$N = 2$ spinning particles and the spin 1 massless particle (the photon)

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The $N = 2$ spinning particle action is characterized by a $N = 2$ extended supergravity on the worldline. The gauge fields $(e, \chi, \bar{\chi}, a)$ of the $N = 2$ supergravity contain in particular the einbein e which gauges worldline translations, complex conjugate gravitinos χ and $\bar{\chi}$ which gauge the $N = 2$ worldline supersymmetry, and a standard gauge field a which gauges the $U(1)$ symmetry that rotates by a phase the worldline fermions and gravitinos. These gauge fields produce constraints that give rise to the Maxwell equations in the vacuum.

describing the wave function of a massless spin 1 particle (the photon).

The action in flat target spacetime is most easily deduced by starting with a model with $N = 2$ extended rigid supersymmetry, and then gauging its symmetries. The rigid model is described in a graded phase space with real bosonic variables (x^μ, p_μ) and complex fermionic variables $(\psi^\mu, \bar{\psi}_\mu)$ ($\bar{\psi}_\mu$ can be taken as the complex conjugate of ψ_μ). It is given by the following real action

$$S = \int dt \Big[p_\mu \dot{x}^\mu + i \bar{\psi}_\mu \dot{\psi}^\mu - \frac{1}{2} \eta_{\mu\nu} p^\mu p^\nu \Big] \tag{1}$$

where the indices $\mu, \nu = 0, 1, 2, 3$ are spacetime indices and $\eta_{\mu\nu} \sim (-, +, +, +)$ is the Lorentz metric used to lower and raise spacetime indices. A dot denotes as usual the time derivative. The graded Poisson brackets are then given by $\{x^\mu, p_\nu\}_{PB} = \delta^\mu_\nu$ and $\{\psi^\mu, \bar{\psi}_\nu\}_{PB} = -i\delta^\mu_\nu$. This action is manifestly Poincaré invariant in target space and thus describes a relativistic model. However to describe the photon one needs some more ingredients. These can be introduced as follows. The action (1) has on the worldline a rigid $N = 2$ supersymmetry generated by the charges

$$H = \frac{1}{2} p_\mu p^\mu \ , \quad Q = p_\mu \psi^\mu \ , \quad \bar{Q} = p_\mu \bar{\psi}^\mu \ , \quad J = \bar{\psi}^\mu \psi_\mu \ . \tag{2}$$

The whole symmetry algebra can be gauged since the charges close under Poisson brackets

$$\{Q, \bar{Q}\}_{PB} = -2iH \ , \quad \{J, Q\}_{PB} = iQ \ , \quad \{J, \bar{Q}\}_{PB} = -i\bar{Q} \tag{3}$$

(other Poisson brackets vanish). Introducing the gauge fields $G = (e, \bar{\chi}, \chi, a)$ which correspond to the constraints $C = (H, Q, \bar{Q}, J)$ gives the action

$$\begin{aligned} S &= \int dt \Big[p_\mu \dot{x}^\mu + i \bar{\psi}_\mu \dot{\psi}^\mu - eH - i\bar{\chi} Q - i\chi \bar{Q} - aJ \Big] \\ &= \int dt \Big[p_\mu \dot{x}^\mu + i \bar{\psi}_\mu \dot{\psi}^\mu - \frac{1}{2} e p_\mu p^\mu - i\bar{\chi} p_\mu \psi^\mu - i\chi p_\mu \bar{\psi}^\mu - a \bar{\psi}^\mu \psi_\mu \Big] \ . \end{aligned} \tag{4}$$

The gauge transformations on the phase space variables are generated through Poisson brackets by the generator $G \equiv \xi H + i\bar{\epsilon} Q + i\epsilon \bar{Q} + \alpha J$, where $\xi, \bar{\epsilon}, \epsilon, \alpha$ are local parameters with appropriate

Grassmann parity,

$$\begin{aligned}
\delta x^\mu &= \{x^\mu, G\}_{PB} = \xi p^\mu + i\bar{\epsilon}\psi^\mu + i\epsilon\bar{\psi}^\mu \\
\delta p_\mu &= \{p_\mu, G\}_{PB} = 0 \\
\delta \psi^\mu &= \{\psi^\mu, G\}_{PB} = -\epsilon p^\mu - i\alpha\psi^\mu \\
\delta \bar{\psi}^\mu &= \{\bar{\psi}^\mu, G\}_{PB} = -\bar{\epsilon} p^\mu + i\alpha\bar{\psi}^\mu
\end{aligned} \tag{5}$$

while on gauge fields the gauge transformations are easily obtained with the help of the constraint algebra (3)

$$\begin{aligned}
\delta e &= \dot{\xi} + 2i\bar{\chi}\epsilon + 2i\chi\bar{\epsilon} \\
\delta \chi &= \dot{\epsilon} + ia\epsilon - i\alpha\chi \\
\delta \bar{\chi} &= \dot{\bar{\epsilon}} - ia\bar{\epsilon} + i\alpha\bar{\chi} \\
\delta a &= \dot{\alpha} \ .
\end{aligned} \tag{6}$$

