QUANTUM COMMUTING TRACES FOR (DYNAMICAL) REFLEXION ALGEBRAS

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① ORIGINAL MOTIVATIONS

② PARTICULAR PROBLEMS

③ COMMUTING QUANTUM TRACES

FOR NON-DYNAMICAL REFLECTION ALGEBRAS

qa - 0305424 w. A. DOIKOU

④ DYNAMICAL REFLECTION (BRAIDED) ALGEBRAS

qa 0307026 w. Z. NAGY & G. ROLLET

1) ORIGINAL MOTIVATIONS

1: Quantum Trace Formula for $RTT = TTR$

J.M. Maillet, Phys. Lett. B **245** (90), 421.

Classical Lax matrix $L(p, q, \lambda)$ has PB structure

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]$$

$$\Rightarrow \quad \{\operatorname{Tr} L^n(\lambda), \operatorname{Tr} L^m(\mu)\} = 0$$

Analog for quantum lax structure (qm group):

$$R_{12} L_1 L_2 = L_2 L_1 R_{12} \qquad ?$$

There:

$$\operatorname{Tr}_{1\ldots N}\left(\check{R}_{12}\check{R}_{23}\ldots\check{R}_{N-1\,N} L_1 \ldots L_N\right) = H_N$$

are commuting operators

and $\lim_{\hbar \to 0} H_N = \operatorname{Tr} L^N$.

⚠ BEWARE IF AFFINE QM GROUP !

2. Quantum Trace Formula for Dynamical Q.Gr.

J. Avan, O. Babelon, E. Billey
Comm. Math. Phys. 178 (1996), 281

Dynamical QG (Felder):

$$R_{12}(x+\gamma h_q)\, L_1(x)\, L_2(x+\gamma h_1)$$

$$= L_2(x)\, L_1(x+\gamma h_2)\, R_{12}(x)$$

where $x \in \mathfrak{h}^*$ $\quad$ $\mathfrak{h}$ commutative algebra

(∃ non-commutative notion, see Ping Xu...)

Rmk: Dynamical QG is quasi-Hopf def'n of QG (Jimbo + al.)
(Arnaudon + al.)

Similar trace formulae written: $\left\{\begin{array}{l}\text{needing "zero-weight"} \\ \text{conditions.}\end{array}\right.$

Explicit examples: CM R-matrix

$\Longrightarrow$ Ruijsenaars-Schneider hamiltonians!

Rmk: uses R-matrix-solution as Lax matrix
PLUS Fusion processes.

II PARTICULAR PROBLEMS :

1. Find quantum trace formula for general braided algebra :

$$A_{12}\, T_1\, B_{12}\, T_2 = T_2\, C_{12}\, T_1\, D_{12}$$

(assume $B = C^{\pi}$, $A^{\pi} = A^{-1}$, $D^{\pi} = D^{-1}$)

see . previous talk by P.P. Kulish

. L. Freidel, J.M. Maillet, PLB **262** (1991), 278 ; P.P. Kulish, R. Sasaki, hepth 9212007)

2. Define "dynamical" braided algebra ?

$\Longrightarrow$ a whole subject in itself .

(structure, consistency, algebraic interpretation, examples ...)

2b. Define + compute quantum traces for DBA .

III QUANTUM TRACES FOR BRAIDED AL'S

PARTICULARIZED HERE TO:

$$R_{12}(\lambda_1-\lambda_2)\, T_1(\lambda_1)\, R_{21}(\lambda_1+\lambda_2)\, T_2(\lambda_2)$$

$$= T_2(\lambda_2)\, R_{12}(\lambda_1+\lambda_2)\, T_1(\lambda_1)\, R_{21}(\lambda_1-\lambda_2)$$

. I. Cherednik, TMΦ 61 (1984), 977

. E.K. Sklyanin, JΦ A21 (1988), 2375

where R obeys
. YB equation
. unitarity $R^{\pi} \propto R^{-1}$
. general crossing $R_{12} = V_1\, R_{12}^{t_2}(-\lambda-\rho)\, V_1$

BASIC INGREDIENTS:

① Introduce "dual" reflection equation:

$$R_{12}(\lambda_2-\lambda_1)\, K_1(\lambda_1)\quad M_1^{-1} R_{21}(-\lambda_1-\lambda_2-2\rho)\, M_1\quad K_2(\lambda_2)\,\cdot$$

$$= K_2(\lambda_2)\quad M_1 R_{12}(-\lambda_1-\lambda_2-2\rho)\, M_1^{-1}\quad K_1(\lambda_1)\quad R_{21}(-\lambda_1+\lambda_2)$$

Notation :

$$R_{NM}(\lambda_N - \lambda_M) \equiv R_{11'}(\lambda_1 - \lambda_{1'})\, R_{21}(\lambda_2 - \lambda_{1'}) \ldots$$
$$\ldots R_{12'}(\lambda_1 - \lambda_{2'})\, R_{22'}(\lambda_2 - \lambda_{2'}) \ldots$$

$N = \{1 \ldots a\}$ $\qquad M = \{1' \ldots a'\}$ $\qquad$ (ordered).

