Path integrals for fermions, susy quantum mechanics, etc..

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Fermions at the classical level can be described by Grassmann variables, also known as anticommuting or fermionic variables. These variables allow to introduce "classical" models that upon quantization produce the degrees of freedom associated to spin. Actually, rather than "classical" models one should call them "pseudoclassical", since the description of spin at the classical level is a formal construction (spin vanish for $\hbar \to 0$). However, quantization of Grassmann variables gives rise to spin degrees of freedom, and for simplicity we keep using the term classical for such models.

In worldline approaches to quantum field theories, one often describes relativistic point particles with spin by using worldline coordinates for the position of the particle in space-time and Grassmann variables to account for the associated spin degrees of freedom. From the worldline point of view these Grassmann variables have equations of motion that are first order in time, and give rise to a Dirac equation in one dimension (the worldline can be considered as a 0+1 dimensional space-time, with the one dimension corresponding to the time direction). They are fermions on the worldline and obey the Pauli exclusion principle, even though there is no real spin in a zero dimensional space. In the following we study the path integral quantization of models with Grassmann variables, and refer to them as path integrals for fermions, or fermionic path integrals. In a hypercondensed notation the resulting formulae describe the quantization of a Dirac field in higher dimensions as well.

Thus, we start introducing Grassmann variables and develop the canonical quantization of mechanical models with Grassmann variables. Using a suitable definition of coherent states that mimics the standard definition of coherent states for the harmonic oscillator, we derive a path integral representation of the transition amplitude for fermionic systems. To exemplify the use of Grassmann variables in mechanical models, we present a class of supersymmetric models, which allow us to introduce supersymmetry and related tools in one of their simplest realization. Supersymmetry is a guiding principle of many worldline descriptions of relativistic fields, as we shall discuss in chapter 5. The $N = 1$ and $N = 2$ superspaces, the former used in worldline description of Dirac fields, are also briefly exposed.

1 Grassmann algebras

A n-dimensional Grassmann algebra $\mathcal{G} = \{\theta_i\}$ is formed by generators θ_i with $i = 1, ..., n$ that satisfy

$$\{\theta_i, \theta_j\} \equiv \theta_i\theta_j + \theta_j\theta_i = 0 \ . \tag{1}$$

In particular any fixed generator squares to zero

$$\theta_i^2 = 0 \tag{2}$$

suggesting already at the classical level the essence of the Pauli exclusion principle, according to which one cannot put two identical fermions in the same quantum state.

Functions of these Grassmann variables have a finite Taylor expansion (in fact they are really defined by the latter). For example, for $n = 1$ there is only one Grassmann variable θ and an arbitrary function is given by

$$f(\theta) = f_0 + f_1\theta \tag{3}$$

where f_0 and f_1 are taken to be either real or complex numbers. Similarly, for $n = 2$ one has

$$f(\theta_1, \theta_2) = f_0 + f_1\theta_1 + f_2\theta_2 + f_3\theta_1\theta_2 \ . \tag{4}$$

Terms with an even number of θ's are called Grassmann even (or equivalently even, commuting, or bosonic). Terms with an odd number of θ's are called Grassmann odd (or equivalently odd, anticommuting, or fermionic).

Derivatives with respect to Grassmann variables are very simple. As any function can be at most linear with respect to a fixed Grassmann variable, one has just to keep track of signs. Left derivatives are defined by removing the variable from the left of its Taylor expansion: for the function $f(\theta_1, \theta_2)$ given above

$$\frac{\partial_L}{\partial\theta_1} f(\theta_1, \theta_2) = f_1 + f_3\theta_2 \ . \tag{5}$$

Similarly, right derivatives are obtained by removing the variable from the right

$$\frac{\partial_R}{\partial\theta_1} f(\theta_1, \theta_2) = f_1 - f_3\theta_2 \tag{6}$$

where a minus sign emerges because one has first to commute θ_1 past θ_2.

Equivalently, using Grassmann increments $\delta\theta$, one may write

$$\delta f = \delta\theta \frac{\partial_L f}{\partial\theta} = \frac{\partial_R f}{\partial\theta} \delta\theta \tag{7}$$

which makes evident how to keep track of signs. If not specified otherwise, we use left derivatives and omit the corresponding subscript.

Integration can be defined, according to Berezin, to be identical with differentiation

$$\int d\theta \equiv \frac{\partial}{\partial\theta} \ . \tag{8}$$

This definition has the virtue of producing a translational invariant measure, that is

$$\int d\theta f(\theta + \eta) = \int d\theta f(\theta) \ . \tag{9}$$

This statement is easily proven by a direct calculation

$$\int d\theta f(\theta + \eta) = \int d\theta \ (f_0 + f_1(\theta + \eta)) = f_1 = \int d\theta f(\theta) \ . \tag{10}$$

Grassmann variables can be defined to be either real or complex. A real variable satisfies

$$\bar{\theta} = \theta \tag{11}$$

with the bar indicating complex conjugation. For products of Grassmann variables the complex conjugate is defined to include an exchange of their ordered position

$$\overline{\theta_1\theta_2} = \bar{\theta}_2\bar{\theta}_1 \tag{12}$$

so that the complex conjugate of the product of two real variables is purely imaginary

$$\overline{\theta_1\theta_2} = -\theta_1\theta_2 \ . \tag{13}$$

It is $i\theta_1\theta_2$ that is real, as the complex conjugate of the imaginary unit carries the additional minus sign to obtain a formally real object

$$\overline{i\theta_1\theta_2} = i\theta_1\theta_2 \ . \tag{14}$$

Complex Grassmann variables η and $\bar{\eta}$ can always be decomposed in terms of two real Grassmann variables θ_1 and θ_2 by setting

$$\eta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \ , \qquad \bar{\eta} = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2) \ . \tag{15}$$

Having defined integration over Grassmann variables, we consider in more details the gaussian integration that is at the core of fermonic path integrals. For the case of a single real Grassmann variable θ the gaussian function is trivial, $e^{-a\theta^2} = 1$, as θ anticommutes with itself. One needs at least two real Grassmann variables θ_1 and θ_2 to have a nontrivial exponential function with an exponent quadratic in Grassmann variables

$$e^{-a\theta_1\theta_2} = 1 - a\theta_1\theta_2 \tag{16}$$

where a is either a real or a complex number. With the above definitions the corresponding "gaussian integral" is computed straghtforwardly

$$\int d\theta_1 d\theta_2 \, e^{-a\theta_1\theta_2} = a \ . \tag{17}$$

Defining the antisymmetric 2×2 matrix

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \tag{18}$$

one may rewrite the result of the Grassmann integration as

$$\int d\theta_1 d\theta_2 \, e^{-\frac{1}{2}\theta_i A^{ij} \theta_j} = \mathrm{Pfaff}(A) \tag{19}$$

where $\mathrm{Pfaff}(A)$ is the pfaffian, the square root of the determinant of A. The latter is necessarily positive definite for antisymmetric matrices. It is easy to see that this formula extends to an even number $n = 2m$ of real Grassmann variables, so that one may write

$$\int d^n\theta \, e^{-\frac{1}{2}\theta_i A^{ij} \theta_j} = \mathrm{Pfaff}(A) \tag{20}$$

with the measure normalized as $d^n\theta \equiv d\theta_1 d\theta_2 \dots d\theta_n$. Indeed with an orthogonal transformation that leaves the integration measure invariant, the antisymmetric matrix A^{ij} can be skew-diagonalized to

$$\begin{pmatrix} 0 & a_1 & & & & & & \\ -a_1 & 0 & & & & & & \\ & & 0 & a_2 & & & & \\ & & -a_2 & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 0 & a_m \\ & & & & & & -a_m & 0 \end{pmatrix} \tag{21}$$

and one immediately gets the above result with $\text{Pfaff}(A) = a_1 a_2 \cdots a_m$.

Similarly, the determinant of a matrix $A_i{}^j$ is obtained by gaussian integration over complex Grassmann variables $(\eta_i, \bar\eta^i)$

$$\int d^n\bar\eta d^n\eta \, e^{-\bar\eta^i A_i{}^j \eta_j} = \det(A) \tag{22}$$

where now we have defined the measure by $d^n\bar\eta d^n\eta \equiv d\bar\eta^1 d\eta_1 d\bar\eta^2 d\eta_2 \dots d\bar\eta^n d\eta_n$.

For applications to dynamical models and subsequent path integral quantization, one needs an infinite dimensional Grassmann algebra ($n \to \infty$), to obtain, for example, a function of time that takes values in the infinite dimensional Grassmann algebra, i.e. $\eta_i \rightsquigarrow \eta(t)$, that may be used to describe suitable mechanical systems.

2 Pseudoclassical models and canonical quantization

To gain some familiarity with the use of Grassmann numbers as dynamical variables let us consider the example of the fermionic harmonic oscillator. It contains the essence of all fermonic systems and is described by functions $\psi(t)$ and $\bar\psi(t)$, complex conjugates of each other, taking values in an infinite dimensional Grassmann algebra (t denotes the time). Of course, the Grassmann property implies relations like $\psi(t)\psi(t) = 0$, $\psi(t)\psi(t') = -\psi(t')\psi(t)$, $\dot\psi(t)\dot\psi(t) = 0$, $\psi(t)\dot\psi(t) = -\dot\psi(t)\psi(t)$, etc., where we used the standard notation for time derivatives $\dot\psi = \frac{d}{dt}\psi$.

The dynamics of the fermionic harmonic oscillator is fixed by the action

$$S[\psi, \bar\psi] = \int dt \, \mathcal{L} \, , \quad \mathcal{L} = i\bar\psi\dot\psi - \omega\bar\psi\psi \tag{23}$$

which is formally real (up to boundary terms). The equations of motion are obtained by extremizing the action and solved easily

$$\dot\psi = -i\omega\psi \quad \Longrightarrow \quad \psi(t) = \psi_0 \, e^{-i\omega t} \tag{24}$$

with ψ_0 a suitable initial datum (a complex Grassmann number). Canonical quantization is obtained by considering the hamiltonian structure of the model. We sketch it now, briefly postponing a general discussion of the phase-space structure associated to Grassmann variables. The momentum π conjugate to ψ is obtained by

$$\pi \equiv \frac{\partial_L \mathcal{L}}{\partial\dot\psi} = -i\bar\psi \tag{25}$$

which shows that the systems is already in a hamiltonian form, the conjugate momenta being $\bar{\psi}$ up to factors. The classical Poisson bracket $\{\pi, \psi\}_{PB} = -1$ can be written as $\{\psi, \bar{\psi}\}_{PB} = -i$. Quantizing with anticommutators (as we shall see, fermionic system must be treated this way) one obtains

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = \hbar \ , \quad \{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0 \tag{26}$$

that is, the classical variables $\psi, \bar{\psi}$ are promoted to linear operators $\hat{\psi}, \hat{\psi}^\dagger$ satisfying anticommutation relations taken to be $i\hbar$ times the value of the classical Poisson brackets. Setting $\hat{\psi} = \sqrt{\hbar}\, \hat{b}$ and $\hat{\psi}^\dagger = \sqrt{\hbar}\, \hat{b}^\dagger$ one finds the fermionic creation/annihilation algebra

$$\{\hat{b}, \hat{b}^\dagger\} = 1 \ , \quad \{\hat{b}, \hat{b}\} = \{\hat{b}^\dagger, \hat{b}^\dagger\} = 0 \ . \tag{27}$$

The latter can be realized in a two dimensional Hilbert space. One may construct it à la Fock, considering $\hat{b}$ as destruction operator and $\hat{b}^\dagger$ as creation operator. One starts defining the Fock vacuum $|0\rangle$, fixed by the condition $\hat{b}|0\rangle = 0$. A second state is obtained acting with $\hat{b}^\dagger$

$$|1\rangle = \hat{b}^\dagger |0\rangle \ . \tag{28}$$

No other states can be constructed acting with the creation operator $\hat{b}^\dagger$ as $(\hat{b}^\dagger)^2 = 0$. Normalizing the Fock vacuum to unity, $\langle 0|0\rangle = 1$, with $\langle 0| = |0\rangle^\dagger$, one finds that these states are orthonomal

$$\langle m|n\rangle = \delta_{mn} \qquad m, n = 0, 1 \tag{29}$$

and span a two-dimensional Hibert space.

