$$\begin{aligned} Z_E[J] &= \int_{PBC} Dx\, \exp\Big[-\int d\tau \left(\frac{1}{2}x(\tau)\Big(-\frac{d^2}{d\tau^2}+\omega^2\Big)x(\tau) - J(\tau)x(\tau)\right)\Big] \\ &= N \exp\Big[\frac{1}{2}\int d\tau d\tau'\, J(\tau)G_E(\tau,\tau')J(\tau')\Big] \end{aligned} \tag{72}$$

where the euclidean Green function G_E is given by

$$G_E(\tau,\tau') = \Big[-\frac{d^2}{d\tau^2}+\omega^2\Big]^{-1} = \int \frac{dp_E}{2\pi}\,\frac{e^{-ip_E(\tau-\tau')}}{p_E^2+\omega^2}\ . \tag{73}$$

This euclidean Green function is unique: there are no poles and related prescriptions to specify how to perform the integration. The inverse Wick rotation implies $\tau \equiv t_E \to it_M \equiv it$ and $p_E \to -ip_M \equiv -ip$, with the latter arising form the requirement that the correct Fourier transform is kept during the analytic deformation. Indeed

$$\begin{aligned} G_E(\tau,\tau') &= \int \frac{dp_E}{2\pi}\,\frac{e^{-ip_E(\tau-\tau')}}{p_E^2+\omega^2} \quad \to \\ &\to \quad -i\int \frac{dp_M}{2\pi}\,\frac{e^{-ip_M(t-t')}}{-p_M^2+\omega^2} = -iG_M(t,t') \equiv D_F(t,t') \end{aligned} \tag{74}$$

where $D_F(t,t')$ is the Feynman propagator in (69) (with $\hbar = 1$). Calculating (73), or Wick rotating directly (69), produces the euclidean propagator

$$\langle x(\tau)x(\tau')\rangle = \frac{1}{2\omega}e^{-\omega|\tau-\tau'|}\ . \tag{75}$$

We recognize that the Feynman propagator is the unique analytical extension of the euclidean two point function. All other Green functions, such as the retarded or advanced ones, correspond to different boundary conditions that can be implemented with different prescriptions for performing the integration around the poles. They cannot be Wick rotated as one would encounter poles in the analytic continuation.

5 Perturbative expansion

The free theory corresponds to a gaussian path integral which is exactly solvable. With interactions one is often unable to compute exactly the path integral, and one must resort to some sort of approximation. The simplest one is the perturbative expansion around a free theory, which consists in expanding the solution in a power series in the coupling constants that parametrize the interactions. If the couplings are small enough, the perturbative expansion might give a good approximation to the searched for solution.

We describe the perturbative expansion taking as guiding example the case of an anharmonic oscillator

$$S[x] = \int dt\Big(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2 - \frac{g}{3!}x^3 - \frac{\lambda}{4!}x^4\Big)\ . \tag{76}$$

When the coupling constants g and λ vanish, the theory is exactly solvable. Thus one may try to include perturbatively the corrections that arise when g and λ are small enough. It is convenient to split the action as the sum of two terms, a free part S_0 and an interacting one S_{int}

$$\begin{aligned} S[x] &= S_0[x] + S_{int}[x] \\ S_0[x] &= \int dt \left(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2\right) \\ S_{int}[x] &= \int dt \left(- \frac{g}{3!}x^3 - \frac{\lambda}{4!}x^4\right) . \end{aligned} \tag{77}$$

Now one can expand in a Taylor series the exponential of the interaction term under the path integral

$$\begin{aligned} Z[J] &= \int \mathcal{D}x\, e^{\frac{i}{\hbar}(S[x]+\int Jx} \\ &= \int \mathcal{D}x\, e^{\frac{i}{\hbar}(S_0[x]+S_{int}[x]+\int Jx)} \\ &= \int \mathcal{D}x\, e^{\frac{i}{\hbar}S_{int}[x]}\, e^{\frac{i}{\hbar}(S_0[x]+\int Jx)} \\ &= \int \mathcal{D}x\, \Big[1 + \frac{i}{\hbar}S_{int}[x] + \frac{1}{2}\Big(\frac{i}{\hbar}S_{int}[x]\Big)^2 + \cdots \\ &\qquad \cdots + \frac{1}{n!}\Big(\frac{i}{\hbar}S_{int}[x]\Big)^n + \cdots\Big]\, e^{\frac{i}{\hbar}(S_0[x]+\int Jx)} \end{aligned} \tag{78}$$

Equivalently, in an obvious notation,

$$Z[J] = \left\langle e^{\frac{i}{\hbar}S_{int}[x]} \right\rangle_{U,0,J} \tag{79}$$

where the subscripts $U, 0, J$ denote un-normalized averaging (U) with the free theory (0) and in the presence of a source (J). This last expression is sometimes called "Dyson formula". It generates immediately the perturbative expansion in terms of Feynman diagrams.

