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General relativity

(notes for "Relativity" a.a. 2020/21)

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1 Foreword

There are many books on General Relativity. For this course I will follow mostly
S. Weinberg: "Gravitation and Cosmology", John Wiley & Sons, 1972
for an introduction to tensor analysis and the derivation of Einstein's equation (see chapters 3,4,5,6,7) and
R. D'Inverno: "Introducing Einstein Relativity", Oxford University Press, 1992
for additional discussions on the classical tests, the Schwarzschild black hole solution, and gravitational waves.

These notes will be used to describe details or fill gaps in the presentation, if necessary, leaving the above textbooks as the main source for studying general relativity.

2 The principle of equivalence of gravitation and inertia

This is described in Chapter 3 of [1].

3 The principle of general covariance and tensor analysis

This is described in Chapter 4 of [1].

4 Effects of gravitation

This is described in Chapter 5 of [1].

5 Curvature

This is described in Chapter 6 of [1].

In class, we used slightly different conventions. They are as follows.

Given the metric $g_{\mu\nu}(x)$, interpreted as the potential of gravitational forces, it defines invariant lenghts on spacetime by

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \ . \tag{1}$$

In particular, the squared proper time of an object travelling for an infinitesimal distance dx^μ in spacetime is given by $d\tau^2 = -g_{\mu\nu}(x)dx^\mu dx^\nu$. The metric is a tensor with transformation properties indicated by its index structure

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \ . \tag{2}$$

Derivatives of tensors are not tensors, so that it is useful to introduce the concept of *covariant derivative*. It is defined by the property that acting on tensors it produces new tensors (with an additional index, of course). The covariant derivative makes use of the affine connection $\Gamma^{\lambda}_{\mu\nu}$ constructed out of the metric (known as the Levi-Civita connection or Christoffel symbols)

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \ . \tag{3}$$

Then, the covariant derivative of vectors is defined by

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}_{\mu\lambda} V^\lambda \tag{4}$$
$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^{\lambda}_{\mu\nu} V_\lambda \tag{5}$$

and similarly for tensors (which will have a connection for each index).

Covariant derivatives do not commute. They may be used to define implicitly the Riemann tensor $R_{\mu\nu}{}^{\lambda}{}_{\rho}$ by

$$[\nabla_\mu, \nabla_\nu] V^\lambda = R_{\mu\nu}{}^{\lambda}{}_{\rho} V^\rho \ . \tag{6}$$

Indeed, the left-hand side is a tensor, so must be the right-hand side, and in particular the Riemann tensor $R_{\mu\nu}{}^{\lambda}{}_{\rho}$. The latter is manifestly antisymmetric under the exchange of the indices μ, ν. A direct calculation shows that

$$R_{\mu\nu}{}^{\lambda}{}_{\rho} = \partial_\mu \Gamma^{\lambda}_{\nu\rho} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\sigma}_{\mu\rho} \ . \tag{7}$$

A useful mnemonic for remembering this structure is to write

$$R_{\mu\nu}{}^{\lambda}{}_{\rho} = \bar{\nabla}_\mu \Gamma^{\lambda}_{\nu\rho} - (\mu \leftrightarrow \nu) \tag{8}$$

where $\bar{\nabla}_\mu$ contains a connection for the upper indices only (in general, covariant derivatives are defined for tensors only).

Algebraic properties of the Riemann tensor are best written lowering the upper index with the metric, and are the following ones

$$R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu} \qquad (\textit{symmetry}) \tag{9}$$
$$R_{\mu\nu\lambda\rho} = -R_{\nu\mu\lambda\rho} = -R_{\mu\nu\rho\lambda} \qquad (\textit{antisymmetry}) \tag{10}$$
$$R_{\mu\nu\lambda\rho} + R_{\lambda\mu\nu\rho} + R_{\nu\lambda\mu\rho} = 0 \qquad (\textit{cyclicity}) \ . \tag{11}$$

A brute force way of proving them is to write them down explicitly in terms of the metric. The additional tensors that can be constructed by index contraction are

$$R_{\mu\nu} = R_{\lambda\mu}{}^{\lambda}\nu \qquad \text{(Ricci tensor)} \tag{12}$$
$$R = g^{\mu\nu} R_{\mu\nu} \qquad \text{(Ricci scalar or curvature scalar)} \tag{13}$$

as other contractions will not give rise to independent structures.