Hamiltonian structure and canonical quantization

Path integrals for fermions can be derived form the canonical formalism, just as in the bosonic case. For this we need to review briefly the hamiltonian formalism and canonical quantization of mechanical systems with Grassmann variables.

The hamiltonian formalism aims to produce equations of motion as first order differential equations in time. For a simple bosonic model with phase space coordinates (x, p), the phase space action is usually written in the form

$$S[x, p] = \int dt\, \Big(p\dot{x} - H(x, p)\Big) \tag{30}$$

The first term with derivatives $(p\dot{x})$ is called the symplectic term, and fixes the Poisson bracket structure of phase space. Up to boundary terms it can be written in a more symmetrical form, with the time derivatives shared equally by x and p,

$$S[x, p] = \int dt\, \Big(\frac{1}{2}(p\dot{x} - x\dot{p}) + H(x, p)\Big) = \int dt\, \Big(\frac{1}{2} z^a (\Omega^{-1})_{ab} \dot{z}^b + H(z)\Big) \tag{31}$$

where we have denoted collectively the phase space coordinates by $z^a = (x, p)$. The symplectic term contains the constant invertible matrix

$$(\Omega^{-1})_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{32}$$

with inverse

$$\Omega^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{33}$$

The latter is used to define the Poisson bracket between generic phase space functions F and G

$$\{F,G\}_{PB} = \frac{\partial F}{\partial z^a}\Omega^{ab}\frac{\partial G}{\partial z^B} \tag{34}$$

which is readily seen to coincide with standard definitions. It satisfies the following properties:

$$\begin{aligned} &\{F,G\}_{PB} = -\{G,F\}_{PB} \qquad \text{(antisymmetry)} \\ &\{F,GH\}_{PB} = \{F,G\}_{PB}H + G\{F,H\}_{PB} \qquad \text{(Leibniz rule)} \\ &\{F,\{G,H\}_{PB}\}_{PB} + \{G,\{H,F\}_{PB}\}_{PB} + \{H,\{F,G\}_{PB}\}_{PB} = 0 \qquad \text{(Jacobi identity)} \end{aligned} \tag{35}$$

These properties make it consistent to adopt the canonical quantization rules of substituting the fundamental variables z^a by linear operators $\hat{z}^a$ acting on a Hilbert space of physical states. The commutation relations of the $\hat{z}^a$ are fixed to be $i\hbar$ times the value of the classical Poisson brackets

$$[\hat{z}^a, \hat{z}^b] = i\hbar\Omega^{ab} \; . \tag{36}$$

More generally, phase space functions $F(z)$ are elevated to operators $\hat{F}(\hat{z})$ (after fixing eventual ordering ambiguities) having commutation relations with other operators of the form

$$[\hat{F}(\hat{z}), \hat{G}(\hat{z})] = i\hbar\{F,G\}_{PB} + \text{ higher order terms in } \hbar \; . \tag{37}$$

These prescriptions are consistent as both sides satisfy the same algebraic properties.

This set up is readily extended to models containing Grassmann variables, after taking care of sign arising from the anticommuting sector. We denote collectively the phase space coordinates by $Z^A = (x^i, p_i, \theta^\alpha)$, with (x^i, p_i) usual Grassmann even phase space variables and θ^α Grassmann odd variables. We consider a phase space action of the form

$$S[z^A] = \int dt \left(\frac{1}{2} z^A (\Omega^{-1})_{AB} \dot{z}^B - H(z)\right) \tag{38}$$

where the symplectic term depends on a constant invertible matrix $(\Omega^{-1})_{AB}$ with inverse Ω^{AB}. Again this term must be written splitting the time derivatives democratically between all variables, as in (31). The symplectic term as well as the hamiltonian are taken to be Grassmann even (i.e. commuting objects). Then, it is seen that Ω^{AB} is a truly antisymmetric matrix along bosonic coordinates, while it is symmetric along anticommuting directions. It is used to define the Poisson bracket by

$$\{F,G\}_{PB} = \frac{\partial_R F}{\partial z^A}\Omega^{AB}\frac{\partial_L G}{\partial z^B} \tag{39}$$

where both right and left derivatives are used. In particular one finds $\{Z^A, Z^B\}_{PB} = \Omega^{AB}$. For phase space functions F with definite Grassmann parity $(-1)^{\epsilon_F}$ ($\epsilon_F = 0$ if F is Grassmann even, and $\epsilon_F = 1$ if F is Grassmann odd), one finds a graded generalization of the properties in eq. (35), namely

$$\begin{aligned} &\{F,G\}_{PB} = (-1)^{\epsilon_F\epsilon_G+1}\{G,F\}_{PB} \\ &\{F,GH\}_{PB} = \{F,G\}_{PB}H + (-1)^{\epsilon_F\epsilon_G}G\{F,H\}_{PB} \\ &\{F,\{G,H\}_{PB}\}_{PB} + (-1)^{\epsilon_F(\epsilon_G+\epsilon_H)}\{G,\{H,F\}_{PB}\}_{PB} + (-1)^{\epsilon_H(\epsilon_F+\epsilon_G)}\{H,\{F,G\}_{PB}\}_{PB} = 0 \end{aligned} \tag{40}$$

These properties make it consistent to adopt the canonical quantization rules of promoting phase space coordinates Z^A to operators $\hat{Z}^A$ with commutation/anticommutation rules fixed by the classical Poisson brackets

$$[\hat{Z}^A, \hat{Z}^B\} = i\hbar\{Z^A, Z^B\}_{PB} = i\hbar\Omega^{AB} \tag{41}$$

where we have employed the compact notation

$$[\cdot\,,\cdot\} = \begin{cases} \{\cdot\,,\cdot\} & \text{anticommutator if both variables are fermionic} \\ [\cdot\,,\cdot] & \text{commutator otherwise} \end{cases} \tag{42}$$

The previous quick exposition becomes clearer by considering the following examples.

Examples

(i) Single real Grassmann variable ψ (single Majorana fermion in one dimension).
Taking as phase space lagrangian

$$\mathcal{L} = \frac{i}{2}\psi\dot{\psi} - H(\psi) \tag{43}$$

one finds $\Omega^{-1} = i$, $\Omega = -i$, and Poisson bracket at equal times $\{\psi, \psi\}_{PB} = -i$. The dynamical variable $\psi(t)$ is often called Majorana fermion in one dimension, as it satisfy a reality condition (akin to the Majorana condition) and the Dirac equation in one dimension. One may notice that the only possible Grassmann even hamiltonian is a real constant. Also one verifies in this example that phase space can be odd dimensional if Grassmann variables are present. This model is quantized by the anticommutator

$$\{\hat{\psi}, \hat{\psi}\} = \hbar\ , \tag{44}$$

however the quantum theory is trivial, as one may realize this algebra in a one dimensional Hilbert space, with the operator $\hat{\psi}$ acting as multiplication by $\sqrt{\hbar/2}$.

(ii) Several real Grassmann variables ψ^i (Majorana fermions in one dimension).
For the case os several real Grassmann variables one may take as phase space lagrangian

$$\mathcal{L} = \frac{i}{2}\psi^i\dot{\psi}^i - H(\psi^i) \qquad i = 1, ..., n \tag{45}$$

and one finds $(\Omega^{-1})_{ij} = i\delta_{ij}$, $\Omega^{ij} = -i\delta^{ij}$. Thus the Poisson brackets at equal times read as $\{\psi^i, \psi^j\}_{PB} = -i\delta^{ij}$. Quantization is obtained by the anticommutator

$$\{\hat{\psi}^i, \hat{\psi}^j\} = \hbar\delta^{ij} \tag{46}$$

which is recognized to be proportional to the Clifford algebra of the gamma matrices of the Dirac equation in n euclidean dimensions. Indeed setting $\hat{\psi}^i = \sqrt{\hbar/2}\,\gamma^i$ one finds $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ which is the defining properties of the gamma matrices appearing in the Dirac equation

$$(\gamma^i\partial_i + m)\Psi(x) = 0\ . \tag{47}$$

It is known that the algebra (46) is realized in a complex vector space of dimension $2^{[\frac{n}{2}]}$, where $[\frac{n}{2}]$ indicates the integer part of $\frac{n}{2}$. The previous case is recognized as a subcase of the present one.

(iii) Complex Grassmann variables ψ and $\bar{\psi}$ (Dirac fermion in one dimension).
Taking as phase space lagrangian

$$\mathcal{L} = i\bar{\psi}\dot{\psi} - H(\psi, \bar{\psi}) \tag{48}$$

one finds $\{\psi,\bar\psi\}_{_{PB}}=-i$ as the only nontrivial Poisson bracket between phase space coordinates $(\psi,\bar\psi)$. It is quantized by the anticommutator $\{\hat\psi,\hat\psi^\dagger\}=\hbar$, producing a fermionic annihilation/creation algebra. It is realized in a two dimensional Fock space, as anticipated for the case of the fermionic harmonic oscillator.

This basic result can be used to understand the statement on the dimensions of the gamma matrices. In even $n=2m$ dimensions one may combine the $2m$ Majorana fermions corresponding to the gamma matrices in m pairs of Dirac fermions, that generate a set of m independent, anticommuting annihilation/creation operators. The latter act on the m-th tensor product of two dimensional fermionic Fock spaces, giving a total Hilbert space of 2^m dimensions, in accord with the assertion given in point *(ii)* above. Adding an extra Majorana fermion to this set up corresponds to a Clifford algebra in odd dimensions $(2m+1)$: the dimension of the Hilbert space does not change and the last Majorana fermion can be realized as proportional to the chirality matrix of the $2m$ dimensional case.

2.1 Coherent states

For a derivation of the path integral for fermionic systems it is useful to introduce coherent states, an overcomplete basis of vectors of the Fock space described previously. It is useful to review the bosonic construction, to have a guide on the construction for the fermionic case.

In the theory of the harmonic oscillator one introduces coherent states defined as eigenstates of the annihilation operator $\hat a$. Let us recall that the algebra of the creation and annihilation operators $\hat a^\dagger$ and $\hat a$ is the following

$$[\hat a,\hat a^\dagger]=1\ ,\quad [\hat a,\hat a]=0\ ,\quad [\hat a^\dagger,\hat a^\dagger]=0\ . \tag{49}$$

It is realized in an infinite dimensional Hilbert space that can be identified with a Fock space through the Fock construction, which goes as follows. A complete orthonormal basis is constructed by starting from the Fock vacuum $|0\rangle$, defined by the condition $\hat a|0\rangle=0$. The other states of the basis are obtained by acting with $\hat a^\dagger$ an arbitrary number of times on the Fock vacuum $|0\rangle$

$$\begin{array}{rcl}
|0\rangle & & \text{such that}\quad \hat a|0\rangle=0\\
|1\rangle &=& \hat a^\dagger|0\rangle\\
|2\rangle &=& \dfrac{\hat a^\dagger}{\sqrt2}|1\rangle=\dfrac{(\hat a^\dagger)^2}{\sqrt{2!}}|0\rangle\\
|3\rangle &=& \dfrac{\hat a^\dagger}{\sqrt3}|2\rangle=\dfrac{(\hat a^\dagger)^3}{\sqrt{3!}}|0\rangle\\
\dots & & \\
|n\rangle &=& \dfrac{\hat a^\dagger}{\sqrt n}|n-1\rangle=\dfrac{(\hat a^\dagger)^n}{\sqrt{n!}}|0\rangle\\
\dots & &
\end{array} \tag{50}$$