An equivalent way of writing the perturbative series is the following one

$$\begin{aligned} Z[J] &= \int \mathcal{D}x\; e^{\frac{i}{\hbar}(S[x]+\int Jx)} = \int \mathcal{D}x\; e^{\frac{i}{\hbar}S_{int}[x]}\, e^{\frac{i}{\hbar}(S_0[x]+\int Jx)} \\ &= e^{\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}]} \int \mathcal{D}q\; e^{\frac{i}{\hbar}(S_0[x]+\int Jx)} \\ &= e^{\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}]}\, Z_0[J] \end{aligned} \tag{80}$$

which presents the solution as a (quite complicated) differential operator acting on the solution of the free theory $Z_0[J]$. In particular, all vacuum diagrams are generated by

$$Z[0] = \int \mathcal{D}x\; e^{\frac{i}{\hbar}S[x]} = e^{\frac{i}{\hbar}S_{int}[\frac{\hbar}{i}\frac{\delta}{\delta J}]}\, Z_0[J]\Big|_{J=0} . \tag{81}$$

The expansion in terms of Feynman diagrams is obtained expanding the interactions potential and using the Wick theorem to compute the correlation functions of the free theory: the vertices generated by the interactions potential contain a coupling constant and quantum variables that are tied together two by two in all possible ways with the free propagators ((graphically each vertex is denoted by a dot and propagators are denoted by lines). This will be exemplified next in the case of vacuum diagrams for the anharmonic oscillator.

5.1 Vacuum diagrams

As a first example we compute perturbatively the corrections to the ground state energy of the harmonic oscillator due to the anharmonic potential terms. As already mentioned, it is often the case that one computes using the euclidean version of the theory and only at the very end performs the inverse Wick rotation to obtain the results in minkowskian time. Thus we will proceed with the euclidean version of the theory, which is the one used later on in presenting worldline applications.

Therefore we wish to compute

$$\begin{aligned} Z_E[J] &= \int \mathcal{D}x\; e^{-S_E[x]+\int Jx} \\ S_E[x] &= \lim_{\beta\to\infty}\int_{-\beta/2}^{\beta/2} d\tau \left(\frac{1}{2}\dot{x}^2+\frac{\omega^2}{2}x^2+\frac{g}{3!}x^3+\frac{\lambda}{4!}x^4\right) \end{aligned} \tag{82}$$

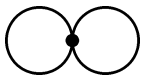
with $\beta\to\infty$. The corrections to the ground state energy can be recognized from the calculation of

$$\begin{aligned} Z_E[0] &= \langle 1\rangle_U = \lim_{\beta\to\infty}\langle 0|e^{-\beta\hat{H}}|0\rangle = \lim_{\beta\to\infty} e^{-\beta E_0} \\ &= \left\langle e^{-S_{E,int}[x]}\right\rangle_{U,0} = \lim_{\beta\to\infty} e^{-\beta(E_0^{(0)}+\Delta E_0)} \end{aligned} \tag{83}$$

where the exact energy E_0 of the ground state $|0\rangle$ differs from the ground state energy of the harmonic oscillator $E_0^{(0)}$ by the term ΔE_0 due to the anharmonic potential. The latter can be obtained perturbatively. We compute now the first non vanishing corrections.