From (9) it follows that the Ricci tensor is symmetric

$$R_{\mu\nu} = R_{\nu\mu} \ . \tag{14}$$

One can compute the number of independent components C_D of the Riemann tensor in arbitrary dimensions D. They are given by

$$\begin{aligned} C_D &= \frac{1}{2}\Big(\frac{1}{2}D(D-1)\Big)\Big(\frac{1}{2}D(D-1)+1\Big) - \frac{D(D-1)(D-2(D-3)}{4!} \\ &= \frac{1}{12}D^2(D^2-1) \end{aligned} \tag{15}$$

with a few values given in the following table

D	D^4	C_D
1	1	0
2	16	1
3	81	6
4	256	20
5	625	50

5.1 Bianchi identities

The Riemann tensors satisfies the following differential Bianchi identities

$$\nabla_\mu R_{\nu\lambda\alpha\beta} + \nabla_\nu R_{\lambda\mu\alpha\beta} + \nabla_\lambda R_{\mu\nu\alpha\beta} = 0 \ . \tag{16}$$

The ciclic sum of the first three indices makes the sum totally antisymmetric in those indices.

One may contract the Bianchi identities on the indices (ν, α) (i.e. multiplying by $g^{\nu\alpha}$) to find

$$\nabla_\mu R_{\lambda\beta} + \nabla^\alpha R_{\lambda\mu\alpha\beta} - \nabla_\lambda R_{\mu\beta} = 0 \tag{17}$$

and contracting once more the indices (λ, β) one finds

$$\nabla_\mu R - 2\nabla^\alpha R_{\mu\alpha} = 0 \qquad \to \qquad \nabla^\mu \Big(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Big) = 0 \ . \tag{18}$$

It is customary to define the Einstein tensor $G_{\mu\nu}$ by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{19}$$

which then is covariantly conserved, i.e. $\nabla^\mu G_{\mu\nu} = 0$.

Exercizes

These exercises help in proving some of the symmetry properties of the Riemann tensor.

Ex.1 *Recalling that the metric is covariantly constant* $(\nabla_\mu g_{\alpha\beta} = 0)$ *use* $[\nabla_\mu, \nabla_\nu] g_{\alpha\beta} = 0$ *to prove the antisymmetry in the last two indices of the Riemann tensor,* $R_{\mu\nu\alpha\beta} = -R_{\mu\nu\beta\alpha}$.

Ex. 2 *Rewriting the Bianchi identities for electromagnetism using covariant derivatives, show the cyclic property of the Riemann tensor.*

Ex. 3 *From the Jacobi identity valid for arbitrary operators* A, B, C

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

(a consequence of the associativity of the multiplication of operators), consider the case with $(A, B, C) = (\nabla_\mu, \nabla_\nu, \nabla_\lambda)$ *acting on a vector field* V^ρ*, i.e.*

$$\Big([\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] + [\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] \Big) V^\rho = 0$$

and prove the Bianchi identities.

6 Einstein's equations of general relativity

This is described in Chapter 7 of [1].

We now come to the Einstein's field equations, writing out the main equations in our notations.

Einstein's equations (the equivalent for the metric of Maxwell's equations for the potential A_μ) can be identified by using the general covariant principle, which embodies the principle of equivalence. We know that any gravitational field can be made sufficiently small in a small region by using a local inertial frame (that in fact makes the gravitational field vanish at a point).

A *weak* and *static* field due to non-relativistic matter with mass density $\rho(x)$ is described by a newtonian potential ϕ, embedded in the component g_{00} of the metric as

$$\nabla^2\phi = 4\pi G\rho \tag{20}$$
$$g_{00} \approx -(1+2\phi) \tag{21}$$

where $G = 6.67\ 10^{-11} Nm^2/Kg^2$ is the Newton gravitational constant. For example, a pointlike particle of mass M at rest has a mass density

$$\rho(x) = M\delta^3(\vec{x}) \tag{22}$$

and it gives rise to a potential that satisfies the equation

$$\nabla^2\phi = 4\pi GM\delta^3(\vec{x}) \qquad \to \qquad \phi(x) = -\frac{GM}{r}\ . \tag{23}$$

