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Notes on worldlines and vertex operators

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To get intuition on the string theory construction of vertex operators, let us look at the particle analogy.

The relativistic scalar particle has an action proportional to the length of the worldline. Using the particle coordinates $x^\mu(\tau)$, defining the embedding of the wordline in flat spacetime with an arbitrary parameter τ, one writes the action in natural units as

$$S[x^\mu] = -m \int |ds| = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu} \; . \tag{1}$$

Introducing an einbein e, which defines the intrinsic geometry of the worldline, one gets a better description in terms of a more general action

$$S_{Mink}[x,e] = \int_0^1 d\tau \left[\frac{1}{2} e^{-1} \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu} - \frac{1}{2} e m^2 \right] \tag{2}$$

that can be Wick rotated ($\tau \to -i\tau$, $x^0 \to -ix^4$, $iS_{Mink} \to -S$) to find the euclidean action

$$S[x,e] = \int_0^1 d\tau \left[\frac{1}{2} e^{-1} \dot{x}^\mu \dot{x}^\nu \delta_{\mu\nu} + \frac{1}{2} e m^2 \right] \; . \tag{3}$$

This action can be path integrated to get the propagator of the particle.

Considering the boundary conditions $x^\mu(0) = x_i^\mu$ and $x^\mu(1) = x_f^\mu$, one finds the particle propagator by computing the path integral on the interval $I = [0,1]$ (i.e. $\tau \in [0,1]$)

$$G(x_i, x_f) = \int_I \frac{DxDe}{\text{vol(Gauge)}} \, e^{-S[x,e]} \tag{4}$$

where the division by the (infinite) volume of the gauge group (reparametrizations of the worldline) is necessary in order not to count the same physical configuration infinitely many times.

To proceed with the calculation, it is necessary to “fix the gauge”. Let us briefly review the main points. The gauge symmetry allows to set the gauge-fixing condition $e(\tau) = 2T$, with T being a suitable constant. The value of T is determined by the length of the worldline, measured with the intrinsic metric defined by the einbein. This length is given by $\int_0^1 d\tau e(\tau) = 2T$ and is gauge invariant (invariant under redefinition of the parameter τ). Then, the integral over all possible einbeins $e(\tau)$ reduces to an integral over all possible lengths, and thus over T. The amplitude therefore takes the form

$$\begin{aligned} G(x_i, x_f) \; &= \; \int_0^\infty dT \, e^{-m^2 T} \int_I Dx \, e^{-S[x]} \\ S[x] \; &= \; \int_0^1 d\tau \frac{1}{4T} \dot{x}^\mu \dot{x}_\mu \; . \end{aligned} \tag{5}$$

Aside from the integral over T, the remaining path integral is identical to that of a free non-relativistic particle with an appropriate mass, whose solution is well-known. The latter can be expressed in terms of its Fourier transform

$$\int_I Dx\, e^{-S[x]} = \frac{1}{(4\pi T)^{\frac{D}{2}}}\, e^{-\frac{(x_f-x_i)^2}{4T}} = \int \frac{d^D p}{(2\pi)^D}\, e^{ip_\mu(x_f^\mu - x_i^\mu)} e^{-p^2 T} \tag{6}$$

so that one finds

$$G(x_i, x_f) = \int_0^\infty dT \int \frac{d^D p}{(2\pi)^D}\, e^{ip_\mu(x_f^\mu - x_i^\mu)} e^{-(p^2+m^2)T} = \int \frac{d^D p}{(2\pi)^D}\, e^{ip_\mu(x_f^\mu - x_i^\mu)}\, \frac{1}{p^2+m^2}\ . \tag{7}$$

With the inverse Wick rotation ($x^4 \to ix^0$ and $p^4 \to -ip^0$), one returns to Minkowski space and obtains the propagator with the correct Feynman-Stückelberg prescription, i.e. the two-point function of the Klein-Gordon field $\phi(x)$

$$\langle \Omega | T\hat{\phi}(x_f)\hat{\phi}^\dagger(x_i)|\Omega\rangle = \int \frac{d^D p}{(2\pi)^D}\, e^{ip_\mu(x_f^\mu - x_i^\mu)}\, \frac{(-i)}{p^2+m^2-i\epsilon} \tag{8}$$

where $|\Omega\rangle$ is the vacuum state of the quantum field theory. This result exemplifies how the method of first quantization of relativistic particles reproduces results in quantum field theory. The purpose of the latter is to formalize and extend the relativistic quantum mechanics to an arbitrary number of identical particles and give it a nonperturbative definition.

Note the role of the modulus T, the so-called Fock-Schwinger proper time, that emerges by quantizing the geometry of the worldline. In a similar way, moduli emerge when quantizing the worldsheets of the string (Riemann surfaces).