Let us consider first the case with $g=0$ and focus on the first correction in λ

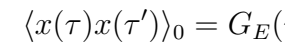
$$\begin{aligned} Z_E[0] &= \langle 1\rangle_U = \left\langle e^{-S_{E,int}[x]}\right\rangle_{U,0} = \left\langle (1-S_{E,int}[x]+\cdots)\right\rangle_{U,0} \\ &= \langle 1\rangle_{U,0} - \frac{\lambda}{4!}\int_{-\beta/2}^{\beta/2} d\tau\,\langle x^4(\tau)\rangle_{U,0}+\cdots \\ &= \langle 1\rangle_{U,0}\left[1-\frac{\lambda}{4!}\int_{-\beta/2}^{\beta/2} d\tau\,\langle x^4(\tau)\rangle_0+\cdots\right] \\ &= \langle 1\rangle_{U,0}\left[1-\frac{\lambda}{4!}\left[3\times\right.\right.\right. \end{aligned}$$



$$\left.\left.\left.\right]+\cdots\right]\right. . \tag{84}$$

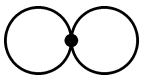
In the last line we have used Wick contractions to calculate normalized correlations functions in the free theory, and used also a graphical representation in terms of Feynman diagrams. In this graphical representation a line denotes a propagator that joins two points in time, while vertices arising form the interactions are denoted by dots. Recalling the euclidean propagator, computed in eq. (75),

$$\langle x(\tau)x(\tau')\rangle_0 = G_E(\tau-\tau') = \frac{1}{2\omega}e^{-\omega|\tau-\tau'|} = \quad \tau$$



$$\tau' \tag{85}$$

one immediately finds



$$= \int_{-\beta/2}^{\beta/2} d\tau\, G_E^2(0) = \frac{\beta}{4\omega^2}\ . \tag{86}$$

Thus, to this perturbative order one gets

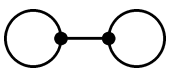
$$Z_E[0] = \langle 1 \rangle_{U,0} \Big[1 - \frac{\lambda}{4!} \Big[3 \frac{\beta}{4\omega^2} \Big] + \cdots \Big] = \langle 1 \rangle_{U,0} \; e^{-\frac{\beta\lambda}{32\omega^2} + \cdots} \tag{87}$$

so that

$$\Delta E_0 = \frac{1}{32} \frac{\lambda}{\omega^2} \; . \tag{88}$$

Similarly one may consider the case with $g \neq 0$ and $\lambda = 0$. The first non vanishing correction is obtained from

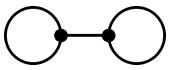
$$\begin{aligned} Z_E[0] &= \langle 1 \rangle_U = \Big\langle \Big(1 - S_{E,int} + \frac{1}{2} S^2_{E,int} + \cdots \Big) \Big\rangle_{U,0} \qquad (89) \\ &= \langle 1 \rangle_{U,0} - \frac{g}{3!} \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \, \langle x^3(\tau) \rangle_{U,0} \\ &\quad + \frac{1}{2} \Big(\frac{g}{3!} \Big)^2 \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} d\tau' \, \langle x^3(\tau) x^3(\tau') \rangle_{U,0} + \cdots \\ &= \langle 1 \rangle_{U,0} \Big[1 + 0 + \frac{1}{2} \Big(\frac{g}{3!} \Big)^2 \Big[3! \times \; + \, 3^2 \times \; \Big] + \cdots \Big] \; . \end{aligned}$$

Now

$$\begin{aligned} \; &= \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' \, G_E^3(\tau - \tau') = \frac{1}{8\omega^3} \int_{-\beta/2}^{\beta/2} d\tau \int_{-\infty}^{\infty} d\sigma \, e^{-3\omega|\sigma|} \\ &= \frac{\beta}{8\omega^3} \frac{2}{3\omega} \end{aligned} \tag{90}$$

and



$$\begin{aligned} &= \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' \, G_E(0) G_E(\tau - \tau') G_E(0) \\ &= \frac{1}{8\omega^3} \int_{-\beta/2}^{\beta/2} d\tau \int_{-\infty}^{\infty} d\sigma \, e^{-\omega|\sigma|} = \frac{\beta}{8\omega^3} \frac{2}{\omega} \end{aligned} \tag{91}$$

where the limit $\beta \to \infty$ has been used suitably to calculate some integrals. Therefore

$$Z_E[0] = \langle 1 \rangle_{U,0} \Big[1 + \frac{1}{2} \Big(\frac{g}{3!} \Big)^2 \Big(3! \frac{\beta}{12\omega^4} + 3^2 \frac{\beta}{4\omega^4} \Big) + \cdots \Big] = \langle 1 \rangle_{U,0} \; e^{\beta \frac{11}{8(3!)^2} \frac{g^2}{\omega^4} + \cdots} \tag{92}$$