Normalizing the Fock vacuum to unit norm, $\langle 0|0\rangle=1$, where $\langle 0|=|0\rangle^\dagger$, one finds that these states are orthonomal (it is enough to use the algebra (49) to verify it)

$$\langle m|n\rangle=\delta_{mn}\qquad m,n=0,1,2,\dots\ . \tag{51}$$

Now, choosing a complex number α, one builds the coherent states $|\alpha\rangle$ as

$$|\alpha\rangle=e^{\alpha\hat a^\dagger}|0\rangle \tag{52}$$

which turn out to be eigenstates of the annihilation operator $\hat{a}$

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \ . \tag{53}$$

A quick way of seeing this is by expanding the exponential and viewing $|\alpha\rangle$ as an infinite sum with suitable coefficients of the basis vectors constructed above

$$\begin{aligned} |\alpha\rangle &= e^{\alpha\hat{a}^\dagger}|0\rangle \\ &= \Big(1+\alpha\hat{a}^\dagger+\frac{1}{2!}(\alpha\hat{a}^\dagger)^2+\frac{1}{3!}(\alpha\hat{a}^\dagger)^3+\cdots+\frac{1}{n!}(\alpha\hat{a}^\dagger)^n+\cdots\Big)|0\rangle \\ &= |0\rangle+\alpha|1\rangle+\frac{\alpha^2}{\sqrt{2!}}|2\rangle+\frac{\alpha^3}{\sqrt{3!}}|3\rangle+\cdots+\frac{\alpha^n}{\sqrt{n!}}|n\rangle+\cdots \ . \end{aligned} \tag{54}$$

In this form it is easy to calculate (using $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$)

$$\begin{aligned} \hat{a}|\alpha\rangle &= \hat{a}\Big(|0\rangle+\alpha|1\rangle+\frac{\alpha^2}{\sqrt{2!}}|2\rangle+\frac{\alpha^3}{\sqrt{3!}}|3\rangle+\cdots+\frac{\alpha^n}{\sqrt{n!}}|n\rangle+\cdots\Big) \\ &= 0+\alpha|0\rangle+\alpha^2|1\rangle+\frac{\alpha^3}{\sqrt{2!}}|2\rangle+\cdots+\frac{\alpha^n}{\sqrt{(n-1)!}}|n-1\rangle+\cdots \\ &= \alpha\Big(|0\rangle+\alpha|1\rangle+\frac{\alpha^2}{\sqrt{2!}}|2\rangle+\cdots+\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle+\cdots\Big) \\ &= \alpha|\alpha\rangle \ . \end{aligned} \tag{55}$$

An even faster way is to realize that the algebra (49) can be realized by

$$\hat{a}^\dagger \ \to \ \bar{a} \ , \qquad \hat{a} \ \to \ \frac{\partial}{\partial\bar{a}} \tag{56}$$

with $\bar{a} \in \mathbb{C}$, and the result follows straighfowardly.

A list of properties that can be easily proven with similar calculations are

$$\begin{array}{llll} (i) & \langle\bar{\alpha}| = |\alpha\rangle^\dagger = \langle\bar{0}|e^{\bar{\alpha}\hat{a}} & \Longrightarrow & \langle\bar{\alpha}|\hat{a}^\dagger = \langle\bar{\alpha}|\bar{\alpha} \\ (ii) & \langle\bar{\alpha}|\alpha\rangle = e^{\bar{\alpha}\alpha} & & \text{(scalar product)} \\ (iii) & \mathbb{1} = \displaystyle\int \frac{d\alpha d\bar{\alpha}}{2\pi i}\, e^{-\bar{\alpha}\alpha}\, |\alpha\rangle\langle\bar{\alpha}| & & \text{(resolution of the identity)} \\ (iv) & \mathrm{Tr}\,\hat{A} = \displaystyle\int \frac{d\alpha d\bar{\alpha}}{2\pi i}\, e^{-\bar{\alpha}\alpha}\, \langle\bar{\alpha}|\hat{A}|\alpha\rangle & & \text{(trace of the operator } \hat{A}) \ . \end{array} \tag{57}$$

One should note that the set of coherent states form an over-complete basis, it is not orthonormal (as $\langle\bar{\beta}|\alpha\rangle = e^{\bar{\beta}\alpha} \neq 0$), but it is useful to keep this redundancy.

A similar construction can be introduced for fermionic systems. Let us consider the algebra of the anticommutators of the fermionic creation and annihilation operators $\hat{\psi}^\dagger$ and $\hat{\psi}$,

$$\{\hat{\psi},\hat{\psi}^\dagger\} = 1 \ , \quad \{\hat{\psi},\hat{\psi}\} = 0 \ , \quad \{\hat{\psi}^\dagger,\hat{\psi}^\dagger\} = 0 \ . \tag{58}$$

This algebra can be realized by 2×2 matrices acting in the two-dimensional fermionic Fock space generated by the vectors $|0\rangle$ and $|1\rangle$, defined by

$$\hat{\psi}|0\rangle = 0 \ , \quad |1\rangle = \hat{\psi}^\dagger|0\rangle \ . \tag{59}$$

Indeed acting again with the fermionic creation operator on the state $|1\rangle$, does not produce additional states $(\hat{\psi}^\dagger|1\rangle = (\hat{\psi}^\dagger)^2|0\rangle = 0$ as a consequence of the algebra (58): one cannot have occupation numbers bigger then one. As already mentioned, this embodies the Pauli exclusion principle.

One defines fermionic coherent states again as eigenstates $|\psi\rangle$ of the annihilation operator $\hat{\psi}$, having as eigenvalues the complex Grassmann number ψ

$$\hat{\psi}|\psi\rangle = \psi|\psi\rangle \ . \tag{60}$$

The Grassmann numbers, such as ψ and its complex conjugate $\bar{\psi}$, anticommute of course between themselves, and we define them to anticommute also with the fermionic operators $\hat{\psi}^\dagger$ and $\hat{\psi}$. No confusion should arise between the operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ that have a hat, and the complex Grassmann variables ψ and $\bar{\psi}$, egeinvalues of the eigenstates $|\psi\rangle$ and $\langle\bar{\psi}|$ respectively, that carry no hat (just as in the previous chapter we indicated position operator, eigenstates and eigenvalues so that $\hat{x}|x\rangle = x|x\rangle$.)

One can prove the following properties

$$\begin{array}{rll}
(i) & |\psi\rangle = e^{\hat{\psi}^\dagger\psi}|0\rangle & \\
(ii) & \langle\bar{\psi}| = \langle 0|e^{\bar{\psi}\hat{\psi}} \quad \Longrightarrow \quad \langle\bar{\psi}|\hat{\psi}^\dagger = \langle\bar{\psi}|\bar{\psi} & \\
(iii) & \langle\bar{\psi}|\psi\rangle = e^{\bar{\psi}\psi} & \\
(iv) & \mathbb{1} = \int d\bar{\psi}d\psi\, e^{-\bar{\psi}\psi}\, |\psi\rangle\langle\bar{\psi}| & \\
(v) & \mathrm{Tr}\,\hat{A} = \int d\bar{\psi}d\psi\, e^{-\bar{\psi}\psi}\, \langle-\bar{\psi}|\hat{A}|\psi\rangle & \\
(vi) & \mathrm{Str}\,\hat{A} = \mathrm{Tr}[(-1)^{\hat{F}}\hat{A}] = \int d\bar{\psi}d\psi\, e^{-\bar{\psi}\psi}\, \langle\bar{\psi}|\hat{A}|\psi\rangle \ . & (61)
\end{array}$$

The proofs can be obtained by explicit calculation. Let us proceed systematically.

(i) First of all one can expand the exponential and write the coherent state in the following fashion

$$\begin{aligned}
|\psi\rangle &= e^{\hat{\psi}^\dagger\psi}|0\rangle \\
&= (1+\hat{\psi}^\dagger\psi)|0\rangle = |0\rangle - \psi\hat{\psi}^\dagger|0\rangle \\
&= |0\rangle - \psi|1\rangle
\end{aligned} \tag{62}$$

so that one may compute

$$\begin{aligned}
\hat{\psi}|\psi\rangle &= \hat{\psi}e^{\hat{\psi}^\dagger\psi}|0\rangle \\
&= \hat{\psi}\Big(|0\rangle - \psi|1\rangle\Big) = -\hat{\psi}\psi|1\rangle = \psi\hat{\psi}|1\rangle = \psi|0\rangle = \psi\Big(|0\rangle - \psi|1\rangle\Big) \\
&= \psi|\psi\rangle
\end{aligned} \tag{63}$$

which proves that $|\psi\rangle$ is a coherent state. Note that terms proportional to ψ^2 can be inserted or eliminated at wish, as they vanish due to Grassmann property $\psi^2 = 0$.

(ii) "Bra" coherent state. To prove this relation for a bra coherent state it is sufficient to take the hermitian conjugate of the ket coherent state $|\psi\rangle$. One must remember that the definition of hermitian conjugate reduces to complex conjugation for Grassmann variables and includes an exchange of the ordered position of both variables and operators. For example

$$(\hat{\psi}^\dagger \psi)^\dagger = \bar{\psi}\hat{\psi} \ . \tag{64}$$

(iii) Scalar product. It is sufficient to compute it directly (recalling that $\psi^2 = 0$, $\bar{\psi}^2 = 0$ and $\psi\bar{\psi} = -\bar{\psi}\psi$)

$$\begin{aligned} \langle \bar{\psi}|\psi\rangle &= \Big(\langle 0| - \langle 1|\bar{\psi}\Big)\Big(|0\rangle - \psi|1\rangle\Big) \\ &= \langle 0|0\rangle + \bar{\psi}\psi\langle 1|1\rangle = 1 + \bar{\psi}\psi \\ &= e^{\bar{\psi}\psi} \ . \end{aligned} \tag{65}$$

(iv) Resolution of the identity. First of all one must recall that the definition of integration over Grassmann variables makes it identical with differentiation. In particular we use left differentiation, that removes the variable form the left (one must pay attention to signs arising from this operation)

$$\int d\psi \equiv \frac{\partial_L}{\partial \psi} \ , \qquad \int d\bar{\psi} \equiv \frac{\partial_L}{\partial \bar{\psi}} \ . \tag{66}$$

Now a direct calculation shows that

$$\begin{aligned} \int d\bar{\psi} d\psi \, e^{-\bar{\psi}\psi} \, |\psi\rangle\langle\bar{\psi}| &= \int d\bar{\psi} d\psi \, (1 - \bar{\psi}\psi)\Big(|0\rangle - \psi|1\rangle\Big)\Big(\langle 0| - \langle 1|\bar{\psi}\Big) \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| \ . \end{aligned} \tag{67}$$

We point out that the Grassmann variables are here defined to commute with the Fock vacuum $|0\rangle$, so that they commute with the coherent states, but anticommute with $|1\rangle = \hat{\psi}^\dagger|0\rangle$ (as they anticommute with $\hat{\psi}^\dagger$).

(v) Trace. Given a bosonic operator $\hat{A}$, that commutes with ψ and $\bar{\psi}$, one can verify that

$$\begin{aligned} \int d\bar{\psi} d\psi \, e^{-\bar{\psi}\psi} \, \langle -\bar{\psi}|\hat{A}|\psi\rangle &= \int d\bar{\psi} d\psi \, (1 - \bar{\psi}\psi)\Big(\langle 0| + \langle 1|\bar{\psi}\Big)\hat{A}\Big(|0\rangle - \psi|1\rangle\Big) \\ &= \int d\bar{\psi} d\psi \, (1 - \bar{\psi}\psi)\Big(\langle 0|\hat{A}|0\rangle - \bar{\psi}\psi\langle 1|\hat{A}|1\rangle + \cdots\Big) \\ &= \langle 0|\hat{A}|0\rangle + \langle 1|\hat{A}|1\rangle \\ &= \mathrm{Tr}\, \hat{A} \ . \end{aligned} \tag{68}$$

(vi) Supertrace. An analogous calculation gives

$$\begin{aligned} \int d\bar{\psi} d\psi \, e^{-\bar{\psi}\psi} \, \langle \bar{\psi}|\hat{A}|\psi\rangle &= \int d\bar{\psi} d\psi \, (1 - \bar{\psi}\psi)\Big(\langle 0| - \langle 1|\bar{\psi}\Big)\hat{A}\Big(|0\rangle - \psi|1\rangle\Big) \\ &= \int d\bar{\psi} d\psi \, (1 - \bar{\psi}\psi)\Big(\langle 0|\hat{A}|0\rangle + \bar{\psi}\psi\langle 1|\hat{A}|1\rangle + \cdots\Big) \\ &= \langle 0|\hat{A}|0\rangle - \langle 1|\hat{A}|1\rangle = \mathrm{Tr}[(-1)^{\hat{F}}\hat{A}] \\ &= \mathrm{Str}\, \hat{A} \ . \end{aligned} \tag{69}$$

Here $\hat{F}$ is the fermion number operator (or occupation number, with egenvalues $F = 0$ for $|0\rangle$ and $F = 1$ for $|1\rangle$). The last but one line indeed gives the definition of the supertrace.