$\bar{N} \equiv \{a \ldots 1\}$

② " fused " T- matrices :

$$T_N \equiv T_1\, R_{21}(\lambda_1 + \lambda_2)\, R_{31}(\lambda_1 + \lambda_3) \ldots T_2\, R_{32}(\lambda_2 + \lambda_3) \ldots$$

They obey GRE :

$$T_N\, R^{+}_{M'N}\, T_{M'}\, R_{\bar{N}M'}(\lambda_{M'} - \lambda_{\bar{N}}) \cdot$$

$$= R_{M'\bar{N}}(\lambda_{M'} - \lambda_{\bar{N}})\, T_{M'}\, R_{NM'}(\lambda_N + \lambda_{M'})\, T_N$$

③ " fused" K- matrices :

$$K_N^{(0)} = K_n\, M_{n-1}\, R_{n-1,n}(-\lambda_n - \lambda_{n-1} - i\ell)\, M_{n-1}^{-1}\, K_{n-1} \ldots$$
$$\ldots K_2\, M_1\, R_{1n}(-\lambda_1 - \lambda_n - i\ell) \ldots R_{12}(-\lambda_1 - \lambda_2 - \ell)\, M_1^{-1} K_1$$

They obey DGRE :

$$K_N\, M_{M'}\, R_{\bar{M}'\bar{N}}\, M_{\bar{M}'}^{-1}\, K_{M'}\, R_{N\bar{M}'}(\lambda_N - \lambda_{\bar{M}'})$$

$$= R_{\bar{M}'N}(\lambda_N - \lambda_{\bar{M}'})\, K_{M'}\, M_{\bar{M}'}^{-1}\, R_{\bar{N}\bar{M}'}\, M_{M'}\, K_N .$$

③ Dressing operator Q_N

= any object Q_N such that:

$$[\, Q_N ,\ R_{MN}\,(\lambda_N \pm \lambda_{M'})\,] = 0$$

Then if K_N solves DGRE, $Q_N K_N$ also does.

NOW GET QUANTUM TRACES AS SCALAR PRODUCT:

$$H_N = \mathrm{Tr}_N\,\left(Q_N K_N^{(0)}\, T_N\,(\lambda_N)\right)$$

$$[\, H_N ,\ H_M\,] = 0$$

for any sets of indices N, M, disjoint.

Remark 1: The importance of Q_N.

If trivial choice $Q_N = 1\!\!1_N$, then:

$$H_N = (H_1)^{\# N}, \quad \text{trivial.}$$

hence needs non-trivial Q_N.

Example in our case:

$$Q_N = \check{R}_{12} \cdots \check{R}_{n-1\,n}$$

where "$\check{R}_{12}(\lambda_1, \lambda_2)$" $\equiv \Pi_{12}\, R_{12}^{(\lambda_1 \to \lambda_2)}\ \delta(\lambda_1/\lambda_2)$

as formal series

delicate problem!

Good since
- $\lim_{\hbar \to 0} T(\lambda) \sim t(\lambda)$
- $K(\lambda) \sim 1\!\!1$ (ok in many cases)
- $\lim_{\hbar \to 0} R_{12}(\lambda_1, \lambda_2) = 1\!\!1_{12} + o(\hbar)$

$$H_N \underset{\hbar \to 0}{=\!=} tr\ t^n(\lambda)$$ classical P. com. trace! good!

but delicate since $(\hbar \neq 0)$ $R(\lambda_1, \lambda_2) \propto \frac{\pi}{\lambda_1 - \lambda_2}$

hence $R_{12}\, \delta(\lambda_1/\lambda_2)$ ...

Pb. Find better Q_N's

or find consistent scheme for singularities

or ...

Remark 2 Kulish - Damin - Mudrov universal coproduct:

In this case, . adds $(\prod R^{-1})$ to LHS of T_N

. modifies writing of fusion relns

. but quantum trace ends up identical.