and one finds

$$\Delta E_0 = -\frac{11}{288} \frac{g^2}{\omega^4} \; . \tag{93}$$

5.2 Heat kernel

A second example to illustrate perturbation theory is the calculation of the heat kernel using path integrals. The heat kernel is the fundamental solution of the heat equation

$$-\frac{\partial}{\partial\beta}\psi = \hat{H}\psi \tag{94}$$

where $\hat{H}$ is a second order elliptic differential operator. It can be obtained by Wick rotating the Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = \hat{H}\psi \tag{95}$$

with $t \to -i\beta$, as discussed previously. We consider the simple case of an hamiltonian with a smooth scalar potential V

$$\hat{H} = -\frac{1}{2}\nabla^2 + V(x) \tag{96}$$

where ∇^2 is the laplacian in cartesian coordinates x^i on $\mathbb{R}^D$.

The heat kernel is the fundamental solution that can be represented in operatorial form by

$$\psi(x, y; \beta) = \langle y|e^{-\beta\hat{H}}|x\rangle \ . \tag{97}$$

It satisfies eq. (94) plus the boundary condition

$$\psi(x, y; 0) = \delta^D(x - y) \ . \tag{98}$$

It is well-known that the solution in the free case (i.e. for $V = 0$) is given by

$$\psi(x, y; \beta) = \frac{1}{(2\pi\beta)^{\frac{D}{2}}}\, e^{-\frac{(x-y)^2}{2\beta}} \ . \tag{99}$$

The path integral which computes the transition amplitude in euclidean time (97) can be written as

$$\psi(x, y; \beta) = \int_{x(0)=x}^{x(\beta)=y} \mathcal{D}x \ e^{-S[x]} \tag{100}$$

where the symbol $\int_{x(0)=x}^{x(\beta)=y} \mathcal{D}x$ indicates the sum over all functions $x^i(t)$ which satisfy the boundary conditions $x^i(0) = x^i$ and $x^i(\beta) = y^i$, whereas the euclidean action $S[x]$ is given by

$$S[x] = \int_0^\beta dt \left(\frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + V(x)\right) \ . \tag{101}$$

No confusion should arise in denoting the paths by $x^i(t)$ and their boundary values at $t = 0$ by $x^i(0) = x^i$. The path integral cannot be computed exactly for an arbitrary potential V, but one can compute it perturbatively for small propagation times β, assuming the potential and its derivatives to be functions that vanish sufficiently fast at infinity.

To start with let us rescale the euclidean time variable $t = \beta\tau$, so that $\tau \in [0, 1]$. The action (101) can now be written as follows

$$S[x] = \frac{1}{\beta}\int_0^1 d\tau \left(\frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j + \beta^2 V(x)\right) \tag{102}$$

where of course dots (as in $\dot{x}^i$) now represent derivatives with respect to τ. This rescaling is useful since we are going to compute the path integral in a perturbative expansion valid for small β.

One can decompose all paths by

$$x^i(\tau) = x^i_{bg}(\tau) + q^i(\tau) \tag{103}$$

where $x^i_{bg}(\tau)$ is a fixed path (sometimes called the background path, or the classical path) which can be taken to satisfy the boundary conditions and the classical equations of motion for $V = 0$. Thus

$$x^i_{bg}(\tau) = x^i + (y^i - x^i)\tau \tag{104}$$

as it satisfies the boundary conditions $x^i_{bg}(0) = x^i$ and $x^i_{bg}(1) = y^i$, as well as the equation of motion $\ddot{x}^i_{bg}(\tau) = 0$ valid for $V = 0$. Indeed for small β the potential can be neglected, as evident from (102). The remaining arbitrary "quantum fluctuations" $q^i(\tau)$ must then have vanishing boundary conditions, $q^i(0) = q^i(1) = 0$.

The inclusion of an arbitrary potential V makes the problem quite difficult to solve in full generality. However it can be treated in perturbation theory. The emerging solution will be of the form

$$\psi(x, y; \beta) = \frac{1}{(2\pi\beta)^{\frac{D}{2}}}\, e^{-\frac{(x-y)^2}{2\beta}} \Big(a_0(x, y) + a_1(x, y)\beta + a_2(x, y)\beta^2 + \dots \Big) \tag{105}$$

where the so-called Seeley-DeWitt coefficients a_n depend on the points x^i and y^i and on the potential V. Instead of y^i it is useful to use the displacement variable

$$\xi^i = (y^i - x^i) \tag{106}$$

whose length (i.e. the distance between the two points) may be considered of order $\sqrt{\beta}$ for the brownian motion. Thus β controls the perturbative expansion. Of course $a_0(x, y) = 1$, as dictated by the free theory.