Vertex operators

One can extend the action (3) by considering a nontrival metric $g_{\mu\nu}(x)$ to represent the coupling of the particle to external gravity

$$S[x,e] = \int_0^1 d\tau \Big[\frac{1}{2} e^{-1}\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x) + \frac{1}{2} e m^2)\Big]\ . \tag{9}$$

Considering a small deformation of the flat metric by a plane wave, one can write

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \kappa\epsilon_{\mu\nu} e^{ik\cdot x} \tag{10}$$

and keep the smallest order term in the coupling constant κ. One finds that the path integral leads to

$$\int_0^\infty dT\, e^{-m^2 T} \int Dx\, V_{grav}[x]\, e^{-S[x]} \tag{11}$$

where $S[x]$ is the action in (5) and V_{grav} is the vertex operator that describe the emission/absorption of a graviton of momentum k^μ and polarization $\epsilon_{\mu\nu}$ by the particle

$$V_{grav}[x] = -\frac{\kappa}{4T} \int_0^1 d\tau\, \epsilon_{\mu\nu}\, \dot{x}^\mu(\tau)\dot{x}^\nu(\tau) e^{ik\cdot x(\tau)}\ . \tag{12}$$

In a similar way, considering more general couplings to tensor/vector/scalar potentials denoted by $g_{\mu\nu}(x), A_\mu(x), \phi(x)$, (i.e. spin 2/1/0 external particles), one finds the general nonlinear sigma model

$$S[x^\mu, e] = \int d\tau \Big[\frac{1}{2} e^{-1} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu - iq A_\mu(x) \dot{x}^\mu + e\Big(\frac{1}{2} m^2 + g\phi(x)\Big)\Big] \ . \tag{13}$$

where q is the electric charge and g a scalar charge. Note that it is reparametrization invariant and can be gauge-fixed as before. Again, plane wave deformations of the background potentials $(g_{\mu\nu}, A_\mu, \phi)$ around empty flat space $(\delta_{\mu\nu}, 0, 0)$

$$\begin{aligned} g_{\mu\nu} &= \delta_{\mu\nu} + \kappa \epsilon_{\mu\nu} e^{ik\cdot x} \\ A_\mu &= \epsilon_\mu e^{ik\cdot x} \\ \phi &= e^{ik\cdot x} \end{aligned} \tag{14}$$

lead to the vertex operators for emission/absorption of particles of spin 2, 1, and 0

$$\begin{aligned} V_{grav}[x] &= -\frac{\kappa}{4T} \int_0^1 d\tau \, \epsilon_{\mu\nu} \, \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) e^{ik\cdot x(\tau)} \ . \\ V_{ph}[x] &= iq \int_0^1 d\tau \, \epsilon_\mu \, \dot{x}^\mu(\tau) e^{ik\cdot x(\tau)} \ . \\ V_{scal}[x] &= g \int_0^1 d\tau \, e^{ik\cdot x(\tau)} \ . \end{aligned} \tag{15}$$

The particle case furnishes intuition and guidance for the introduction of vertex operators in string theory.

Topology of the worldline and one-loop amplitudes

Let us review also the path integral on the circle, that leads to one-loop amplitudes, and its relation to the Schwinger-DeWitt heat kernel method.

For worldlines with the topology of the circle S^1, the path integral computes one-loop amplitudes

$$\begin{aligned} Z &= \int_{S^1} \frac{DxDe}{\text{vol(Gauge)}} \, e^{-S[x,e]} \\ &= \int_0^\infty \frac{dT}{T} \int_{PBC} Dx \, e^{-S[x,e=2T]} \\ &= \int_0^\infty \frac{dT}{T} \text{Tr}\Big[e^{-HT}\Big] = -\text{Tr} \ln H = -\ln(\text{Det}\, H) = \ln(\text{Det}^{-1} H) \\ &= \ln \int D\phi^* D\phi \, e^{-\int d^D x \, \phi^* H \phi} \end{aligned} \tag{16}$$

where the path integral on the circle S^1 is gauge-fixed by setting $e(\tau) = 2T$ (second line), with the modulus T acquiring the non trivial measure $\frac{1}{T}$ on the circle. The path integral on the circle is obtained by choosing periodic boundary conditions (PBC) in τ. It computes a trace in the Hilbert space of the first quantized theory. The modulus T is recognized as the Fock-Schwinger proper time and leads to a well-known representation of the logarithm[1] (third line). Finally, the Hamiltonian of the free relativistic particle is $H = p^2 + m^2 = -\Box + m^2$ and is recognized as the Klein-Gordon operator of the free scalar QFT, whose gaussian path integral leads to the one-loop effective action.

Once again, we observe how first quantization serves as a framework for representing quantities in Quantum Field Theory (QFT), such as the one-loop effective action.

Schwinger [1] pioneered the representation of the one-loop effective action in terms of the proper time already in 1951, initially for particles interacting with external electromagnetic fields. Later, DeWitt extended this representation to include curved spaces, accounting for coupling to external gravity.

The introduction of proper time proved crucial in providing a useful representation for the logarithm, as well as for the inverse kinetic term in the propagator. Initially conceived as a mathematical tool to derive expressions related to the quantum mechanical transition amplitude e^{-HT} within a "fictitious" quantum mechanical model with hamiltonian H, it eventually became apparent that this quantum mechanical model corresponded to the first quantized description of the particle associated with the quantum field. The proper time emerges then as a consequence of gauge-fixing the einbein $e(\tau)$.

Going back to the path integral in (16), one can extend it by introducing couplings to external backgrounds and deduce the vertex operators that define interactions with external particles of fixed momenta, as in the case of the propagator.

References

[1] J. S. Schwinger, "On gauge invariance and vacuum polarization," Phys. Rev. **82** (1951), 664-679 doi:10.1103/PhysRev.82.664

[1]Consider

$$\ln \frac{a}{b} = -\int_0^\infty \frac{dT}{T}(e^{-aT} - e^{-bT}) \ .$$

Both sides have the same derivative in a and have the same value for $a = b$, so they define the same function. In the Schwinger application to operators, one drops an infinite constant (the term with b), which is however taken care of by renomalization.