Non-Perturbative Effects in Matrix Models and D-branes

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Motivation

- $c \leq 1$ non-critical strings have a dual description in terms of double scaled matrix models. These models were shown to reproduce the perturbative genus expansion of closed string amplitudes:

$$\mathcal{A} \simeq \sum_{h=0}^{\infty} g_s^{2h-2} c_h$$

- The genus h contributions c_h grow with genus as $c_h \simeq a^h (2h)!$ with a positive parameter a. This implies that the amplitudes can get non-perturbative corrections. The general form of the leading corrections is fixed by the large order growth:

$$\mathcal{A}_{\text{np}} \sim g_s^{f_A} e^{-f_D/g_s}$$

- It was conjectured that the same large order behaviour is a feature of any string theory. [Shenker '90] Thus, the leading non-perturbative effects (NPE) should be always of order e^{-1/g_s}.

- The origin of these NPE was identified with D-instantons (D(-1)-branes). [Polchinski '94] The closed string amplitudes get contributions from strings attached to these instantons. At the leading order they are represented by disk partition sums with Dirichlet boundary conditions. Summing over multiple strings exponentiates the single disk contribution:

$$\mathcal{A}_{\text{D}} \sim e^{Z_{\text{disk}}}, \qquad Z_{\text{disk}} \sim 1/g_s$$

$$
\begin{array}{lcccc}
\text{non-critical strings} & \boxed{(2h)!\ \text{growth}} \rightarrow & \boxed{\mathcal{A}_{\rm np} \sim e^{-f_D/g_s}} & \longleftarrow & \boxed{?} \\
& \downarrow & \text{Shenker} \quad \downarrow & & \uparrow \\
\text{critical strings} & \boxed{(2h)!\ \text{growth}} \rightarrow & \boxed{\mathcal{A}_{\rm np} \sim e^{-f_D/g_s}} & \overset{\text{Polchinski}}{\longleftarrow} & \boxed{\text{D(-1)-branes}}
\end{array}
$$

- What are D(-1)-branes in non-critical string theory?
  The problem is to find localized D-branes in Liouville theory.

$$
\begin{array}{c}\text{The energy of}\\ \text{a Liouville D}-\text{brane}\end{array} \sim 1/g_s(\phi) \sim e^{-Q\phi}
$$

$$
\Rightarrow \begin{array}{c}\text{D}-\text{brane is localized}\\ \text{at } \phi = \infty\end{array}
$$

  The perturbation theory breaks down and there is no sensible description.

- Explicit construction of Liouville D-branes.
  [A.&Al. Zamolodchikov '01]
  There is a two-parameter family of (m, n) branes.

- A subset of D-branes discovered by Zamolodchikovs is responsible for NPE in non-critical strings. [Martinec '03]

We identify the Liouville D-branes which give the leading NPE and show that the obtained results are in excellent agreement with matrix model results.

Also we derive new predictions for NPE in 2D string theory with a winding perturbation. By duality, some of them should correspond to NPE in string theory on the black hole background.

NPE in $c < 1$ non-critical strings

Matrix model results

From matrix models of $c < 1$ non-critical strings ((p,q) rational CFT coupled with world sheet gravity) it was shown that the free energy satisfies the KP differential equations. They are written in terms of susceptibility $u(\mu) = -\partial_\mu^2 F$, where the cosmological constant μ is related to the string coupling: $\mu = g_s^{-(p+q-1)/(p+q)}$. The KP equations allow to restore the full perturbative expansion of the susceptibility.

Example: pure gravity $(p = 2, q = 3)$

$$u^2(\mu) - \frac{1}{6}u''(\mu) = \mu$$

$$u(\mu) = \mu^{1/2} \sum_{h=0}^{\infty} c_h \mu^{-5h/2}, \; c_0 = 1, c_1 = -1/48, \ldots$$

The non-perturbative corrections $\epsilon(\mu)$ can be found linearizing the KP equations for $u(\mu) + \epsilon(\mu)$.

Example: pure gravity $(p = 2, q = 3)$

$$\varepsilon'' = 12u\varepsilon \quad \Rightarrow \quad \varepsilon \propto \mu^{-1/8} e^{-\frac{8\sqrt{3}}{5}\mu^{5/4}}$$

The coefficient in the exponent was found for arbitrary unitary $(p, p + 1)$ model. [Eynard, Zinn-Justin '93]
It is expressed through the dimensionless quantity:

$$\boxed{r_{m,n} = \frac{\partial_\mu \log \varepsilon_{m,n}}{\sqrt{u}} = -4 \sin \frac{\pi m}{p} \sin \frac{\pi n}{p+1}}$$

The indices (m, n) vary over the Kac table and label different non-perturbative corrections.