5.2.1 Perturbative expansion

The perturbative expansion is based on the gaussian averages of the path integral with the free quadratic action S_0 (i.e. the one with $V = 0$), namely

$$\begin{aligned} A &= \int \mathcal{D}q\, e^{-S_0[q]} = \frac{1}{(2\pi\beta)^{\frac{D}{2}}} \\ \langle q^i(\tau) \rangle &= \frac{1}{A} \int \mathcal{D}q\, q^i(\tau)\, e^{-S_0[q]} = 0 \\ \langle q^i(\tau) q^j(\sigma) \rangle &= \frac{1}{A} \int \mathcal{D}q\, q^i(\tau) q^j(\sigma)\, e^{-S_0[q]} = -\beta\delta^{ij}\Delta(\tau, \sigma) \end{aligned} \tag{107}$$

and so on. Here $\Delta(\tau, \sigma)$ is the Green function of the operator $\frac{\partial^2}{\partial\tau^2}$ on the space of functions $f(\tau)$ with vanishing boundary conditions at $\tau = 0$ and $\tau = 1$

$$\frac{\partial^2}{\partial\tau^2}\Delta(\tau, \sigma) = \delta(\tau - \sigma)\ . \tag{108}$$

It reads (for τ and σ in $[0,1]$)

$$\Delta(\tau,\sigma) = (\tau-1)\sigma\,\theta(\tau-\sigma) + (\sigma-1)\tau\,\theta(\sigma-\tau) \tag{109}$$

where $\theta(x)$ is the standard step function ($\theta(x)=1$ for $x>0$, $\theta(0)=1/2$ for $x=0$, and $\theta(x)=0$ for $x<0$). Indeed it satisfies the differential equation with the correct boundary conditions. The two point function $\langle q^i(\tau)q^j(\sigma)\rangle$ is the propagator in the free theory.

Recalling that we indicate the normalized average of an arbitrary functional $F[q]$ by

$$\langle F[q]\,\rangle = \frac{1}{A}\int \mathcal{D}q\, F[q]\, e^{-S_0[q]} \tag{110}$$

we compute the perturbative expansion as follows. The action is written as

$$S[x] = S_0[x] + S_{int}[x] \tag{111}$$

where the free part is given by

$$S_0[q] = \frac{1}{\beta}\int_0^1 d\tau\, \frac{1}{2}\delta_{ij}\dot{x}^i\dot{x}^j \tag{112}$$

and the interaction part by

$$S_{int}[x] = \beta\int_0^1 d\tau\, V(x)\ . \tag{113}$$

The path integral can now be manipulated as follows

$$\begin{aligned}
&\int_{x(0)=x}^{x(\beta)=y}\mathcal{D}x\ e^{-S[x]} = \int_{x(0)=x}^{x(\beta)=y}\mathcal{D}x\ e^{-(S_0[x]+S_{int}[x])}\\
&= e^{-S_0[x_{bg}]}\int_{q(0)=0}^{q(\beta)=0}\mathcal{D}q\,\Big(e^{-S_{int}[x_{bg}+q]}\Big)e^{-S_0[q]}\\
&= A\,e^{-S_0[x_{bg}]}\Big\langle e^{-S_{int}[x_{bg}+q]})\Big\rangle\\
&= \frac{1}{(2\pi\beta)^{\frac{D}{2}}}\,e^{-\frac{(x-y)^2}{2\beta}}\Big\langle\Big(1-S_{int}[x_{bg}+q]+\frac{1}{2}S^2_{int}[x_{bg}+q]+\cdots\Big)\Big\rangle\ .
\end{aligned} \tag{114}$$

The transition from the first to the second line is due to the translation invariance of the path integral measure, $\mathcal{D}x = \mathcal{D}(x_{bg}+q) = \mathcal{D}q$, since $x_{bg}(\tau) = x^i + \xi^i\tau$ is a fixed function. Then we used the notation in (110) to indicate normalized averages with the free path integral. Finally, the perturbative expansion is generated by expanding the interaction part, as shown in the last line.