In special relativity mass and energy are equivalent, so that one can take $\rho(x)$ as the energy density, which appears as the T_{00} component of the energy-momentum tensor (also named stress tensor) of the matter system, and rewrite the equation for the gravitational potential as

$$\nabla^2 g_{00} = -8\pi GT_{00}\ . \tag{24}$$

Then, special relativity implies that there must be a tensor $G_{\alpha\beta}$ (tensor under Lorentz transformations) with component $G_{00} = -\nabla^2 g_{00}$ (the minus sign is conventional) that can be constructed with second derivatives of the metric, so that the Lorentz invariant form of (24) becomes

$$G_{\alpha\beta} = 8\pi GT_{\alpha\beta} \tag{25}$$

where the complete energy-momentum tensor $T_{\alpha\beta}$ appears on the right-hand side. So far, this is just as a consequence of special relativity. Finally, general relativity is obtained by searching for a general covariant extension that must take the general covariant form

$$G_{\mu\nu} = 8\pi GT_{\mu\nu}\ . \tag{26}$$

The conservation of $T_{\alpha\beta}$, namely $\partial^\alpha T_{\alpha\beta} = 0$ is covariantized to $\nabla^\mu T_{\mu\nu} = 0$, so that by consistency also $G_{\mu\nu}$ must be covariantly conserved, i.e. $\nabla^\mu G_{\mu\nu}$. The weak and static limit identifies it uniquely with the Einstein tensor.

These considerations lead to the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} \tag{27}$$

which are generally covariant field equations for the metric $g_{\mu\nu}$. The tensor $T_{\mu\nu}$ is the energy-momentum tensor of the matter that gravitates. An equivalent way of writing these equations is to first take the trace (by multiplying with $g^{\mu\nu}$) to find (in four spacetime dimensions)

$$R - 2R = 8\pi G T^{\mu}{}_{\nu} \qquad \rightarrow \qquad R = -8\pi G T^{\mu}{}_{\mu}$$

so that Einstein's equations take the form

$$R_{\mu\nu} = 8\pi G\Big(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda}\Big) . \tag{28}$$

In vacuum, these equations reduce to

$$R_{\mu\nu} = 0 . \tag{29}$$

An additional term with a dimensionful coupling constant Λ with positive mass dimensions, the so-called cosmological constant, can be added to the equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} . \tag{30}$$

Originally introduced and then rejected by Einstein, nowadays it allows to describe the presence of dark energy in the universe.

Finally, reintroducing by dimensional analysis the speed of light c, Einstein's equations takes the form

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}} \tag{31}$$

however, we will continue to use units with $c = 1$.

7 Harmonic gauge

The gauge symmetry associated to the arbitrary change of coordinates can be used to simplify the analysis of Einstein's equations.

The gauge symmetry implies that given a solution $g_{\mu\nu}(x)$, also $g'_{\mu\nu}(x)$ will be a solution if the functions in $g'_{\mu\nu}$ are obtained by a change of coordinates

$$g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} . \tag{32}$$

Infinitesimally, under the change of coordinates $x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$, the metric varies as

$$\begin{aligned} \delta g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) &= \xi^{\alpha}\partial_{\alpha} g_{\mu\nu} + (\partial_{\mu}\xi^{\alpha}) g_{\alpha\nu} + (\partial_{\nu}\xi^{\alpha}) g_{\mu\alpha} \\ &= \nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} \end{aligned} \tag{33}$$

The previuos gauge symmetries can be fixed by requiring the harmonic gauge (or De Donder gauge) conditions

$$\Gamma^{\mu} \equiv g^{\nu\lambda}\Gamma^{\mu}_{\nu\lambda} = 0 \qquad \leftrightarrow \qquad \partial_{\nu}(\sqrt{g} g^{\nu\mu}) = 0 \tag{34}$$

These four conditions specify a gauge in which the coordinates are harmonic functions, just like the cartesian coordinates of flat spacetime, and are sometimes called quasi-cartesian coordinates.