NPE in $c < 1$ non-critical strings

Results from D-branes

The leading non-perturbative corrections come from the disk partition function with Dirichlet boundary conditions. Instead of considering the disk partition function we calculate the dimensionless quantity:

$$r = \frac{\frac{\partial}{\partial \mu} \log \varepsilon}{\sqrt{u}} = \frac{\frac{\partial}{\partial \mu_L} Z_{\text{disk}}}{\sqrt{-\partial^2_{\mu_L} F}}$$

The normalization is not important!

Factorization of the disk amplitude:

$$\frac{\partial}{\partial \mu_L} Z_{\text{disk}} = Z_{m,n} \times \langle V_b \rangle$$

$Z_{m,n}$ — matter disk partition function
$\langle V_b \rangle$ — one-point correlator of the cosmological constant operator $e^{2b\phi}$ on the disk in Liouville theory

Minimal model D-branes are in one-to-one correspondence with the Kac table. All of them contribute to NPE.

The matter partition function of $(p, p+1)$-model with boundary conditions corresponding to (m, n)-brane [Cardy '89]:

$$Z_{m,n} = \left(\frac{8}{p(p+1)} \right)^{1/4} \frac{\sin \frac{\pi m}{p} \sin \frac{\pi n}{p+1}}{\left(\sin \frac{\pi}{p} \sin \frac{\pi}{p+1} \right)^{1/2}}$$

There is an infinite (two-parameter (m,n), $m,n\in\mathbf{N}$) set of Liouville D-branes. [A.&Al. Zamolodchikov '01]

Conjecture: at the leading order only the $(1,1)$ brane contributes to NPE

$$\begin{matrix} c<1 \\ \mathrm{D-instanton} \end{matrix} = \begin{matrix} (m,n) \text{ minimal} \\ \text{model brane} \end{matrix} \times \begin{matrix} (1,1) \text{ Liouville} \\ \text{brane} \end{matrix}$$

The one-point correlator on the disk with boundary at $(1,1)$ D-brane: $(\gamma(x)=\Gamma(x)/\Gamma(1-x))$

$$\langle V_b(0)\rangle_{(1,1)} = -C\,\frac{2^{1/4}\sqrt{\pi}[\pi\mu_L\gamma(b^2)]^{\frac{1}{2}(1/b^2-1)}}{b\Gamma(1-b^2)\Gamma(1/b^2)}$$

C is a constant independent of all parameters.

The sphere two-point correlator:

$$\partial^2_{\mu_L}F = \frac{1/b^2-1}{\pi b}\left[\pi\mu_L\gamma(b^2)\right]^{1/b^2-1}\gamma(b^2)\gamma(1-1/b^2)$$

Plugging $Z_{m,n}$, $\langle V_b(0)\rangle_{(1,1)}$ and $\partial^2_{\mu_L}F$ into r, one obtains:

$$\boxed{r_{m,n} = -2C\sin\tfrac{\pi m}{p}\sin\tfrac{\pi n}{p+1}}$$

This agrees with the matrix model result for all (m,n) if we choose

$$\boxed{C=2}$$

NPE in compactified 2D string theory

Matrix quantum mechanics with winding perturbations

World sheet description: compactified scalar field $x \simeq x + 2\pi R$ coupled to world sheet gravity with Sine–Gordon perturbation: $(\tilde{x} = x_L - x_R)$

$$\mathcal{L}_{\rm SL} = \frac{1}{4\pi}\left[(\partial x)^2 + (\partial\phi)^2 + 2\hat{R}\phi + \mu_{_L}\phi e^{2\phi} + \lambda e^{(2-R)\phi}\cos[R\tilde{x}]\right]$$

The relevant radii: $1 < R < 2$

MQM description: time dependent matrix $M(t)$ in the inverted oscillator potential. The Sine–Gordon perturbation introduces windings which are described by higher representations of SU(N) (angular degrees of freedom of $M(t)$).