Remark 3: General situation

Same formulae extended to:

$R_{12}(\lambda_1 - \lambda_2) \to A_{12}$
$R_{21}(\lambda_1 + \lambda_2) \to B_{12}$
$R_{12}(\lambda_1 + \lambda_2) \to C_{12}$
$R_{21}(\lambda_1 - \lambda_2) \to D_{12}$

General algebra:

$$A_{12}\, T_1\, B_{12}\, T_2 = T_2\, C_{12}\, T_1\, D_{12}$$

$A_{12} = A_{21}^{-1}$
$D_{12} = D_{21}^{-1}$
$B_{12} = C_{21}$

Our fusion formulae:

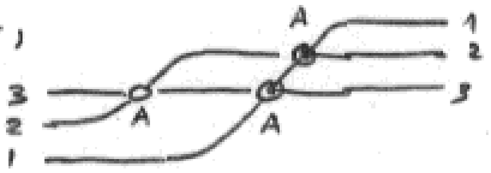
$$T_{N \cup \{a\}} = T_a\, B_{aN}\, T_N$$

$A \to A_{M\bar{N}}$
$B \to B_{MN}$
$C \to C_{MN}$
$D \to D_{M\bar{N}}$

KDM formulae

$$T_{N \cup \{c\}} = A^{-1}_{1\bar{M}}\, T_1\, B_{1M}\, T_M$$

$A \to A_{\bar{M}N}$
$B \to B_{M\bar{N}}$
$C \to C_{\bar{M}N}$
$D \to D_{M\bar{N}}$

Connected to ours by l.h.s. action of A-factors (graphically permuting of lines: 

)

Rmk: our fusion formula not adapted to RAZ (KDM) since $A \neq C$ after fusion.

IV DYNAMICAL BRAIDED ALGEBRAS

(.. an example of ...)

1. the abstract structure

- exchange relations:

$$A_{12}(x)\, T_1(x)\, B_{12}(x)\, T_2(x+\gamma h_1)$$
$$= T_2(x)\, C_{12}(x)\, T_1(x+\gamma h_2)\, D_{12}(x)$$

with : $A_{12} = A_{21}^{-1}$ $\quad D_{12} = D_{21}^{-1}$

$B_{12} = C_{21}$

$[h_1+h_2, D_{12}] = 0 \qquad [h_1, B_{12}] = [h_2, C_{12}] = 0$

zero-weight conditions, required.

- consistency conditions (DYBE)

exact YB non-dynamical $\leftarrow$ $A_{12}\, A_{13}\, A_{23} = A_{23}\, A_{13}\, A_{12}$

$$A_{12}\, C_{13}\, C_{23} = C_{23}\, C_{13}\, A_{12}(x+\gamma h_3)$$

GNF equation (DYBE) $\leftarrow$ $D_{12}(x+\gamma h_3)\, D_{13}\, D_{23}(x+\gamma h_1)$

$$= D_{23}(x)\, D_{13}(x+\gamma h_2)\, D_{12}(x)$$

$$D_{12}\, B_{13}\, B_{23}(x+\gamma h_1) = B_{23}\, B_{13}(x+\gamma h_2)\, D_{12}$$

② Realization

(G.E. Arutyunov, L. Chekov, S. Frolov
q-alg 9612032)

A construction for it's own sake of Lax matrix + ABCD matrices obeying this structure. From quantum Ruijsenaars-Schneider model:

$$L(\lambda, x) = \sum_{i,j=1}^{n} \Phi(z, q_i - q_j + \gamma)\, b_j\, E_{ij}$$

$$b_j = \prod_{a \neq j} \Phi(\gamma, q_a - q_j)$$

$\Phi \sim \dfrac{\Theta(t+s)}{\Theta(t)\,\Theta(s)}$

Θ = Jacobi Θ-function

D_{12} = Felder's elliptic $sl(n)$ R-matrix

$A_{12} \equiv B_{12}\, D_{12}\, C_{12}^{-1}$

B, C are elliptic-functions ...

⚠ Relation $BDC^{-1} = A$ generally not true
(not consistent ...)

③ <u>Other realizations: "coproduct"</u>

If T_{1q} representation on some quantum space q

Then add evaluation space V of matrices ABCD:

$$T_{1\,q\otimes V} \equiv A_{1V}\, T_{1q}\, B_{1V}$$

or $$T_{1\,q\otimes V} \equiv \left(C_{1V}^{t_V}\right)^{-1} T_{1q} \left(D_{1V}^{-1}\right)^{t_V}$$

which would help to go beyond situation of "trivial" quantum space of Lax matrix for scalar RS.