8 Linearized Einstein's equations

To study the Einstein's equations in a linearized apporximation around flat spacetime, one sets the metric as

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{35}$$

and considers $|h_{\mu\nu}(x)| \ll 1$. Then one may raise and lower indices with the Minkowski metric

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta} \tag{36}$$

and define for simplicity the "trace" of $h_{\mu\nu}$

$$h = \eta^{\mu\nu}h_{\mu\nu} \ . \tag{37}$$

Then, one may compute at the linear order in $h_{\mu\nu}$

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x) \ , \quad g = |\det g_{\mu\nu}| = 1 + h \ , \quad \sqrt{g} = 1 + \frac{1}{2}h \tag{38}$$

The Christoffel symbols linearize as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) = \frac{1}{2}(\partial_\mu h_\nu{}^\rho + \partial_\nu h_\mu{}^\rho - \partial^\rho h_{\mu\nu}) \ , \tag{39}$$

the Riemann tensor as

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + ... = \frac{1}{2}\partial_\sigma(\partial_\mu h_\nu{}^\rho - \partial_\nu h_\mu{}^\rho) - \frac{1}{2}\partial^\rho(\partial_\mu h_{\nu\sigma} - \partial_\nu h_{\mu\sigma}) \tag{40}$$

and the Ricci tensor

$$R_{\nu\sigma} = R_{\mu\nu}{}^{\mu}{}_{\sigma} = \frac{1}{2}(\partial_\nu \partial^\mu h_{\sigma\mu} + \partial_\sigma \partial^\mu h_{\nu\mu} - \partial_\nu \partial_\sigma h - \Box h_{\nu\sigma}) \tag{41}$$

where now $\Box = \partial^\mu \partial_\mu = \eta^{\mu\nu}\partial_\mu \partial_\nu$.

Then, Einstein's equations in vacuum take the linearized form

$$\Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial^\sigma h_{\sigma\nu} - \partial_\nu \partial^\sigma h_{\sigma\mu} = 0 \ . \tag{42}$$

One can verify that they are gauge invariant under the linearized gauge symmetry

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{43}$$

where the four components of ξ_μ are arbitrary functions. These symmetries can be used to set four gauge-fixing conditions, that may be take to be the linearzied harmonic (De Donder) gauge

$$\partial^\sigma h_{\sigma\mu} = \frac{1}{2}\partial_\mu h \tag{44}$$

which simplify Einstein's equations to

$$\Box h_{\mu\nu} = 0 \tag{45}$$

which evidently support plane waves solutions (gravitational waves).

It can be shown that only two independent polarizations of the gravitational waves can exist, just like the electromagnetic waves.

8.1 Electromagnetic waves and physical polarizations

Let us first review the case of the electromagnetic waves. The introduction of the four-potential A_μ solves half of the Maxwell equations. The remaining ones in vacuum take the form

$$\partial^\mu F_{\mu\nu} = \partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) = 0 \tag{46}$$

and are gauge invariant under

$$\delta A_\mu = \partial_\mu \theta \tag{47}$$

with θ an arbitrary function of spacetime. The gauge freedom allows to set the Lorenz gauge $\partial^\mu A_\mu = 0$, and in this gauge the equations simplify to

$$\begin{aligned} \Box A_\mu &= 0 \\ \partial^\mu A_\mu &= 0 \;. \end{aligned} \tag{48}$$

Plane wave solution are found using the ansatz (up to an overall normalization) by setting

$$A_\mu(x) = \epsilon_\mu(k)\, e^{ik\cdot x} + c.c. \tag{49}$$

where $\epsilon_\mu(k)$ is an arbitrary polarization depending on the wave vector k^μ, and the exponent contains the Lorentz invariant phase $k\cdot x = k_\mu x^\mu = \eta_{\mu\nu}k^\mu x^\nu = -k^0x^0 + \vec{k}\cdot\vec{x}$. The notation *c.c.* stands for complex conjugation, and makes the solution real. Plugging this ansatz into the equations (48), one finds a solution if

$$k^\mu k_\mu = 0\;, \qquad k^\mu \epsilon_\mu(k) = 0\;. \tag{50}$$

Thus, only three polarizations $\epsilon_\mu(k)$ are possible. However, one of these polarizations, the one with $\epsilon_\mu(k) \sim k_\mu$ is not physical, and can be removed by a gauge transformation (it does not carry electric and magnetic fields, and thus no energy and momentum). The gauge transformations that removes it has the form in (47), but with θ of the form

$$\theta(x) \sim e^{ik\cdot x} \tag{51}$$

which satisfies $\Box\theta(x) = 0$, and thus does not ruin the Lorenz gauge condition. The gauge transformation becomes

$$\delta A_\mu = \partial_\mu\theta \sim ik_\mu\, e^{ik\cdot x} \tag{52}$$

and shows that the polarization $\epsilon_\mu(k) \sim k_\mu$ is not physical, as can be removed by an appropriate gauge transformation. Thus, only two physical polarizations remain.