A matrix model for $\mathcal{L}_{\rm SL}$ is known. [Kazakov, Kostov, Kutasov '01] It is integrable: the partition function $\mathcal{F}(\mu,\lambda)$ satisfies Toda equation:

$$\frac{1}{4}\lambda^{-1}\partial_\lambda\lambda\partial_\lambda\mathcal{F}(\mu,\lambda) + \exp\left[-4\sin^2\left(\frac{1}{2}\frac{\partial}{\partial\mu}\right)\mathcal{F}(\mu,\lambda)\right] = 1$$

The initial condition at $\lambda = 0$:

$$\mathcal{F}(\mu,0) = \frac{R}{4}\,{\rm Re}\int\limits_{\Lambda^{-1}}^{\infty}\frac{ds}{s}\frac{e^{-i\mu s}}{\sinh\frac{s}{2}\sinh\frac{s}{2R}} = -\frac{R}{2}\mu^2\log\frac{\mu}{\Lambda} - \frac{1}{24}(R+\frac{1}{R})\log\frac{\mu}{\Lambda}$$
$$+R\sum_{h=2}^{\infty}\mu^{2-2h}c_h(R) + O(e^{-2\pi\mu}) + O(e^{-2\pi R\mu})$$

Scaling variables:

$$y = \mu\xi, \qquad \xi = (\lambda\sqrt{R-1})^{-\frac{2}{2-R}}$$

Sphere partition function: $\partial_\mu^2\mathcal{F}_0 = R\log\xi + X(y)$

$$y = e^{-\frac{1}{R}X} - e^{-\frac{R-1}{R}X}$$

NPE in compactified 2D string theory

Results from matrix quantum mechanics

$\lambda = 0$: Two types of corrections:

$$1)\ e^{-2\pi\mu} \qquad \text{and} \qquad 2)\ e^{-2\pi R\mu}$$

$\lambda \neq 0$: 1) $\Delta\mathcal{F} = \sum_{n=1}^{\infty} C_n e^{-2\pi n\mu}$ — exact solution

2) The correction of the form:

$$\varepsilon(\mu, y) = A(\mu, y) e^{-\mu f(y)}$$

$$f(y \to \infty) = 2\pi R \text{ — initial condition}$$

The scaling function $f(y)$ is found by plugging $\mathcal{F}(\mu,\lambda) + \varepsilon(\mu,\lambda)$ into Toda equation and linearizing w.r.t. ε. The resulting equation is integrable! The solution:

$$\boxed{\begin{aligned} f(y) &= 2\phi(y) + \frac{2(2-R)}{\sqrt{R-1}} e^{-\frac{1}{2}X(y)} \sin\phi(y) \\ e^{\frac{2-R}{2R}X(y)} &= \sqrt{R-1}\,\frac{\sin\left(\frac{1}{R}\phi\right)}{\sin\left(\frac{R-1}{R}\phi\right)} \end{aligned}}$$

Small perturbation limit $\lambda \to 0$:

$$f(y) = 2\pi R + 4\sin(\pi R)\,\mu^{-\frac{2-R}{2}}\,\lambda + R\sin(2\pi R)\,\mu^{-(2-R)}\,\lambda^2 + O(\lambda^3)$$

Black hole limit $\mu \to 0$, $R = 3/2$:

$$\mu f(y) = \frac{1}{2}\lambda^4 + \frac{3\pi}{2}\mu + O(\mu^2)$$

For $R = 3/2$ the whole $f(y)$ is known explicitly.

$c = 0$ limit $\mu \to \mu_c = -(2-R)(R-1)^{\frac{R}{2-R}}\lambda^{\frac{2}{2-R}}$:

$$r = \frac{\partial_\mu \log\varepsilon}{\sqrt{-\partial_\mu^2 \mathcal{F}_0}} = -2\sqrt{3}$$

The result agrees with NPE found from Painleve-I equation. The NPE appears to be the same as in pure gravity.

NPE in compactified 2D string theory

Results from D-branes in $c = 1$ theory

$\lambda = 0$: Two types of corrections (T-duality):

1) $e^{-2\pi\mu} \longleftrightarrow c = 1$ Dirichlet brane $\times$ $(1,1)$ Liouville brane

2) $e^{-2\pi R\mu} \longleftrightarrow c = 1$ Neumann brane $\times$ $(1,1)$ Liouville brane

The comparison is done again for

$$r = \frac{\frac{\partial}{\partial \mu_L} Z_{\text{disk}}}{\sqrt{\partial^2_{\mu_L} \mathcal{F}_0}} = \lim_{b\to 1} \frac{Z_{\text{Neum}} \times \langle V_b(0)\rangle_{(1,1)}}{\sqrt{\partial^2_{\mu_L}\mathcal{F}_0}} = -\frac{2\pi\sqrt{R}}{\sqrt{\log\frac{\Lambda}{\mu}}}$$

For the same value of the constant $C = 2$, the result agrees with the one of the matrix model!