3 Fermionic path integrals

We now have all the tools to find a path integral representation of the transition amplitude between coherent states $\langle\bar\psi_f|e^{-i\hat H T}|\psi_i\rangle$. We consider an hamiltonian $\hat H = \hat H(\hat\psi^\dagger,\hat\psi)$ written in such a way that all creation operators are on the left of the annihilation operators, something that is always possible to achieve using the fundamental anticommutation relations in (58); indeed for a single pair of fermionic creation and annihilation operators the most general (bosonic) hamiltonian is of the form $\hat H = \omega\hat\psi^\dagger\hat\psi + \omega_0$.

First of all one divides the total propagation time T in N elementary steps of duration $\epsilon = \frac{T}{N}$, so that $T = N\epsilon$. Using $N-1$ times the decomposition of identity in terms of coherent states one obtains the following equalites

$$\begin{aligned}
\langle\bar\psi_f|e^{-i\hat H T}|\psi_i\rangle &= \langle\bar\psi_f|\Big(e^{-i\hat H\epsilon}\Big)^N|\psi_i\rangle = \langle\bar\psi_f|\underbrace{e^{-i\hat H\epsilon}e^{-i\hat H\epsilon}\cdots e^{-i\hat H\epsilon}}_{N\ \text{volte}}|\psi_i\rangle \\
&= \langle\bar\psi_f|e^{-i\hat H\epsilon}\,\mathbb{1}\,e^{-i\hat H\epsilon}\,\mathbb{1}\cdots\mathbb{1}\,e^{-i\hat H\epsilon}|\psi_i\rangle \\
&= \int\Big(\prod_{k=1}^{N-1} d\bar\psi_k d\psi_k\, e^{-\bar\psi_k\psi_k}\Big)\prod_{k=1}^{N}\langle\bar\psi_k|e^{-i\hat H\epsilon}|\psi_{k-1}\rangle
\end{aligned} \tag{70}$$

where we have defined $\psi_0 \equiv \psi_i$ and $\bar\psi_N \equiv \bar\psi_f$. For $\epsilon \to 0$ one can approximate the elementary transition amplitudes as follows

$$\begin{aligned}
\langle\bar\psi_k|e^{-i\hat H(\hat\psi^\dagger,\hat\psi)\epsilon}|\psi_{k-1}\rangle &= \langle\bar\psi_k|\Big(1 - i\hat H(\hat\psi^\dagger,\hat\psi)\epsilon + \cdots\Big)|\psi_{k-1}\rangle \\
&= \langle\bar\psi_k|\psi_{k-1}\rangle - i\epsilon\langle\bar\psi_k|\hat H(\hat\psi^\dagger,\hat\psi)|\psi_{k-1}\rangle + \cdots \\
&= \Big(1 - i\epsilon H(\bar\psi_k,\psi_{k-1}) + \cdots\Big)\langle\bar\psi_k|\psi_{k-1}\rangle \\
&= e^{-i\epsilon H(\bar\psi_k,\psi_{k-1})}\langle\bar\psi_k|\psi_{k-1}\rangle \\
&= e^{-i\epsilon H(\bar\psi_k,\psi_{k-1})}e^{\bar\psi_k\psi_{k-1}}\ .
\end{aligned} \tag{71}$$

The substitution $\hat H(\hat\psi^\dagger,\hat\psi) \to H(\bar\psi_k,\psi_{k-1})$ follows form the ordering of the hamiltonian that we have specified previously ($\hat\psi^\dagger$ on the left and $\hat\psi$ on the right). This allows one to act with the creation operator on the left on one of his eigenstates, and with the annihilation operator on the right on one of his eigenstates, so that all operators in the hamitonian gets substituted by the respective eigenvalues, producing a function of these Grassmann numbers. In this way the hamiltonian operator $\hat H(\hat\psi^\dagger,\hat\psi)$ is substituted by the hamiltonian function $H(\bar\psi_k,\psi_{k-1})$. These approximations are valid for $N\to\infty$, i.e. $\epsilon \to 0$. Substituting (71) in (70) one gets

$$\begin{aligned}
\langle\bar\psi_f|e^{-i\hat H T}|\psi_i\rangle &= \lim_{N\to\infty}\int\Big(\prod_{k=1}^{N-1} d\bar\psi_k d\psi_k\, e^{-\bar\psi_k\psi_k}\Big)e^{i\sum_{k=1}^{N}[-i\bar\psi_k\psi_{k-1} - H(\bar\psi_k,\psi_{k-1})\epsilon]} \\
&= \lim_{N\to\infty}\int\Big(\prod_{k=1}^{N-1} d\bar\psi_k d\psi_k\Big)e^{i\sum_{k=1}^{N}[i\bar\psi_k(\psi_k-\psi_{k-1}) - H(\bar\psi_k,\psi_{k-1})\epsilon]}e^{\bar\psi_N\psi_N} \\
&= \lim_{N\to\infty}\int\Big(\prod_{k=1}^{N-1} d\bar\psi_k d\psi_k\Big)e^{i\sum_{k=1}^{N}[i\bar\psi_k\frac{(\psi_k-\psi_{k-1})}{\epsilon} - H(\bar\psi_k,\psi_{k-1})]\epsilon + \bar\psi_N\psi_N} \\
&= \int\mathcal{D}\bar\psi\mathcal{D}\psi\, e^{i\int_0^T dt[i\bar\psi\dot\psi - H(\bar\psi,\psi)] + \bar\psi(T)\psi(T)} = \int\mathcal{D}\bar\psi\mathcal{D}\psi\, e^{iS[\bar\psi,\psi]}\ .
\end{aligned} \tag{72}$$

This the path integral for one complex fermionic degree of freedom. We recognize in the exponent a discretization of the classical action

$$
\begin{aligned}
S[\bar\psi,\psi] &= \int_0^T dt\,[i\bar\psi\dot\psi - H(\bar\psi,\psi)] - i\bar\psi(T)\psi(T) \\
&\to \sum_{k=1}^{N-1}[i\bar\psi_k\frac{(\psi_k-\psi_{k-1})}{\epsilon} - H(\bar\psi_k,\psi_{k-1})]\epsilon - i\bar\psi_N\psi_N
\end{aligned} \tag{73}
$$

where $T = N\epsilon$ is the total propagation time. The last way of writing the amplitude in (72) is symbolic and indicates the formal sum over all paths $\bar\psi(t),\psi(t)$ such that $\psi(0)=\psi_0\equiv\psi_i$ and $\bar\psi(T)=\bar\psi_N\equiv\bar\psi_f$, weighed by the exponential of i times the classical action $S[\bar\psi,\psi]$ that contains also the boundary value $-i\bar\psi(T)\psi(T)$.

Trace

We can bow calculate the trace of the transition amplitude $e^{-i\hat H T}$. Using coherent states and the path integral representation of the transition amplitude one has

$$
\begin{aligned}
\mathrm{Tr}[e^{-i\hat H T}] &= \int d\bar\psi_0 d\psi_0\, e^{-\bar\psi_0\psi_0}\,\langle-\bar\psi_0|e^{-i\hat H T}|\psi_0\rangle \\
&= \lim_{N\to\infty}\int\Big(\prod_{k=1}^N d\bar\psi_k d\psi_k\Big)e^{i\sum_{k=1}^N[i\bar\psi_k\frac{(\psi_k-\psi_{k-1})}{\epsilon}-H(\bar\psi_k,\psi_{k-1})]\epsilon} \\
&= \int_{ABC}\mathcal{D}\bar\psi\mathcal{D}\psi\, e^{iS[\bar\psi,\psi]}
\end{aligned} \tag{74}
$$

where we have identified $\bar\psi_N = -\bar\psi_0$ and $\psi_N = -\psi_0$, and used that the exponential $e^{-\bar\psi_0\psi_0}$ present in the measure for calculating the trace cancels the boundary term $e^{\bar\psi_N\psi_N}$. In the continuum limit it emerges the sum on all antiperiodic paths (ABC, antiperiodic boundary conditions), i.e. such that $\psi(T)=-\psi(0)$ and $\bar\psi(T)=-\bar\psi(0)$.

Supertrace

Similarly, the supertrace is calculated by

$$
\begin{aligned}
\mathrm{Str}[e^{-i\hat H T}] &= \int d\bar\psi_0 d\psi_0\, e^{-\bar\psi_0\psi_0}\,\langle\bar\psi_0|e^{-i\hat H T}|\psi_0\rangle \\
&= \lim_{N\to\infty}\int\Big(\prod_{k=1}^N d\bar\psi_k d\psi_k\Big)e^{i\sum_{k=1}^N[i\bar\psi_k\frac{(\psi_k-\psi_{k-1})}{\epsilon}-H(\bar\psi_k,\psi_{k-1})]\epsilon} \\
&= \int_{PBC}\mathcal{D}\bar\psi\mathcal{D}\psi\, e^{iS[\bar\psi,\psi]}
\end{aligned} \tag{75}
$$

where now we have identified $\bar\psi_N = \bar\psi_0$ and $\psi_N = \psi_0$. Again the exponential $e^{-\bar\psi_0\psi_0}$ present in the measure for calculating the supertrace cancels the boundary term $e^{\bar\psi_N\psi_N}$. In there continuum limit the sum is over all periodic trajectories (PBC, periodic boundary conditions) defined by the boundary conditions $\psi(T)=\psi(0)$ and $\bar\psi(T)=\bar\psi(0)$.

We have derived the path integral for fermionic systems from the operatorial formulation using a time slicing of the total propagation time. This produces a discretization of the continuum classical action which allows to define concretely the path integral when written directly in the continuum. This corresponds to the time slicing regularization of the path integral. Other regularizations will be described later on in the book. We have discussed just a simple model with one complex degree of freedom, $\psi(t)$ and its complex conjugate $\bar{\psi}(t)$. This may be called a Dirac fermion in one dimension. The extension of the path integral to additional complex degrees of freedom is immediate.

A more subtle situation arises if one imposes a reality condition on the fermionic variable $\psi(t)$. In a sense this is the minimal model for fermions. Since $\bar{\psi}(t) = \psi(t)$, it is referred to as a Majorana fermion in one dimension, because of similar definitions for fermionic fields of spin 1/2 in four-dimensions. Its action is of the form

$$S[\psi] = \int dt \, \frac{i}{2} \psi \dot{\psi} \, . \tag{76}$$

It is formally real and non vanishing because of the Grassmann character of the variable, but no nontrivial even term can be written down for an hamiltonian. For a number $n > 1$ of Majorana fermions ψ^i, $i = 1, ..., n$, one can write down instead a nontrivial hamiltonian

$$S[\psi^i] = \int dt \left(\frac{i}{2} \psi^i \dot{\psi}^i - H(\psi^i) \right) \, . \tag{77}$$

Note that for a single Majorana fermion ψ the canonical quantization gives rise to the anticommutation relation

$$\{\psi, \psi\} = 1 \tag{78}$$

which may be realized by a number, $\psi = \frac{1}{\sqrt{2}}$. The model is empty, and indeed it carries no hamiltonian. For an even number of Majorana fermions, say $n = 2m$, one has instead

$$\{\psi^i, \psi^j\} = \delta^{ij} \tag{79}$$

so that one can realize the algebra by pairing together the Majorana fermions to obtain m complex Dirac fermions, rewrite the anticommutator algebra in this basis, and realize it as a set of independent fermionic creation and annihilation operators on a corresponding fermionic Fock space, as in (58). Then one may proceed in constructing the path integral. This procedure is sometimes called "fermion halving", as one halves the number of fermions at the price of making them complex. However, this procedure may hide symmetries manifest in the Majorana basis.