Let us exemplify this considering the motion along the z axis. We can take

$$k^\mu = (k^0, \vec{k}) = (\omega, 0, 0, \omega) \tag{53}$$

which solves $k^\mu k_\mu = 0$ and producing the phase $e^{ik\cdot x} = e^{i\omega(z-t)}$. The two expected polarizations can be taken as

$$\begin{aligned} \epsilon^1_\mu &= (0,1,0,0) \\ \epsilon^2_\mu &= (0,0,1,0) \end{aligned} \tag{54}$$

which indeed satisfy

$$k^\mu \epsilon^i_\mu = 0\;, \qquad \epsilon^i_\mu \neq \alpha k_\mu\;. \tag{55}$$

Considering for example the solution with ϵ^1_μ, plugging it into (49), and multiplying with an arbitrary amplitude A_0 one finds

$$\begin{aligned}\vec{A} &= A_0 \cos(\omega z - \omega t)\, \hat{x} \\ \vec{E} &= -\frac{\partial \vec{A}}{\partial t} = E_0 \sin(\omega z - \omega t)\, \hat{x} \\ \vec{B} &= \vec{\nabla} \times \vec{A} = B_0 \sin(\omega z - \omega t)\, \hat{y}\end{aligned} \tag{56}$$

where $E_0 = B_0 = \omega A_0$, and $\hat{x}, \hat{y}, \hat{z}$ the usual unit vectors.

The above plane waves do not carry angular momentum. Plane waves carrying angular momentum are obtained using the circular polarization defined by

$$\epsilon^\pm_\mu = \epsilon^1_\mu \pm i\epsilon^2_\mu \ . \tag{57}$$

They are also said to correspond to the helicity $h = \pm 1$, as in a quantum interpretation they are related to photons carrying angular momentum $\pm\hbar$ along the direction of motion (helicity), and with a wavefunction of the form

$$A_\mu(x) = \epsilon^\pm_\mu(k) e^{ik_\nu x^\nu} = \epsilon^\pm_\mu(k) e^{\frac{i}{\hbar} p_\nu x^\nu} \tag{58}$$

where $p^\mu = \hbar k^\mu$ is the 4-momentum of the photon.

8.2 Gravitational waves and physical polarizations

We can now consider in a similar way the gravitational waves. We have seen that they satisfy the equations

$$\begin{aligned}\Box h_{\mu\nu} &= 0 \\ \partial^\mu h_{\mu\nu} &= \frac{1}{2}\partial_\nu h\end{aligned} \tag{59}$$

with the second one describing the harmonic gauge. Plane wave solution can be found using the ansatz (up to a normalization) by setting

$$h_{\mu\nu}(x) = \epsilon_{\mu\nu}(k)\, e^{ik\cdot x} + c.c. \tag{60}$$

where $\epsilon_{\mu\nu}$ is an arbitrary polarization tensor depending on the wave vector k^μ, and the exponent contains the Lorentz invariant phase $k \cdot x = k_\mu x^\mu = \eta_{\mu\nu} k^\mu x^\nu = -k^0 x^0 + \vec{k} \cdot \vec{x}$. The notation *c.c.* stands for complex conjugation, and makes the solution real. Plugging this ansatz into the equations (59), one finds a solution if

$$k^\mu k_\mu = 0 \ , \qquad k^\mu \epsilon_{\mu\nu}(k) = \frac{1}{2} k_\nu \epsilon^\sigma{}_\sigma \ . \tag{61}$$