$\lambda \neq 0$: 1) Matrix model prediction:

$$\langle \mathcal{T}^n \rangle_{(1,1)\times\text{Dirichlet}} = 0 \qquad \mathcal{T} = \int d^2 z e^{(2-R)\phi} \cos[R(x_L - x_R)]$$

Explanation: In the T-dual picture the x-field is localized in the strong coupling region $\phi = \infty$ whereas the D-brane should be wrapped around the circle.

2) The matrix model predicts the whole function $f(y)$ which contains information about all disk correlators $\langle \mathcal{T}^n \rangle_{(1,1)\times\text{Neumann}}$.

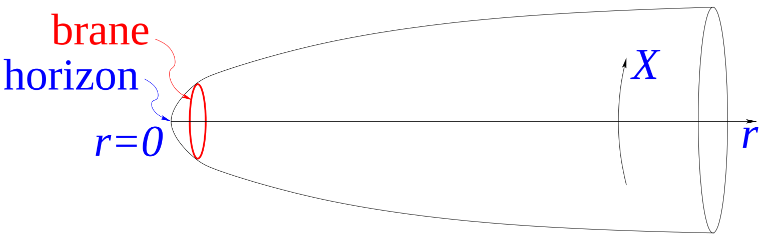
We compare the one-point correlator again using the dimensionless combination:

$$\rho = \left.\frac{\frac{\partial}{\partial\lambda} Z_{\text{disk}}}{\sqrt{-\partial^2_\lambda \mathcal{F}_0}}\right|_{\lambda=0} = \lim_{b\to 1} \frac{B_{\mathcal{T}} \times \langle V_{b-\frac{R}{2}}\rangle_{(1,1)}}{\sqrt{-\langle \mathcal{T}^2\rangle_{\text{sphere}}}} = -2\sqrt{2}\sin(\pi R)$$

The result is in agreement with the matrix model!

Conclusions

- We showed excellent agreement of NPE in $c \leq 1$ matrix models with instanton calculations in the continuum approach.
- We identified D-branes in Liouville theory responsible for the leading non-perturbative effects. These are $(1,1)$ Zamolodchikovs' branes.
- We obtained an infinite set of matrix model predictions for 2D string theory with winding perturbation:
  - All correlators of the Sine–Gordon perturbation on the disk with Dirichlet boundary conditions (for the T-dual $\lambda \cos(x/R)$ — with Neumann b.c.) must vanish.
  - The universal function $f(y)$ predicts all disk correlators with Neumann boundary conditions.
  - In the black hole limit we obtain the prediction for NPE related to the D-brane sitting at the tip of the cigar describing the Euclidean black hole.



Prospects

Subleading corrections

The structure of the perturbative expansion around D-instantons?

$$\mathcal{A}_{\rm np} \sim C g_s^{f_A} e^{-f_D/g_s}\left[1+\sum_\chi g_s^\chi d_\chi\right]$$

Next terms should come from contributions of world sheets with more handles and holes. The correct combinatorics may be captured by matrix models.

The subleading correction is given by f_A. It is associated with genus 1 contribution $\sim \log g_s$.

Conjecture: f_A arises from the annulus diagram with $(1,1)$ boundary conditions on the Liouville field.

Matrix model results:

$c < 1$: For the partition sum of minimal models one always has

$$\boxed{f_A = 1/2}$$

$c = 1$: For $\lambda = 0$ one has $f_A = 0$.

For $\lambda \neq 0$ f_A is still vanishing. It is possible to find the prefactor as a function of dimensionless parameter:

$$\mathcal{F}(\mu, y) = A(y) e^{-\mu f(y)}$$

$$\boxed{A(y) = C\left\{\sin^2\tfrac{\phi}{R}\sin^2\left(\tfrac{R-1}{R}\phi\right)\left((R-1)\cot\left(\tfrac{R-1}{R}\phi\right)-\cot\tfrac{\phi}{R}\right)\right\}^{-1/2}}$$

Consistency check: it gives the correct $c = 0$ behaviour near the critical point: $A \sim (y - y_c)^{-5/8}$.