To avoid this last problem, one may instead add a second set of (free) Majorana fermions χ^i to be able to make n complex combinations $\Psi^i = \frac{1}{\sqrt{2}}(\psi^i + i\chi^i)$ and with these Dirac fermions proceed again as before in the construction of the path integral. The fermions χ^i are just free spectators, and the hamiltonian does not contain them. In this case one must be careful to correct appropriately the overall normalization of the path integral, especially when computing traces, as the physical Hilbert space of the ψ^i has dimension $2^{\frac{n}{2}}$, which differs from the dimensions of the full unphysical Hilbert space of the Ψ^i that is 2^n. In this sense this procedure is sometimes called "fermion doubling", as the number of Majorana fermions is doubled.

Finally if the number of Majorana fermions is odd, say $n = 2m + 1$, one can treat the first $2m$ fermions in one of the two ways, but for the last one one must necessarily use the fermion doubling procedure.

An important application of Majorana fermions is in the worldline description of spin 1/2 particles in spacetimes of dimensions $D > 1$, $D = 4$ in particular.

To summarize, we have discussed the path integral quantization of fermionic theories with one-dimensional Majorana fermions ψ^i

$$S[\psi] = \int dt \left(\frac{i}{2} \psi^i \dot{\psi}^i - H(\psi^i) \right) \quad \to \quad \int \mathcal{D}\psi \; e^{iS[\psi]} \tag{80}$$

and one-dimensional Dirac fermions $\psi^i, \bar{\psi}_i$

$$S[\psi, \bar{\psi}] = \int dt \left(i \bar{\psi}_i \dot{\psi}^i - H(\psi^i, \bar{\psi}_i) \right) \quad \to \quad \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \; e^{iS[\psi, \bar{\psi}]} \tag{81}$$

whose path integral is concretely made sense of by using the time slicing discretizations. Other regularizations will be discussed later on.

One can again perform a Wick rotation to an euclidean time τ by $t \to -i\tau$, and derive the euclidean path integrals of the form

$$\int \mathcal{D}\psi \; e^{-S_E[\psi]} \qquad \text{and} \qquad \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \; e^{-S_E[\psi, \bar{\psi}]} \tag{82}$$

where now

$$S[\psi] = \int dt \left(\frac{1}{2} \psi^i \dot{\psi}^i + H(\psi^i) \right) \qquad \text{and} \qquad S[\psi, \bar{\psi}] = \int dt \left(\bar{\psi}_i \dot{\psi}^i + H(\psi^i, \bar{\psi}_i) \right). \tag{83}$$

In this form they will be used in the subsequent parts of the book.

3.1 Correlation functions

Correlation functions are defined as normalized averages of the dynamical variables with the path integral. Again, one may introduce a generating functional by adding sources to the path integral. Using an hypercondensed notation by denoting all fermions by ψ^i and the corresponding sources by η_i (taking values in a Grassmann algebra) one may write down the generating functional of correlation functions

$$Z[\eta] = \int \mathcal{D}\psi \; e^{iS[\psi] + i\eta_i \psi^i} \tag{84}$$

so that, as an example, the two point function is given by

$$\langle \psi^i \psi^j \rangle = \frac{\int \mathcal{D}\psi \; \psi^i \psi^j \; e^{iS[\psi]}}{\int \mathcal{D}\psi \; e^{iS[\psi]}} = \frac{1}{Z[0]} \left(\frac{1}{i} \right)^2 \frac{\delta^2 Z[\eta]}{\delta \eta_i \delta \eta_j} \bigg|_{\eta=0}. \tag{85}$$

For a free theory, identified by a quadratic action of the form $S[\psi] = -\frac{1}{2} \psi^i K_{ij} \psi^j$ with K_{ij} an antisymmetric matrix, one may formally compute the path integral with sources by gaussian integration (after completing squares and making use of the transitional invariance of the measure), thus obtaining an answer of the form

$$Z[\eta] = \text{Pfaff}(K_{ij}) \; e^{-\frac{i}{2} \eta_i G^{ij} \eta_j} \tag{86}$$

where G^{ij} is the inverse of K_{ij}, also an antisymmetric matrix. One finds

$$\langle \psi^i \psi^j \rangle = -iG^{ij} \tag{87}$$

where, of course, G^{ij} is interpreted as a Green function in quantum mechanical applications. To get the correct overall normalization one must be careful with signs arising from the anti-commuting character of the Grassmann variables as well as from the antisymmetric properties of K_{ij} and G^{ij}.

Similar formulae may be written down for complex fermions (though they are contained in the above formula as well) and in a euclidean time obtained after the Wick rotation. Eventually one must take into account the chosen boundary conditions on the path integral and use the corresponding Green functions. The whole set of generating functions described for the bosonic case may be introduced here as well. They are not used in this book, so that we leave their derivation as an exercize to the reader.

4 Supersymmetric quantum mechanics

Very special quantum mechanical models containing bosons and fermions may exhibit the property of supersymmetry, a symmetry that finds many applications in theoretical physics. In quantum field theories supersymmetry relates particles with integer spins, the bosons, to particles with half-integer spins, the fermions. It is employed to describe possible extensions of the standard model of elementary particles that softens hierarchical problems related to the mass of the Higgs boson. More theoretically, it helps improving the ultraviolet behavior of gravitational theories (supergravities) and it is a fundamental ingredient of string theory.

At the quantum mechanical level, supersymmetry appears in worldline models for particles with spin. It also finds beautiful applications in mathematical physics, as in producing simple and elegant proof of index theorems that relate topological properties of differential manifolds to local properties.

In the following section we present a class of models that exemplifies several aspects of supersymmetry in the simple setting of quantum mechanics. It includes as a particular case the supersymmetric harmonic oscillator. In particular we will describe the path integral quantization and its use to calculate the so called Witten index. As cautionary note, once should observe that in the description of particles in first quantization two spaces are present: the worldline, considered as a one dimensional space-time, and the target space-time on which the particle propagates and which is the embedding space of the worldline. One may have supersymmetries on both spaces: (i) on the worldline, which is the case we are going to analyze and which is useful to describe a spinning particle, that is a particle with intrinsic spin, and (ii) on target space-time, in which case the model describes a multiplet of particles containing bosons and fermions, that is particles with integer spin and particles with half integer spin, in equal numbers. These models are often called superparticles, and will not be discussed in this book.

N=2 supersymmetric model

We introduce now a supersymmetric model for the motion of a particle of unit mass on a line. The particle is described by hermitian bosonic operators, the position $\hat{x}$ and the momentum $\hat{p}$, satisfying the usual commutation relation

$$[\hat{x}, \hat{p}] = i \tag{88}$$

augmented by fermionic operators ψ and $\psi^\dagger$ that satisfy anti commutation relations

$$\{\psi, \psi^\dagger\} = 1 \ . \tag{89}$$

As we shall see, the latter can be identified with fermionic annihilation and creation operators. Bosonic operators commute with the fermionic ones.

The dynamics is specified by an hamiltonian that depends on a function $W(x)$, called the prepotential, through its first and second derivatives $W'(x)$ and $W''(x)$

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\Big(W'(\hat{x})\Big)^2 + \frac{1}{2}W''(\hat{x})(\hat{\psi}^\dagger\hat{\psi} - \hat{\psi}\hat{\psi}^\dagger) \; . \tag{90}$$

In addition the model has conserved charges

$$\begin{aligned} \hat{Q} &= (\hat{p} + iW'(\hat{x}))\hat{\psi} \\ \hat{Q}^\dagger &= (\hat{p} - iW'(\hat{x}))\psi^\dagger \\ \hat{F} &= \hat{\psi}^\dagger\hat{\psi} \end{aligned} \tag{91}$$

that indeed commute with the hamiltonian, as can be verified by an explicit calculation

$$[\hat{Q}, \hat{H}] = [\hat{Q}^\dagger, \hat{H}] = [\hat{F}, \hat{H}] = 0 \; . \tag{92}$$

The charges $\hat{Q}$ and $\hat{Q}^\dagger$ are the supercharges that generate supersymmetry transformations. They are fermionic operators. There are two of them, $\hat{Q}$ and $\hat{Q}^\dagger$, or equivalently the real and the imaginary part of $\hat{Q}$, so that there are two supersymmetries, that is $N = 2$ supersymmetry. The hamiltonian may look complicated, but in supersymmetric models it is related to the anticommutator of the supercharges, which are more basic objects. In the present case one has

$$\{\hat{Q}, \hat{Q}^\dagger\} = 2\hat{H} \; , \quad \{\hat{Q}, \hat{Q}\} = \{\hat{Q}^\dagger, \hat{Q}^\dagger\} = 0 \; . \tag{93}$$

These relations contain the hallmark of supersymmetry: the hamiltonian, which is the generator of time translations, is related to the anticommutator of the supersymmetry charges, that is a time translation is obtained by the composition of two supersymmetry transformations.

In the present case there is also the conserved fermion number operator $\hat{F}$, that is seen to satisfy

$$[\hat{F}, \hat{Q}] = -\hat{Q} \; , \quad [\hat{F}, \hat{Q}^\dagger] = \hat{Q}^\dagger \; . \tag{94}$$

To summarize, the $N = 2$ supersymmetric model is characterized by a $N = 2$ supersymmetry algebra, a superalgebra with bosonic ($\hat{H}$, $\hat{F}$) and fermionic ($\hat{Q}$, $\hat{Q}^\dagger$) charges with the following non vanishing graded commutation relations

$$\{\hat{Q}, \hat{Q}^\dagger\} = 2\hat{H} \; , \quad [\hat{F}, \hat{Q}] = -\hat{Q} \; , \quad [\hat{F}, \hat{Q}^\dagger] = \hat{Q}^\dagger \; . \tag{95}$$

The operators of the model are realized explicitly in a Hilbert spaces constructed as follows. The operators $\hat{x}$ and $\hat{p}$ are realized in the usual way on the Hilbert space of square integrable functions by multiplication $\hat{x} \to x$ and differentiation $\hat{p} \to -i\frac{\partial}{\partial x}$. As for the fermionic operators $\hat{\psi}^\dagger$ e $\hat{\psi}$, their algebra identifies them as fermionic creation and annihilation operators that that can be realized on a two dimensional Fock space with basis $|0\rangle$ and $|1\rangle$, defined by

$$\begin{aligned} &\hat{\psi}|0\rangle = 0, \qquad \langle 0|0\rangle = 1 \qquad \text{(Fock vacuum)} \\ &|1\rangle = \hat{\psi}^\dagger|0\rangle \qquad \text{(state with one fermionic exitation).} \end{aligned} \tag{96}$$

One cannot add additional excitations, as the creation operator satisfies $\{\psi^\dagger, \psi^\dagger\} = 0$, so that $|2\rangle = \hat{\psi}^\dagger|1\rangle = (\hat{\psi}^\dagger)^2|0\rangle = 0$. Thus, one has a Pauli exclusion principle for the states created by the fermionic operators.

The full Hilbert space is thus the direct product of the two spaces described above, so that one may realize the states by a wave function with two components

$$\Psi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \tag{97}$$

on which the basic operators act as follows

$$\begin{aligned} \hat{x} &\longrightarrow x \\ \hat{p} &\longrightarrow -i\frac{\partial}{\partial x} \\ \hat{\psi} &\longrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \hat{\psi}^\dagger &\longrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \end{aligned} \tag{98}$$

As a consequence, one finds the following explicit realization of the conserved charges of the model

$$\begin{aligned} \hat{Q} &= \begin{pmatrix} 0 & -i\partial_x + iW'(x) \\ 0 & 0 \end{pmatrix} \\ \hat{Q}^\dagger &= \begin{pmatrix} 0 & 0 \\ -i\partial_x - iW'(x) & 0 \end{pmatrix} \\ \hat{H} &= \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\partial_x^2 + \frac{1}{2}(W'(x))^2 - \frac{1}{2}W''(x) & 0 \\ 0 & -\frac{1}{2}\partial_x^2 + \frac{1}{2}(W'(x))^2 + \frac{1}{2}W''(x) \end{pmatrix} \\ \hat{F} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{99}$$

The hamiltonian is diagonal in the Fock space basis and contains the two hamiltonians $H_\mp$ with potentials $V_\mp = \frac{1}{2}(W'(x))^2 \mp \frac{1}{2}W''(x)$. We assume them to have a suitable asymptotic behavior that confine the particle, so that the resulting bound states have normalized wave functions.