Thus, only 6 polarizations $\epsilon_{\mu\nu}(k)$ are possible. However, 4 of these polarizations, the ones with $\epsilon_{\mu\nu}(k) \sim k_\mu \epsilon_\nu(k) + k_\nu \epsilon_\mu(k)$ for some $\epsilon_\mu(k)$ are not physical, and can be removed by gauge transformations. The latter have the form in (43), but with ξ_μ of the form

$$\xi_\mu(x) \sim \epsilon_\mu(k) e^{ik\cdot x} \tag{62}$$

which satisfies $\Box\xi_\mu(x) = 0$, and thus does not ruin the harmonic gauge condition (43). The gauge transformation becomes

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \sim i(k_\mu \epsilon_\nu(k) + k_\mu \epsilon_\nu(k))\, e^{ik\cdot x} \tag{63}$$

and shows that these types of polarizations are not physical, and can be removed by an appropriate gauge transformations. Thus, only two physical polarizations remain.

Let us exemplify this again by considering the motion along the z axis. We can take

$$k^\mu = (k^0, \vec{k}) = (\omega, 0, 0, \omega) \tag{64}$$

which solves $k^\mu k_\mu = 0$ and gives the phase $e^{ik\cdot x} = e^{i\omega(z-t)}$. The two expected polarizations can be taken as (using the previous em polarizations)

$$\begin{aligned} \epsilon^{\oplus}_{\mu\nu} &= \epsilon^1_\mu \epsilon^1_\nu - \epsilon^2_\mu \epsilon^2_\nu \\ \epsilon^{\otimes}_{\mu\nu} &= \epsilon^1_\mu \epsilon^2_\nu + \epsilon^2_\mu \epsilon^1_\nu \end{aligned} \tag{65}$$

which indeed satisfy

$$k^\mu \epsilon^i_{\mu\nu} = 0 \ , \qquad \epsilon^i_{\mu\nu} \neq \alpha(k_\mu \epsilon_\nu + k_\nu \epsilon_\mu) \tag{66}$$

for $i = (\oplus, \otimes)$. Considering for example the solution with $\epsilon^{\oplus}_{\mu\nu}$, plugging it into (60), and multiplying with an arbitrary amplitude h_0 one finds

$$h_{\mu\nu}(z-t) = h_0 \cos(\omega z - \omega t)\, \epsilon^{\oplus}_{\mu\nu} \tag{67}$$

which inserted into the linearized metric $g_{\mu\nu}(x)$ give the line element

$$\begin{aligned} ds^2 &= (\eta_{\mu\mu} + h_{\mu\nu}(z-t)) dx^\mu dx^\nu \\ &= -dt^2 + (1 + h_{11}(z-t)) dx^2 + (1 - h_{11}(z-t)) dy^2 + dz^2 \end{aligned} \tag{68}$$

which is interpretable as deforming periodically invariant lengths as in the figure 1 (from [3]).

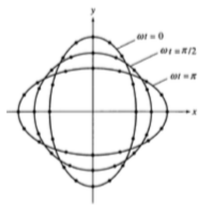

Figure 1: Polarization $\epsilon^{\oplus}_{\mu\nu}$

The polarization $\epsilon^{\otimes}_{\mu\nu}$, does much of the same, but rotated by 45 degrees, see fig. 2

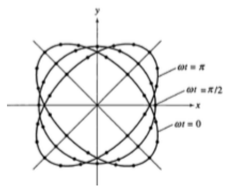

Figure 2: Polarization $\epsilon^{\otimes}_{\mu\nu}$

9 The Schwarzschild solution

Finding exact solutions of the Einstein's field equations is very difficult. One strategy is to use conjectured symmetries of possible solutions, and use these symmetries to restrict the functional form of the metric that is expected to solve the equations. This simplifies Einstein's equations, which then become more tractable and hopefully solvable.