Let us compute systematically the various terms appearing in the last line of eq. (114). The first one is trivial

$$\langle 1\rangle = 1 \tag{115}$$

since correlations functions are normalized. Next we have to consider $\langle S_{int}[x_{bg}+q]\rangle$. We can Taylor expand the potential around the initial point x^i

$$\begin{aligned}
S_{int}[x_{bg}+q] &= \beta\int_0^1 d\tau\, V(x_{bg}+q) \\
&= \beta\int_0^1 d\tau\,\Big(V(x) + [\xi^i\tau + q^i(\tau)]\partial_i V(x)\\
&+ \frac{1}{2}[\xi^i\tau+q^i(\tau)][\xi^j\tau+q^j(\tau)]\,\partial_i\partial_j V(x) + \cdots\Big)
\end{aligned} \tag{116}$$

from which one obtains

$$\begin{aligned}\langle -S_{int}[x_{bg}+q]\rangle &= -\beta V(x) - \frac{\beta}{2}\xi^i\partial_i V(x) - \frac{\beta}{6}\xi^i\xi^j\partial_i\partial_j V(x) \\ &- \frac{\beta}{2}\partial_i\partial_j V(x)\int_0^1 d\tau\,\langle q^i(\tau)q^j(\tau)\rangle + \cdots \end{aligned} \tag{117}$$

with the last term computed using the free propagator in (107) and (109)

$$\begin{aligned}\int_0^1 d\tau\,\langle q^i(\tau)q^j(\tau)\rangle &= \int_0^1 d\tau\,(-\beta\delta^{ij}\Delta(\tau,\tau)) \\ &= -\beta\delta^{ij}\int_0^1 d\tau\,\tau(\tau-1) = \frac{\beta}{6}\delta^{ij}\end{aligned} \tag{118}$$

so that

$$\begin{aligned}\langle -S_{int}[x_{bg}+q]\rangle &= -\beta V(x) - \frac{\beta}{2}\xi^i\partial_i V(x) - \frac{\beta}{6}\xi^i\xi^j\partial_i\partial_j V(x) \\ &- \frac{\beta}{12}\nabla^2 V(x) + \cdots\ .\end{aligned} \tag{119}$$

Similarly, at lowest order in β one gets for the next term in (114)

$$\left\langle \frac{1}{2}S_{int}^2[x_{bg}+q]\right\rangle = \frac{\beta^2}{2}V^2(x) + \cdots\ . \tag{120}$$

Collecting all the terms, we find that at this order the heat kernel is given by

$$\begin{aligned}\psi(x,y;\beta) &= \frac{1}{(2\pi\beta)^{\frac{D}{2}}}\,e^{-\frac{(x-y)^2}{2\beta}}\Big[1-\beta V(x) - \frac{\beta}{2}\xi^i\partial_i V(x) - \frac{\beta}{6}\xi^i\xi^j\partial_i\partial_j V(x) \\ &- \frac{\beta^2}{12}\nabla^2 V(x) + \frac{\beta^2}{2}V^2(x) + \cdots\Big]\end{aligned} \tag{121}$$

from which one reads off the expansion around the point x of the Seeley-DeWitt coefficients a_0, a_1 and a_2

$$\begin{aligned}a_0(x,y) &= 1 \\ a_1(x,y) &= -V(x) - \frac{1}{2}\xi^i\partial_i V(x) - \frac{1}{6}\xi^i\xi^i\partial_j\partial_j V(x) + \cdots \\ a_2(x,y) &= \frac{1}{2}V^2(x) - \frac{1}{12}\nabla^2 V(x) + \cdots\ .\end{aligned} \tag{122}$$

In particular, their values at coinciding points $y^\mu = x^\mu$ (i.e. for $\xi^\mu = 0$) are given exactly by

$$\begin{aligned}a_0(x,x) &= 1 \\ a_1(x,x) &= -V(x) \\ a_2(x,x) &= \frac{1}{2}V^2(x) - \frac{1}{12}\nabla^2 V(x)\ .\end{aligned} \tag{123}$$

This calculation exemplifies again perturbation theory and the use of the path integrals to compute heat kernels.

5.3 Gaussian formulae

We collect here some gaussian formulae used in the previous examples plus some additional ones to be used later on in the book. We use the hypercondensed notation for quantum mechanics and quantum field theory in euclidean time, and setting $\mathcal{D}\phi = \frac{d^n\phi}{(2\pi)^{\frac{n}{2}}}$ one also recovers the finite dimensional case.