Having defined $\hat{F}$ as the fermion number, we see that the states of the form

$$\Psi(x) = \begin{pmatrix} \varphi_1(x) \\ 0 \end{pmatrix}$$

are bosonic as they have fermion number 0, while states of the form

$$\Psi(x) = \begin{pmatrix} 0 \\ \varphi_2(x) \end{pmatrix}$$

are fermionic as they have fermion number 1. This a worldline nomenclature that allows us to exemplify the properties of supersymmetry. From the target space perspective there is no supersymmetry, and the model describes a particle on a line which may have two polarizations of its intrinsic spin.

A fundamental property that emerges form supersymmetry is that the hamiltonian operators is necessarily positive definite, as consequence of the algebra $\{\hat{Q}, \hat{Q}^\dagger\} = 2\hat{H}$. The associated Schrödinger equation takes the standard form

$$i\frac{\partial}{\partial t}\Psi = \hat{H}\Psi \tag{100}$$

and in the following we analyze some properties of the energy eigenstates that follows form suspersymmetry.

General properties of supersymmetry

Let us discuss general properties of supersymmetry that follows form its algebraic structure. We assume that the supersymmetric model has an hamiltonian $\hat{H}$, one hermitian supersymmetry charge $\hat{Q}$, and the parity operator $(-1)^{\hat{F}}$ satisfying the following algebra

$$\{\hat{Q}, \hat{Q}\} = \hat{H}, \qquad \{(-1)^{\hat{F}}, \hat{Q}\} = 0, \qquad [(-1)^{\hat{F}}, \hat{H}] = 0 \; . \tag{101}$$

This algebra is general enough to derive properties of many supersymmetric systems.

In the previous case we have presented quantum mechanical theories with an hamiltonian $\hat{H}$, two supersymmetry charges $\hat{Q}$ and $\hat{Q}^\dagger$, and the fermion number operator $\hat{F}$. As one may equivalently use the two hermitian supersymmetry charges $\hat{Q}_1$ and $\hat{Q}_2$, where $Q = \frac{1}{\sqrt{2}}(\hat{Q}_1 + i\hat{Q}_2)$, we satisy the above requirements using one of its hermitian charges, say Q_1. Also one may exponentiate the fermion number operator to obtain the unitary operator $e^{i\alpha\hat{F}}$ that perform phase rotations, so that for $\alpha = \pi$ one has the parity operator$(-1)^{\hat{F}}$ needed to meet the above requirements.

We also assume a discrete energy spectrum to guarantee that the energy eigenstes are normalizable, and of course the existence of an Hilbert space $\mathcal{H}$ with a positive definite inner product.

One can prove the following properties:

Property 1: *The hamiltonian $\hat{H}$ is positive definite.*

That means that for any vector $|\Psi\rangle$ of the Hilbert space $\mathcal{H}$

$$\langle\Psi|\hat{H}|\Psi\rangle \geq 0 \; . \tag{102}$$

Indeed using $\hat{H} = \{\hat{Q}, \hat{Q}\}$ with a hermitian Q one calculates

$$\langle\Psi|\hat{H}|\Psi\rangle = 2\,\langle\Psi|\hat{Q}\hat{Q}|\Psi\rangle = 2\,||\hat{Q}|\Psi\rangle|^2 \geq 0 \; . \tag{103}$$

Moreover, this shows that an energy eigenstate with energy E must have $E \geq 0$. This implies the following property.

Proprety 2: *Any state $|\Psi_0\rangle$ with $E = 0$ is necessarily a ground state.*

Indeed, that is the lowest energy possible. In addition, from $\langle\Psi_0|\hat{H}|\Psi_0\rangle = 0$, recalling property 1 and the fact that the Hilbert space has a positive definite norm, one finds that

$$\hat{Q}|\Psi_0\rangle = 0 \tag{104}$$

i.e. $|\Psi_0\rangle$ is a supersymmetric state, a state invariant under supersymmetry transformations.

Property 3: *Energy levels with $E \neq 0$ are degenerate.*

Indeed, for any "bosonic" state ($F = 0$) with $E \neq 0$ there must exists a "fermionic" state ($F = 1$) with the same energy, and viceversa. To prove this statement let us consider a bosonic state $|b\rangle$ (that we assume to be normalizable) with energy $E \neq 0$

$$\begin{aligned} \hat{H}|b\rangle &= E|b\rangle \ , \quad E \neq 0 \ , \quad \langle b|b\rangle = ||b\rangle|^2 = 1 \\ (-1)^{\hat{F}}|b\rangle &= |b\rangle \ . \end{aligned} \tag{105}$$

Then one can construct

$$|f'\rangle = \hat{Q}|b\rangle \tag{106}$$

that is an energy eigenstate of opposite fermionic number

$$\hat{H}|f'\rangle = E|f'\rangle \ , \qquad (-1)^{\hat{F}}|f'\rangle = -|f'\rangle \ . \tag{107}$$

Indeed, using $[\hat{H}, \hat{Q}] = 0$, one finds

$$\hat{H}|f'\rangle = \hat{H}\hat{Q}|b\rangle = \hat{Q}\hat{H}|b\rangle = \hat{Q}E|b\rangle = E|f'\rangle \tag{108}$$

and similarly, using $\{(-1)^{\hat{F}}, \hat{Q}\} = 0$, one finds

$$(-1)^{\hat{F}}|f'\rangle = (-1)^{\hat{F}}\hat{Q}|b\rangle = -\hat{Q}(-1)^{\hat{F}}|b\rangle = -\hat{Q}|b\rangle = -|f'\rangle \ . \tag{109}$$

In addition, the state can be properly normalized by $|f\rangle = \frac{1}{\sqrt{E}}|f'\rangle$ since $E \neq 0$, so that $\langle f|f\rangle = 1$.

Thus, the states $|b\rangle$ and $|f\rangle$ are degenerate in energy and they have opposite fermionic parity: they form a two dimensional representation of the supersymmetry algebra. Of course, the same procedure above can be repeated starting from a fermionic eigenstes with $E \neq 0$ and construct the degenerate bosonic energy eigenstate.

One deduces that all energy levels with $E \neq 0$ are degenerate and with an equal number of bosonic and fermionic states, while only the states with $E = 0$ are supersymmetric and if they exists are ground states of the model. This property allows to introduce the Witten index as the number of bosonic minus the number of fermionic ground states. One can write this as $\mathrm{Tr}(-1)^{\hat{F}}$, with the trace extended over the full Hilbert space $\mathcal{H}$, as positive energy states cancels pairwise in the sum and do not contribute

$$\mathrm{Tr}(-1)^{\hat{F}} = n_b^{(E=0)} - n_f^{(E=0)} \ . \tag{110}$$

To make sure that the cancellation of positive energy states is achieved orderly one should regulate the Witten index as

$$\mathrm{Tr}[(-1)^F e^{-\beta\hat{H}}] \tag{111}$$

that is verified to be independent of the regulating parameter β. Indeed, using the complete basis of energy eigenstates one calculates

$$\begin{aligned} \mathrm{Tr}[(-1)^{\hat{F}} e^{-\beta\hat{H}}] &= \sum_k \langle k|(-1)^{\hat{F}} e^{-\beta E_k}|k\rangle \\ &= n_b^{(E=0)} - n_f^{(E=0)} + e^{-\beta E_1} - e^{-\beta E_1} + e^{-\beta E_2} - e^{-\beta E_2} + \ldots \\ &= n_b^{(E=0)} - n_f^{(E=0)} \\ &= \mathrm{Tr}(-1)^{\hat{F}} \ . \end{aligned} \tag{112}$$

The Witten index has topological properties in the sense that it is invariant under reasonable deformations of the parameters of the theory (e.g. by varying the coupling constants without modifying the asymptotic behavior of the potential): only pairs of states can leave the zero energy level by acquiring a small value of the energy, and this pair must necessarily form a supersymmetry doublet containing one bosonic and one fermionic state. Viceversa, states with $E \neq 0$ are paired and by varying the parameters of the theory they can join the set of ground states without modifying the Witten index.

Witten index

Let us calculate the Witten index in the class of models with $N = 2$ supersymmetries presented above. A vacuum state $|\Psi_0\rangle$ with $E = 0$ satisfies $\hat{H}|\Psi_0\rangle = 0$ and must necessarily be supersymmetric

$$\hat{Q}|\Psi_0\rangle = 0 \, , \qquad \hat{Q}^\dagger|\Psi_0\rangle = 0 \, . \tag{113}$$

The solutions of these equations have the following general form

$$\Psi_0(x) = \begin{pmatrix} c_- e^{-W(x)} \\ c_+ e^{+W(x)} \end{pmatrix} \tag{114}$$

where of course the wave function is given by $\Psi_0(x) = \langle x|\Psi_0\rangle$. Indeed,

$$\begin{aligned} \hat{Q}\Psi_0(x) &= \begin{pmatrix} 0 & -i\partial_x + iW'(x) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \star \\ c_+ e^{W(x)} \end{pmatrix} \\ &= \begin{pmatrix} -ic_+(W' - W')e^{W(x)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \end{aligned} \tag{115}$$

Similarly $\hat{Q}^\dagger\Psi_0(x) = 0$, as

$$\begin{aligned} \hat{Q}^\dagger\Psi_0(x) &= \begin{pmatrix} 0 & 0 \\ -i\partial_x - iW'(x) & 0 \end{pmatrix} \begin{pmatrix} c_- e^{-W(x)} \\ \star \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -ic_-(-W' + W')e^{-W(x)} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \, . \end{aligned} \tag{116}$$

Requiring the wave functions to be normalizable fixes the constants c_- and c_+. There are three different cases:

1. Case $W(x) \stackrel{x\to\pm\infty}{\longrightarrow} \infty$.

In such a situation (114) must have $c_+ = 0$ to be normalizable. It has fermionic parity $(-1)^F = 1$ as

$$\hat{F}\Psi_0(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_- e^{-W(x)} \\ 0 \end{pmatrix} = 0 \tag{117}$$

and the Witten index is calculated to be $\mathrm{Tr}(-1)^{\hat{F}} = 1$.

2. Case $W(x) \stackrel{x\to\pm\infty}{\longrightarrow} -\infty$

Now (114) must have $c_- = 0$ to be normalizable. It has fermionic number $F = 1$ and parity $(-1)^F = -1$, so that the Witten index is $\mathrm{Tr}(-1)^{\hat{F}} = -1$.

3. Case $W(x) \stackrel{x\to\pm\infty}{\longrightarrow} \pm\infty$ oppure $W(x) \stackrel{x\to\pm\infty}{\longrightarrow} \mp\infty$

In this case $c_- = 0$ and $c_+ = 0$, so that there are no normalizables solutions with $E = 0$. The Witten index vanishes $\text{Tr}(-1)^{\hat{F}} = 0$.

Typically one says that supersymmetry is spontaneously broken, as the ground state is not invariant under supersymmetry transformations. The concept of spontaneous breaking of symmetries is of fundamental importance in QFT applications.

We can verify in our model the topological properties of the Witten index: one can modify at will the prepotential $W(x)$ without altering its asymptotic behavior, but these deformations do not modify the value of $\text{Tr}(-1)^{\hat{F}}$.

The vanishing of the Witten index does not guarantee that supersymmetry is spontaneously broken (for example there might exist two ground states with $E = 0$ with opposite fermionic parity), but one may be sure that if the Witten index is different form zero, then supersymmery cannot be spontaneously broken: there will be always zero energy supersymmetric ground states.

Classical action

One needs the classical action to be able to study the model through a path integral. The action makes use of Grassmann variables for treating the worldline fermions at the classical level. We first presentthe action, and then show that its canonical quantization givers rise precisely to the previous model.

The action invariant under two supersymetries is given by

$$S[x, \psi, \bar{\psi}] = \int dt \left[\frac{1}{2}\dot{x}^2 + i\bar{\psi}\dot{\psi} - \frac{1}{2}\Big(W'(x)\Big)^2 - W''(x)\bar{\psi}\psi \right] \tag{118}$$

where: $x(t)$ is the coordinate of the particles in one dimension, $\psi(t)$ and $\bar{\psi}(t)$ are the complex Grassmann variables ($\bar{\psi}$ denotes the complex conjugate of ψ) that describe the additional degrees of freedom related to the spin. Alternatively, one may use the real Grassmann variables ψ_1 and ψ_2, representing the real and imaginary part of ψ, $\psi = (\psi_1 + i\psi_2)/\sqrt{2}$.