Eliminating algebraically the momenta p_μ by using their equations of motion

$$p^\mu = \frac{1}{e}(\dot{x}^\mu - i\bar{\chi}\psi^\mu - i\chi\bar{\psi}^\mu) \tag{7}$$

one obtains the action in configuration space

$$S = \int dt \Big[\frac{1}{2}e^{-1}(\dot{x}^\mu - i\bar{\chi}\psi^\mu - i\chi\bar{\psi}^\mu)^2 + i\bar{\psi}_\mu\dot{\psi}^\mu - a\bar{\psi}^\mu\psi_\mu\Big] \ . \tag{8}$$

The corresponding gauge invariances can be easily deduced from the phase space ones using (7).

Let us now study the canonical quantization of this model. The phase space variables are turned into operators satisfying the following (anti)commutation relations (we use $\hbar = 1$)

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu \ , \qquad \{\hat{\psi}^\mu, \hat{\psi}^\dagger_\nu\} = \delta^\mu_\nu \ . \tag{9}$$

States of the full Hilbert space can be described by functions of the coordinates x^μ and ψ^μ. By x^μ we denote the eigenvalues of the operator $\hat{x}^\mu$, while for the fermionic variables we use bra coherent states defined by

$$\langle\psi|\hat{\psi}^\mu = \langle\psi|\psi^\mu \ = \psi^\mu\langle\psi| \ . \tag{10}$$

Any state $|\phi\rangle$ can then be described by the wave function

$$\phi(x,\psi) \equiv (\langle x| \otimes \langle\psi|)|\phi\rangle \tag{11}$$

and since the ψ's are Grassmann variables the wave function has the following general expansion

$$\begin{aligned}
\phi(x,\psi) &= F(x) + F_\mu(x)\psi^\mu + \frac{1}{2}F_{\mu_1\mu_2}(x)\psi^{\mu_1}\psi^{\mu_2} + F_{\mu_1\mu_2\mu_3}(x)\psi^{\mu_1}\psi^{\mu_2}\psi^{\mu_3} \\
&+ \frac{1}{4!}F_{\mu_1\mu_2\mu_3\mu_4}(x)\psi^{\mu_1}\psi^{\mu_2}\psi^{\mu_3}\psi^{\mu_4} \ .
\end{aligned} \tag{12}$$

Higher powers in the ψ^μ cannot be present because they are Grassmann variables and anticommute between themselves.

The classical constraints C now become operators $\hat{C}$ which are used to select the physical states through the requirement $\hat{C}|\phi_{phys}\rangle = 0$. In the above representation they take the form of differential operators

$$\hat{H} = -\frac{1}{2}\partial_\mu\partial^\mu \ , \quad \hat{Q} = -i\psi^\mu\partial_\mu \ , \quad \hat{Q}^\dagger = -i\partial_\mu\frac{\partial}{\partial\psi_\mu} \ , \quad \hat{J} = -\frac{1}{2}\Big[\psi^\mu, \frac{\partial}{\partial\psi^\mu}\Big] \tag{13}$$

where we have antisymmetrized $\hat{\psi}^\mu$ and $\hat{\psi}^\dagger_\mu$ to resolve an ordering ambiguity. Thus the operator J can be written also as follows:

$$\begin{aligned}\hat{J} &= -\frac{1}{2}\Big(\psi^\mu\frac{\partial}{\partial\psi^\mu} - \frac{\partial}{\partial\psi^\mu}\psi^\mu\Big) \\ &= -\psi^\mu\frac{\partial}{\partial\psi^\mu} + \frac{1}{2}\frac{\partial\psi^\mu}{\partial\psi^\mu} = -\psi^\mu\frac{\partial}{\partial\psi^\mu} + 2\end{aligned} \tag{14}$$

The constraint $\hat{J}|\phi_{phys}\rangle = 0$ selects states with only 2 ψ's, namely

$$\phi_{phys}(x,\psi) = \frac{1}{2!}F_{\mu_1\mu_2}(x)\psi^{\mu_1}\psi^{\mu_2} \ . \tag{15}$$

The constraints $\hat{Q}|\phi_{phys}\rangle = 0$ gives the Bianchi identities

$$\partial_{\mu_1}F_{\mu_2\mu_3}(x) + \partial_{\mu_2}F_{\mu_3\mu_1}(x) + \partial_{\mu_3}F_{\mu_1\mu_2}(x) = 0 \tag{16}$$

and the constraint $\hat{Q}^\dagger|\phi_{phys}\rangle = 0$ produces the Maxwell equations

$$\partial^{\mu_1}F_{\mu_1\mu_2}(x) = 0 \ . \tag{17}$$

The constraint $\hat{H}|\phi_{phys}\rangle = 0$ is automatically satisfied as a consequence of the algebra $\{\hat{Q}, \hat{Q}^\dagger\} = 2\hat{H}$.

Thus we see that the $N = 2$ spinning particle produces the Maxwell equation in empty space.