This strategy is the one adopted for finding the Schwarzschild solution. The Schwarzschild metric is obtained by asking for a *static* and *isotropic* solution of the Einstein equations in vacuum, a situation that is realized outside a source that is supposed to be spherical symmetric and static. To implement the required symmetries, time translation and rotational invariance, one assumes the existence of coordinates $x^\mu = (t, \vec{x})$ such that the metric takes the form

$$ds^2 = -F(r)\, dt^2 + 2E(r)\, dt\, \vec{x} \cdot \vec{dx} + D(r)\, (\vec{x} \cdot d\vec{x})^2 + C(r)\, d\vec{x} \cdot d\vec{x} \tag{69}$$

where $r = \sqrt{\vec{x} \cdot \vec{x}}$. This is the most general ansatz consistent with the symmetries. The form of the metric can be further simplified by making changes of coordinates. First of all, one may pass to spherical coordinates (r, θ, ϕ) for $\vec{x}$, and using $\vec{x} \cdot \vec{dx} = rdr$ one rewrites

$$ds^2 = -F(r)\, dt^2 + 2E(r)r\, dtdr + D(r)r^2dr^2 + C(r)\, [dr^2 + r^2d\theta^2 + r^2 \sin^2\theta\, d\phi^2]. \tag{70}$$

Then, one may redefine the time by

$$t \to t' = t + \Phi(r) \tag{71}$$

so that

$$dt' = dt + \frac{d\Phi(r)}{dr} dr \tag{72}$$

and the first two terms inside ds^2 become

$$ds^2 = -F(r) \left(dt' - \frac{d\Phi(r)}{dr} dr\right)^2 + 2E(r)r \left(dt' - \frac{d\Phi(r)}{dr} dr\right) dr + \ldots \tag{73}$$

that rearranges to

$$ds^2 = -F(r)\, dt'^2 + 2\Big[rE(r) + F(r)\frac{d\Phi(r)}{dr}\Big] dt'dr - \Big[F(r)\Big(\frac{d\Phi(r)}{dr}\Big)^2 + 2rE(r)\frac{d\Phi(r)}{dr}\Big] dr^2 + \ldots \tag{74}$$

Now one can fix the function $\Phi(r)$ to satisfy

$$\frac{d\Phi(r)}{dr} = -\frac{rE(r)}{F(r)} \tag{75}$$

so that the mixed term $dt'dr$ vanishes, and the remaining part takes the form

$$ds^2 = -F(r)\, dt'^2 + G(r)dr^2 + C(r)\, [dr^2 + r^2d\theta^2 + r^2 \sin^2\theta\, d\phi^2] \tag{76}$$

where

$$G(r) = r^2 \Big(D(r) + \frac{E^2(r)}{F(r)}\Big)\,. \tag{77}$$

Now one could redefine the radius $r \to r'$ by setting

$$r'^2 = C(r)r^2 \tag{78}$$

so that one gets the so-called *standard form* of the metric

$$ds^2 = -B(r')\, dt'^2 + A(r')dr'^2 + r'^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{79}$$

with

$$
\begin{aligned}
B(r') &= F(r) \\
A(r') &= \Big(1 + \frac{G(r)}{C(r)}\Big)\Big(1 + \frac{r}{2C(r)}\frac{dC(r)}{dr}\Big)^{-2} .
\end{aligned}
\tag{80}
$$

Dropping the primes one finds the static and isotropic metric in the standard form

$$
\boxed{ds^2 = -B(r)\,dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)} . \tag{81}
$$

It is put into Einstein's equations, which are solved to produce the Schwarzschild solution

$$
\boxed{ds^2 = -\Big(1 - \frac{2GM}{r}\Big)dt^2 + \Big(1 - \frac{2GM}{r}\Big)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)} . \tag{82}
$$

The same solution is obtained by relaxing the hypothesis of time invariance (staticity). A more general ansatz for the solution still lead to the same Schwarzschild metric. This is captured by *Birkhoff's theorem*, that states that any spherically symmetric solution of the vacuum field equations must be static and asymptotically flat. This theorem guarantees that the assumption of staticity may be dropped, and still the exterior solution for the spacetime metric outside of a spherical, nonrotating, gravitating body must be given by the Schwarzschild metric.

10 Black holes

The Schwarzschild solution indicates the existence of an event horizon and leads to the concept of a black hole. The recommended treatment is the one presented in [3], see chapter 8.

References

[1] S. Weinberg, "Gravitation and Cosmology", John Wiley & Sons, 1972.

[2] R. D'Inverno, "Introducing Einstein Relativity", Oxford University Press, 1992.

[3] H. Ohanian and R. Ruffini, "Gravitation and Spacetime", Cambridge University Press, 2013.