The required formulae arise from the following gaussian integrals

$$\begin{aligned} Z &\equiv \int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi^i K_{ij}\phi^j} = (\det K_{ij})^{-\frac{1}{2}} \\ Z[J] &\equiv \int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi^i K_{ij}\phi^j + J_i\phi^i} = (\det K_{ij})^{-\frac{1}{2}}\, e^{\frac{1}{2}J_i G^{ij} J_j} \end{aligned} \tag{124}$$

where G^{ij} is the inverse of the kinetic matrix K_{ij} (i.e. $K_{ij}G^{jk} = \delta_i^k$) and gives the propagator. It corresponds to a Green function when K_{ij} is a differential operator. The first one is the standard gaussian integral and the second one is obtained by "completing the square" and shifting integration variables.

The normalized correlation functions follow from differentiating $Z[J]$

$$\begin{aligned} \langle \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n}\rangle &= \frac{1}{Z}\int D\phi\ \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n} e^{-\frac{1}{2}\phi^i K_{ij}\phi^j} \\ &= \frac{1}{Z}\frac{\delta^n}{\delta J_{i_1} J_{i_2}\cdots J_{i_n}} Z[J]\Big|_{J=0} \\ &= \frac{\delta^n}{\delta J_{i_1} J_{i_2}\cdots J_{i_n}} e^{\frac{1}{2}J_i G^{ij} J_j}\Big|_{J=0} \end{aligned} \tag{125}$$

obtaining in particular

$$\begin{aligned} &\langle 1\rangle = 1 \\ &\langle \phi^i\rangle = 0 \\ &\langle \phi^i\phi^j\rangle = G^{ij} \\ &\langle \phi^i\phi^j\phi^k\rangle = 0 \\ &\langle \phi^i\phi^j\phi^k\phi^l\rangle = G^{ij}G^{kl} + G^{ik}G^{jl} + G^{il}G^{jk} \end{aligned} \tag{126}$$

and so on. In particular, correlation functions of an odd number of fields ϕ^i vanish. Those with an even number of fields are given by sums of products of two-point functions. The two-point function is also known as the Feynman propagator.

In the second part of the book we will need more general correlation functions containing the insertion of an exponential $e^{ip_i\phi^i}$. Denoting by $Z[J,p]$ the following functional

$$Z[J,p] = \int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi^i K_{ij}\phi^j + (J_i + ip_i)\phi^i} = (\det K_{ij})^{-\frac{1}{2}}\, e^{\frac{1}{2}(J_i+ip_i)G^{ij}(J_j+ip_j)} \tag{127}$$

with $Z[0,0] = Z$, we find again that the desired correlation functions are obtained by differentiating with respect to the source J and then setting it to zero

$$\begin{aligned} \langle \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n} e^{ip_k\phi^k}\rangle &= \frac{1}{Z}\int D\phi\ \phi^{i_1}\phi^{i_2}\cdots\phi^{i_n} e^{ip_k\phi^k}\, e^{-\frac{1}{2}\phi^i K_{ij}\phi^j} \\ &= \frac{1}{Z}\frac{\delta^n}{\delta J_{i_1} J_{i_2}\cdots J_{i_n}} Z[J,p]\Big|_{J=0} \\ &= \frac{\delta^n}{\delta J_{i_1} J_{i_2}\cdots J_{i_n}} e^{\frac{1}{2}(J_m+ip_m)G^{mn}(J_n+ip_n)}\Big|_{J=0} \end{aligned} \tag{128}$$

obtaining in particular

$$\begin{aligned}
&\langle e^{ip_k\phi^k}\rangle = e^{-\frac{1}{2}p_m G^{mn} p_n} \\
&\langle \phi^i e^{ip_k\phi^k}\rangle = iG^{ij}p_j\, e^{-\frac{1}{2}p_m G^{mn} p_n} \\
&\langle \phi^i\phi^j e^{ip_k\phi^k}\rangle = (G^{ij} - p_k G^{ki} p_l G^{lj})\, e^{-\frac{1}{2}p_m G^{mn} p_n}
\end{aligned} \tag{129}$$

and so on. The exponential $e^{ip_k\phi^k}$ correspond to vertex "operators" appearing in the worldline formalism.