The prepotential $W(x)$ satisfies suitable asymptotic conditions to guarantee bound states at the quantum level. The choice $W(x) = \frac{1}{2}\omega x^2$ identified the so called supersymmetric harmonic oscillator.

The equations of motion are easily obtainable by extremizing the action

$$\begin{aligned}
\frac{\delta S}{\delta x(t)} = 0 \quad &\Rightarrow \quad \ddot{x} + W'(x)W''(x) + W'''(x)\bar{\psi}\psi = 0 \\
\frac{\delta S}{\delta \bar{\psi}(t)} = 0 \quad &\Rightarrow \quad i\dot{\psi} - W''(x)\psi = 0 \\
\frac{\delta S}{\delta \psi(t)} = 0 \quad &\Rightarrow \quad i\dot{\bar{\psi}} + W''(x)\bar{\psi} = 0 \, .
\end{aligned} \tag{119}$$

The action in invariant under infinitesimal supersymmetry transformations (with ϵ and $\bar{\epsilon}$ Grassmann constants) given by

$$\begin{aligned}
\delta x &= i\epsilon\bar{\psi} + i\bar{\epsilon}\psi \\
\delta \psi &= -\epsilon(\dot{x} - iW'(x)) \\
\delta \bar{\psi} &= -\bar{\epsilon}(\dot{x} + iW'(x))
\end{aligned} \tag{120}$$

and under $U(1)$ phase transformation with infinitesimal parameter α

$$\begin{aligned} \delta x &= 0 \\ \delta\psi &= i\alpha\psi \\ \delta\bar{\psi} &= -i\alpha\bar{\psi} \,. \end{aligned} \tag{121}$$

The corresponding conserved Noether charges are easily identified to be

$$\begin{aligned} H &= \frac{1}{2}\dot{x}^2 + \frac{1}{2}\Big(W'(x)\Big)^2 + W''(x)\bar{\psi}\psi \\ Q &= (\dot{x} + iW'(x))\psi \\ \bar{Q} &= (\dot{x} - iW'(x))\bar{\psi} \\ F &= \bar{\psi}\psi \end{aligned} \tag{122}$$

where we have added the energy H arising as a consequence of time translations invariance.

One may note that the commutator of two supersymmetry transformation generates a time translation. In fact, calculating it onto the dynamical variable x one obtains

$$[\delta(\epsilon_1), \delta(\epsilon_2)]x = -a\dot{x} \tag{123}$$

with $a = 2i(\epsilon_1\bar{\epsilon}_2 - \epsilon_2\bar{\epsilon}_1)$. This is the characterizing property of supersymmetry: the composition of two supersymmetries generate a time translation. The same property can be verified on the variables ψ and $\bar{\psi}$ (in this case on the right hand side there appears also terms proportional to the equations of motion)

Hamiltomian formalism

To perform canonical quantization we have to reformulate the model in phase space. The conjugate momentum to the variable x is obtained as usual by

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x} \tag{124}$$

As for the Grassmann variables, they have equations of motions that are first order in time, so that they are already in a hamiltonian form. The momentum conjugate to ψ is proportional to $\bar{\psi}$ as

$$\pi = \frac{\partial_L L}{\partial \dot{\psi}} = -i\psi \tag{125}$$

where ∂_L denotes left differentiation (one commutes the variable to the left and then removes it). The corresponding hamiltonian is given by the Legendre transform of the lagrangian

$$\begin{aligned} H &= \dot{x}p + \dot{\psi}\pi - L \\ &= \frac{1}{2}p^2 + \frac{1}{2}\Big(W'(x)\Big)^2 + W''(x)\bar{\psi}\psi \,. \end{aligned} \tag{126}$$

Thus the phase space action is given by

$$S[x, p, \psi, \bar{\psi}] = \int dt(\dot{x}p + i\bar{\psi}\dot{\psi} - H) \,. \tag{127}$$

The basic Poisson brackets are given by

$$\{p, x\}_{PB} = -1 \,, \quad \{\pi, \psi\}_{PB} = -1 \tag{128}$$

or equivalently, recalling that they are antisymmetric for commuting variables and symmetric for anticommuting ones,

$$\{x,p\}_{PB} = 1 \ , \quad \{\psi,\bar\psi\}_{PB} = -i \ . \tag{129}$$

In general, for arbitrary phase space functions A and B Grassmann parity ϵ_A ed ϵ_B ($\epsilon = 0$ for commuting functions and $\epsilon = 1$ for anticommuting ones) the Poisson brackets are defined by

$$\{A,B\}_{PB} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - i\frac{\partial_R A}{\partial \psi}\frac{\partial_L B}{\partial \bar\psi} - i\frac{\partial_R A}{\partial \bar\psi}\frac{\partial_L B}{\partial \psi} \tag{130}$$

wher ∂_R and ∂_L denote right and left derivatives, respectively. One must recall that these brackets satisfy the properties

$$\begin{aligned} \{A,B\}_{PB} &= (-1)^{\epsilon_A\epsilon_B+1}\{B,A\}_{PB} \\ \{A,BC\}_{PB} &= \{A,B\}_{PB}C + (-1)^{\epsilon_A\epsilon_B}B\{A,C\}_{PB} \end{aligned} \tag{131}$$

The Noether charges calculated previously can be transcribed in phase space as

$$\begin{aligned} H &= \frac{1}{2}p^2 + \frac{1}{2}\Big(W'(x)\Big)^2 + W''(x)\bar\psi\psi \\ Q &= (p + iW'(x))\psi \\ \bar Q &= (p - iW'(x))\bar\psi \\ F &= \bar\psi\psi \end{aligned} \tag{132}$$

and satisfy the following a Poisson bracket algebra whose non vanishing terams are given by

$$\{Q,\bar Q\}_{PB} = -2iH \ , \quad \{J,Q\}_{PB} = iQ \ , \quad \{J,\bar Q\}_{PB} = -i\bar Q \ . \tag{133}$$

The first one shows that the composition of two supersymmetries generate a translation in time.

Path integrals and Witten index

One can now give a path integral representation of the Witten index. The regulated form of the index $\mathrm{Tr}[(-1)^{\hat F}e^{-\beta\hat H}]$ can be obtained by a Wick rotation $t \to -i\beta$ of the transition amplitude $e^{-it\hat H}$. The corresponding path integral has an euclidean action S_E, again obtained from (118) by a Wick rotation

$$iS[x,\psi,\bar\psi] \to -S_E[x,\psi,\bar\psi] \tag{134}$$

that produces

$$S_E[x,\psi,\bar\psi] = \int_0^\beta d\tau \left[\frac{1}{2}\dot x^2 + \bar\psi\dot\psi + \frac{1}{2}\Big(W'(x)\Big)^2 + W''(x)\bar\psi\psi\right] . \tag{135}$$

Now one must recall that a trace in the Hilbert space is calculated by a path integral with periodic boundary conditions (PBC) for the bosonic variables, and antiperiodic boundary conditions (ABC) for the fermionic ones

$$\mathrm{Tr}\, e^{-\beta\hat H} = \int_{PCB}\mathcal{D}x \int_{PCB}\mathcal{D}\psi\mathcal{D}\bar\psi \ e^{-S_E[x,\psi,\bar\psi]} \tag{136}$$

The insertion of the operator $(-1)^{\hat{F}}$ creates instead a supertrace and has the effect of changing the boundary conditions on the fermions form antiperiodic to periodic, so that

$$\mathrm{Tr}\,[(-1)^{\hat{F}} e^{-\beta \hat{H}}] = \int_{PCB} \mathcal{D}x\mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{-S_E[x,\psi,\bar{\psi}]}\;. \tag{137}$$

We can sketch the calculation of the Witten index using this path integral representation. It is simplified by using the fact that the Witten index is invariant under continuous deformations of the parameters of the theory. We can deform the prepotential as

$$W(x) \to \lambda W(x) \tag{138}$$

with positive λ, and the take the limit $\lambda \to \infty$. As the result must be independent of λ, one finds, as we will soon show, that the index is computed by

$$\mathrm{Tr}\,[(-1)^{\hat{F}} e^{-\beta \hat{H}}] = \sum_{\{x_0\}} \frac{W''(x_0)}{|W''(x_0)|} \tag{139}$$

where the sum is over all critical points $\{x_0\}$, defined as the points of local maxima and minima that satisfy $W'(x_0) = 0$. This result exemplify how the index connects topological properties to local properties (the critical points).

Sketch of the calculation

One must sum over all periodic trajectories, i.e. trajectories that close on themselves after an euclidean time β. We indicate such trajectories by $[x(\tau), \psi(\tau), \bar{\psi}(\tau)]$. The leading contribution to the path integral for $\lambda \to \infty$ is associated to the trajectories $[x_0, 0, 0]$ that are certainly periodic, where the constants x_0 are the critical points of the prepotential $W(x)$, defined by the equation $W'(x_0) = 0$. This corresponds to the leading classical approximation to the path integral

$$\mathrm{Tr}\,[(-1)^{\hat{F}} e^{-\beta \hat{H}}] \sim \sum_{\{x_0\}} e^{-S_E[x_0,0,0]} = \sum_{\{x_0\}} e^{-\beta \frac{\lambda^2}{2}(W'(x_0))^2} = \sum_{\{x_0\}} 1\;.$$

Then one must add the semiclassical corrections due to the quantum fluctuations around the "classical vacua" $[x_0 + \delta x(\tau), \psi(\tau), \bar{\psi}(\tau)]$ that are identified by considering the quadratic part in $[\delta x(\tau), \psi(\tau), \bar{\psi}(\tau)]$ from the expansion of the action around $[x_0, 0, 0]$. These corrections correspond to calculating gaussian path integrals and give rise to functional determinants. Let us try to calculate them. Expanding the action in a Taylor series around the trajectory $[x_0, 0, 0]$ and keeping only the quadratic part in $[\delta x(\tau), \psi(\tau), \bar{\psi}(\tau)]$, one finds

$$\begin{aligned} S_E[x,\psi,\bar{\psi}] &= \int_0^\beta d\tau \left[\frac{1}{2}\dot{\delta x}^2 + \bar{\psi}\dot{\psi} + \frac{\lambda^2}{2}\Big(W'(x_0) + W''(x_0)\delta x + \cdots\Big)^2 + \lambda W''(x_0)\bar{\psi}\psi + \cdots\right] \\ &= \int_0^\beta d\tau \left[\frac{1}{2}\delta x[-\partial_\tau^2 + \lambda^2 (W''(x_0))^2]\delta x + \bar{\psi}[\partial_\tau + \lambda W''(x_0)]\psi\right] + \cdots \end{aligned} \tag{140}$$

and the gaussian path integral over $[\delta x(\tau), \psi(\tau), \bar{\psi}(\tau)]$ produces

$$\begin{aligned} \mathrm{Tr}\,[(-1)^{\hat{F}} e^{-\beta \hat{H}}] &= \sum_{\{x_0\}} \int_{PCB} \mathcal{D}\delta x\mathcal{D}\psi\mathcal{D}\bar{\psi}\; e^{-S_E[x_0+\delta x,\psi,\bar{\psi}]} \\ &= \sum_{\{x_0\}} \frac{\mathrm{Det}\,[\partial_\tau + \lambda W''(x_0)]}{\mathrm{Det}^{1/2}\,[-\partial_\tau^2 + \lambda^2 (W''(x_0))^2]}\;. \end{aligned} \tag{141}$$