These identities are also proven in QFT by using Wick's theorem: in the form adapted to QFT they were derived by Gian-Carlo Wick using operators. The mnemonics to write them down, without computing again the derivatives with respect to the source is as follows.

In the absence of the exponential (i.e. with $p_i = 0$), one connects any two quantum fields in all possible ways, and each pair of fields that is connected (called a Wick-contraction, or simply contraction) is substituted by the propagator G^{ij}. As an example, in the 4-point function in (126) one contracts the first field with any other (it can be done in three ways) and substitutes this first pair with the propagator, which is a function that is brought outside the correlation function. The remaining last two fields necessarily gets paired together and substituted again by the propagator. Thus we get a sum of 3!! terms containing the products of 2 free propagators. Similarly, the correlation function of $2m$ fields is given by the sum of $(2m-1)!!$ terms containing each the products of m free propagators.

In the presence of the exponential, one again connects any field not in the exponent with any other in all possible ways, but in addition it can be Wick-contracted with the exponential as well. In the latter case it acts as a derivation: it regenerates the exponential (that remains in the correlation function and participates to further contractions) multiplied by the contraction of the field with the exponent, that produces a propagator times the factor associated to the field in the exponent. One continues this procedure iteratively with all the fields left which are not in the exponent. Eventually, there is only the exponential left, that gives rise to the first line in (129). Again, an example can clarify this recipe. Let us consider the 2-point function in (129). The first term in the bracket is due to the Wick-contraction of the two fields together, the second one corresponds to the Wick-contraction of each field with the exponential.

Finally, let us collect here in the hypercondensed notation the formulae defining the various generating functionals in euclidean conventions. They find their most useful application in path integrals for QFTs. The generating functional for connected correlation functions $W[J]$ is obtained from the path integral with sources by

$$Z[J] = e^{W[J]} = \int D\phi\; e^{-S[\phi]+J_i\phi^i} \tag{130}$$

so that

$$W[J] = \ln Z[J]\ . \tag{131}$$

The effective action is defined by the Legendre transform

$$\Gamma[\phi] = \min_J \Big\{ J_i\phi^i - W[J] \Big\}\ . \tag{132}$$

The functionals $S[\phi]$, $Z[J]$, $W[J]$, $\Gamma[\phi]$ are very useful especially in quantum field theory. For

a free theory (without gauge invariances) one has

$$\begin{aligned} S[\phi] &= \frac{1}{2}\phi^i K_{ij}\phi^j \\ Z[J] &= (\det K)^{-\frac{1}{2}} e^{\frac{1}{2}J_i G^{ij} J_j} \\ W[J] &= \frac{1}{2}J_i G^{ij} J_j - \frac{1}{2}\mathrm{tr}\,\ln K \\ \Gamma[\phi] &= \frac{1}{2}\phi^i K_{ij}\phi^j + \frac{1}{2}\mathrm{tr}\,\ln K\ . \end{aligned} \tag{133}$$

The constant tr $\ln K$ generically depends on the coupling constants of the theory and on eventual background fields . When computing one loop effective actions for QFT in the presence of external fields, but without the source for the quantum variable itself, this constant identifies the (one-loop) effective action. Indeed, setting $\phi^i = 0$ in there last formula one gets

$$\Gamma \equiv \Gamma[0] = \frac{1}{2}\mathrm{tr}\,\ln K\ . \tag{134}$$

Using the identity

$$\ln\frac{a}{b} = -\int_0^\infty \frac{dT}{T}\left(e^{-aT} - e^{-bT}\right)\ , \tag{135}$$

proven by verifying that both sides identify the same function (both sides have the same derivative in a and the same value for $a = b$), and formally extending it to operators, one finds the representation of the effective action in terms of the Fock-Schwinger proper time T

$$\Gamma = \frac{1}{2}\mathrm{tr}\,\ln K = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\ \mathrm{tr}\,\left(e^{-KT} - e^{-T}\right)\ . \tag{136}$$

Dropping the last (infinite) constant, that eventually is taken care of in the QFT renormalization procedure, one finds the famous Schwinger formula

$$\Gamma = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\ \mathrm{tr}\,e^{-KT}\ . \tag{137}$$

In subsequent parts of the book, this one-loop QFT effective action is reinterpreted and analyzed in terms of worldline path integrals.