These determinants are defined by the product of the eigenvalues. A basis of periodic functions with period β is given by

$$f_n(\tau) = e^{\frac{2\pi i n\tau}{\beta}} \qquad n \in Z\ . \tag{142}$$

They are also the eigenfunctions of the differential operators that appear in (141)

$$[\partial_\tau + \lambda W''(x_0)]f_n(\tau) = \lambda_n f_n(\tau) \quad \Rightarrow \quad \lambda_n = \frac{2\pi i n}{\beta} + \lambda W''(x_0)$$

$$[-\partial_\tau^2 + (\lambda W''(x_0))^2]f_n(\tau) = \Lambda_n f_n(\tau) \quad \Rightarrow \quad \Lambda_n = \Big(\frac{2\pi n}{\beta}\Big)^2 + (\lambda W''(x_0))^2$$

so that

$$\begin{aligned}
\frac{\mathrm{Det}\,[\partial_\tau + \lambda W''(x_0)]}{\mathrm{Det}^{1/2}\,[-\partial_\tau^2 + \lambda^2 (W''(x_0))^2]} &= \prod_{n\in Z} \frac{\lambda_n}{\Lambda_n^{1/2}} \\
&= \prod_{n\in Z} \frac{\frac{2\pi i n}{\beta} + \lambda W''(x_0)}{\Big[\Big(\frac{2\pi n}{\beta}\Big)^2 + (\lambda W''(x_0))^2\Big]^{1/2}} \\
&= \frac{\lambda W''(x_0)}{|\lambda W''(x_0)|} \prod_{n>0} \frac{[\frac{2\pi i n}{\beta} + \lambda W''(x_0)][-\frac{2\pi i n}{\beta} + \lambda W''(x_0)]}{\Big(\frac{2\pi n}{\beta}\Big)^2 + (\lambda W''(x_0))^2} \\
&= \frac{W''(x_0)}{|W''(x_0)|}\ .
\end{aligned} \tag{143}$$

The result is indeed independent of λ, and we have obtained the following value of the Witten index

$$\mathrm{Tr}\,[(-1)^{\hat F} e^{-\beta \hat H}] = \sum_{\{x_0\}} \frac{W''(x_0)}{|W''(x_0)|}\ . \tag{144}$$

It is easily seen to reproduce the values obtained by the canonical analysis: for case 1 there is always one more minimum than maxima, so that $\mathrm{Tr}\,(-1)^{\hat F} = 1$, for case 2 there is always one more maximum than minima, so that $\mathrm{Tr}\,(-1)^{\hat F} = -1$, for case 3 the number of maxima and minima concide and $\mathrm{Tr}\,(-1)^{\hat F} = 0$.

$N = 2$, $d = 1$ superspace

The superspace is a useful construction that gives a geometrical interpretation of supersymmetry. It allows to formulate theories that are manifestly supersymmetric. It is constructed by adding anticommuting coordinates to the usual space time coordinates, and supersymmetry transformations will be intepreted as arising from translations in the anticommuting directions. We exemplify this construction for the previous mechanical model with $N = 2$ supersymmetry, considered as a field theory in $(0 + 1)$ space-time dimensions. The extension to $(3 + 1)$ space-time dimensions is conceptually similar, but algebraically more demanding. The $N = 1$, $d = 1$ superspace is also presented, as it is often used in worldline description of spin 1/2 fields.

Time translational invariance

To appreciate the ideas underlying the construction of superspace, it is useful to review in a critical way the elements that guarantee the constructions of actions invariant under translations in time. These considerations will then be extended to translations along anticommuting directions, interpreted as supersymmetry transformations.

In the case of a one degree of freedom carried by the variable $x(t)$, an infinitesimal time translation ($t \to t + a$) contains only a transport term

$$\delta_{T} x(t) \equiv x'(t) - x(t) = -a\dot{x}(t) = (iaH)x(t) \tag{145}$$

which we have rewritten using the differential operator $H \equiv i\frac{\partial}{\partial t}$. This operator is interpreted as the generator of a one parameter Lie group of abelian transformations, the group of time translations isomorphic to R, the group of real numbers with the addition as group product. A finite transformation can be obtained by exponentiation $x'(t) = e^{iaH}x(t) = x(t-a)$. Time derivatives of $x(t)$ transform in the same way

$$\delta_{T} \dot{x}(t) = \frac{\partial}{\partial t}(\delta_{T} x(t)) = -a\ddot{x}(t) = (iaH)\dot{x}(t) \ . \tag{146}$$

Invariant actions can be obtained as integrals in time of a lagrangian that depends on time only implicitly, i.e. through the dynamical variables and their derivatives only,

$$S[x] = \int dt\, L(x, \dot{x}) \tag{147}$$

since as consequence of (145) and (146) it follows that

$$\delta_{T} L(x, \dot{x}) = -a\frac{\partial}{\partial t} L(x, \dot{x}) = \frac{\partial}{\partial t}\left(-aL(x, \dot{x})\right) \tag{148}$$

and thus

$$\delta_{T} S[x] = \int dt\, \delta_{T} L(x, \dot{x}) = \int dt\, \frac{\partial}{\partial t}\left(-aL(x, \dot{x})\right) = 0 \tag{149}$$

up to boundary terms, which is enough to prove invariance.

$N = 2$ superspace

Let us now define the superspace $R^{1|2}$ ($N = 2$ superspace in $d = 1$), as defined by the coordinates

$$(t, \theta, \bar{\theta}) \in R^{1|2} \tag{150}$$

where $(\theta, \bar{\theta})$ are complex Grassmann variables. The generator of time translations is again the differential operator

$$H = i\frac{\partial}{\partial t} \ . \tag{151}$$

Then one introduces the differential operators

$$Q = \frac{\partial}{\partial \theta} + i\bar{\theta}\frac{\partial}{\partial t} \ , \qquad \bar{Q} = \frac{\partial}{\partial \bar{\theta}} + i\theta\frac{\partial}{\partial t} \tag{152}$$

that are seen to realize the algebra of $N = 2$ supersymmetry

$$\{Q, \bar{Q}\} = 2H \ , \qquad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0 \ , \quad [Q, H] = [\bar{Q}, H] = 0 \ . \tag{153}$$

Translations generated by these differential operators on functions of superspace produce supersymmetry transformations in a geometrical way.

Superfields

Superfields are functions of superspace, and are used as dynamical variables in the construction of supersymmetric actions. Let us consider the example of a scalar superfield $X(t, \theta, \bar{\theta})$, taken to be Grassmann even. It may be expanded in components (that is in a Taylor expansion in the anticommuting variables)

$$X(t,\theta,\bar\theta) = x(t) + i\theta\psi(t) + i\bar\theta\bar\psi(t) + \theta\bar\theta F(t) \ . \tag{154}$$

Supersymmetry transformations are by definition generated by the differential operators Q and $\bar Q$

$$\delta X(t,\theta,\bar\theta) = (\bar\epsilon Q + \epsilon\bar Q) X(t,\theta,\bar\theta) \tag{155}$$

where ϵ and $\bar\epsilon$ are Grassmann parameters. In fact, expanding in components one obtains

$$\begin{aligned} \delta x &= i\epsilon\bar\psi + i\bar\epsilon\psi \\ \delta\psi &= -\epsilon(\dot x + iF) \\ \delta\bar\psi &= -\bar\epsilon(\dot x - iF) \\ \delta F &= \bar\epsilon\dot\psi - \epsilon\dot{\bar\psi} \ . \end{aligned} \tag{156}$$

Covariant derivatives and supersymmetric actions

To identify invariant actions it is useful to introduce the covariant derivatives (covariant under supersymmetry transformations), given by

$$D = \frac{\partial}{\partial\theta} - i\bar\theta\frac{\partial}{\partial t} \ , \qquad \bar D = \frac{\partial}{\partial\bar\theta} - i\theta\frac{\partial}{\partial t} \tag{157}$$

and characterized by the fundamental property of anticommuting with the generators of supersymmetry in (152)

$$\{D, Q\} = \{D, \bar Q\} = \{\bar D, Q\} = \{\bar D, \bar Q\} = 0 \ . \tag{158}$$

One may note that D and $\bar D$ differ from the operators Q and $\bar Q$ only by the sign of the second term, and satisfy the algebra

$$\{D, \bar D\} = -2i\partial_t \ , \qquad \{D, D\} = \{\bar D, \bar D\} = 0 \ . \tag{159}$$

Thanks to these properties, covariant derivatives of superfields are again superfields, i.e. they transform under supersymmerty as the original superfield in (155)

$$\delta(DX) = D(\delta X) = D\big((\bar\epsilon Q + \epsilon\bar Q)X\big) = (\bar\epsilon Q + \epsilon\bar Q) DX \ , \tag{160}$$

and similarly for $\bar D X$. As a consequence a lagrangian $L(X, DX, \bar D X)$, function of the superspace point only implicitly through the dependence on the dynamical superfields and their covariant derivatives, that is without any explicit dependence on the point of superspace, transforms as

$$\delta L(X, DX, \bar D X) = (\bar\epsilon Q + \epsilon\bar Q) L(X, DX, \bar D X) \ . \tag{161}$$

Thus actions defined by an integration over the whole superspace

$$S[X] = \int dt d\bar\theta d\theta \, L(X, DX, \bar D X) \tag{162}$$

are manifestly invariant (up to boundary terms), as they transform as integrals of total derivatives.

In particular, the model described by

$$S[X] = \int dt d\bar{\theta} d\theta \left(\frac{1}{2} DX\bar{D}X + W(X) \right) \tag{163}$$

is manifestly supersymmetric. A direct integration over the anticommuting coordinates of superspace shows that it reproduces the model in eq. (118). In a more elegant fashion, one may use the algebra of covariant derivatives to obtain the same result. Let us show this in a telegraphic way. Up to total derivatives one may write

$$\begin{aligned} S[X] &= \int dt d\bar{\theta} d\theta \left(\frac{1}{2} DX\bar{D}X + W(X) \right) \\ &= \int dt \bar{D} D \left(\frac{1}{2} DX\bar{D}X + W(X) \right) \Big|_{\theta,\bar{\theta}=0} \\ &= \int dt \left(-\frac{1}{2} \bar{D}DXD\bar{D}X - iDX\bar{D}\dot{X} + W''(X)\bar{D}XDX + W'(X)\bar{D}DX \right) \Big|_{\theta,\bar{\theta}=0} \end{aligned}$$

and using the projections on the first component of the various superfields

$$\begin{aligned} X\Big|_{\theta,\bar{\theta}=0} &= x(t) \\ DX\Big|_{\theta,\bar{\theta}=0} &= i\psi(t) \\ \bar{D}X\Big|_{\theta,\bar{\theta}=0} &= i\bar{\psi}(t) \\ D\bar{D}X\Big|_{\theta,\bar{\theta}=0} &= -(F + i\dot{x}) \\ \bar{D}DX\Big|_{\theta,\bar{\theta}=0} &= F - i\dot{x} \end{aligned}$$

one finds (up to total derivatives)

$$S[X] = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}F^2 + i\bar{\psi}\dot{\psi} + W'(x)F - W''(x)\bar{\psi}\psi \right) . \tag{164}$$

Eliminating the auxiliary field F through its algebraic equation of motion ($F = -W'(x)$) one obtain the action in (118), which is guaranteed to be supersymmetric by the superspace construction.

<u>$N = 1$, $d = 1$ superspace</u>

A superspace can also be constructed for $N = 1$ supersymmetry, with a real (hermitian) supercharge. It is often used in worldline descriptions of spin 1/2 fields, where the γ^i matrices appearing in the Dirac equations are realized on the worldline by real Grassmann variables ψ^i (worldline Majorana fermions). Indeed, the simple model

$$S[x, \psi] = \int dt \left(\frac{1}{2}\dot{x}^i\dot{x}^i + \frac{i}{2}\psi^i\dot{\psi}^i \right) \tag{165}$$

has $N = 1$ supersymmetry and appears in worldline description of the Dirac field. One can give a superspace construction to prove its supersymmetry. The $N = 1$ supersymmetry algebra $\{Q, Q\} = 2H$ is realized by the differential operators

$$Q = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t} , \qquad H = i\frac{\partial}{\partial t} \tag{166}$$

that act on functions of superspace with coordinates (t, θ). The susy covariant derivative is given by

$$D = \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t} \tag{167}$$

and anticommutes with Q. Using the superfields

$$X^i(t,\theta) = x^i(t) + i\theta\psi^i(t) \tag{168}$$

one can construct in superspace the manifestly supersymmetric action

$$S[X] = \frac{i}{2}\int dt d\theta D X^i \dot{X}^i \tag{169}$$

which reduces to (165) when passing to the superfield components.