TBA equations for excited states in the $O(3)$ and $O(4)$ σ-models

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INTEGRABLE MODELS AND APPLICATIONS
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- **FS effects and LWW coupling**
- **TBA equations and Y-systems**
- **Excited state TBA in the SG model**
- **Sausage model and SS-model**

Introduction

FS effects and LWW coupling (Lüscher, Weisz, Wolff)

- $g^2(R) = R \cdot m(R)$
- $R \to 0$ (PT) $\Longleftrightarrow$ $R \to \infty$ (Lüscher)
- interpolation via lattice MC (NLS + QCD)
- 2-dim NLS integrable: exact result for LWW ($m(R)$) ?
- results for $O(3)$ and $O(4)$ NLS

FS physics in integrable models: TBA

- scattering data $\Longrightarrow$ thermodynamics $\Longrightarrow$ free energy $\sim$ ground state energy $E_0(R)$
- Modular transformation does not work for higher states: $m(R) = E_1(R) - E_0(R)$

- Complete FS spectrum: Destri-deVega NLIE
  microscopic Bethe Ansatz solution:
  SG model (+ restrictions + higher rank gener.)

TBA for excited states

- analytic continuation (Dorey, Tateo)

- integrable lattice models (Pearce, Chim, Ahn)

- BLZ program (Bazhanov, Lukyanov, A. Zamolodchikov)

$O(3)$ and $O(4)$ NLS models

- $O(3)$ and $O(4)$ models: limits of sausage model and SS-model
  ground state TBA simple

- strategy: study SG first and generalize (DdV $\Longrightarrow$ excited state TBA)

TBA and Y-systems

TBA equations : $$\varepsilon_a(\theta) + \sum_b \left(\Psi_{ab} * L_b\right)(\theta) = \frac{\nu_a(\theta)}{T}$$

$$(f * g)(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\theta' f(\theta - \theta')g(\theta')$$

$$L_a(\theta) = \ln\left[1 + e^{-\varepsilon_a(\theta)}\right]$$

pressure : $$\mathcal{P}(T) = \frac{T}{2\pi}\sum_a \int_{-\infty}^{\infty} d\theta\, \nu_a(\theta) L_a(\theta)$$

Modular transformation:

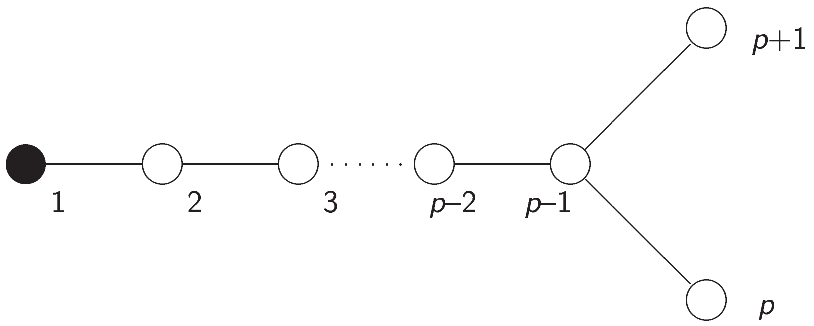
$$E_0(R) = -R\,\mathcal{P}\left(\frac{1}{R}\right)$$

1-Dynkin class: (1 particle + magnons)

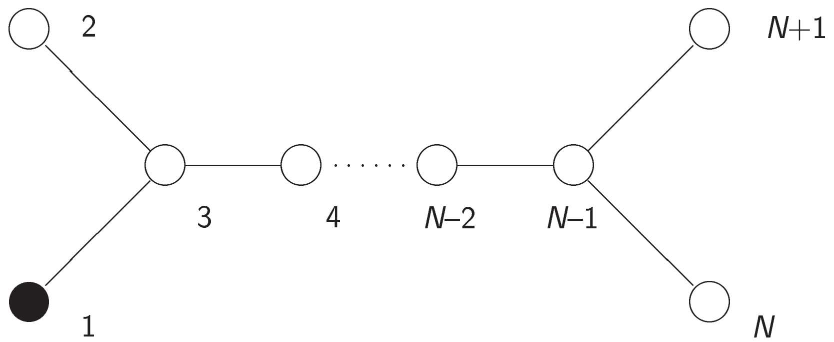
$$\nu_a(\theta) = \delta_{a1}\, M \cosh\theta \qquad\qquad \Psi_{ab}(\theta) = \frac{I_{ab}}{\cosh\theta}$$

I_{ab}: incidence matrix of some (Dynkin-)diagram

Sine-Gordon model: $(\beta^2 = \frac{8\pi p}{p+1})$

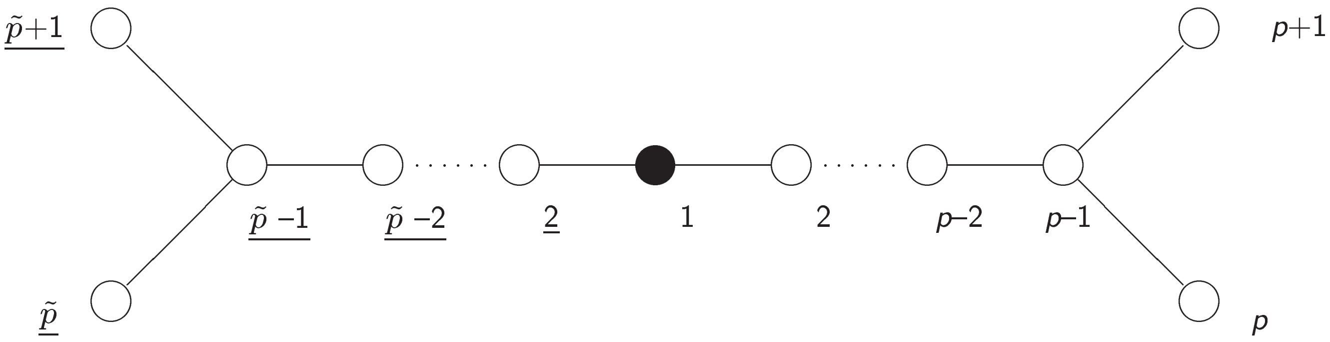


sausage-model: (Fateev, Onofri, Al. Zamolodchikov)



$N \to \infty$: $O(3)$ NLS model

SS-model: (Fateev)



$p, \tilde{p} \to \infty$: $O(4)$ NLS model

TBA integral equations $\Longrightarrow$ Y-system

$$Y_a\left(\theta + \frac{i\pi}{2}\right) Y_a\left(\theta - \frac{i\pi}{2}\right) = \prod_b \left[1 + Y_b(\theta)\right]^{I_{ab}}$$

$$Y_a(\theta) = e^{-\varepsilon_a(\theta)}$$

Y-system + B. C. + anal. $\Longrightarrow$ TBA integral equations

B.C. : $Y_a(\theta) \sim e^{-\delta_{a1} MR \cosh\theta}$ $\qquad \theta \to \infty$

anal. : no poles, no zeroes $\qquad |\mathrm{Im}\,\theta| < \frac{\pi}{2}$

Excited state TBA in the SG-model

$p \geq 2$ integer, H solitons (holes)

DdV equation

$$
\begin{aligned}
Z(\theta) &= MR\sinh\theta + \sum_{\alpha=1}^{H} \chi(\theta - h_\alpha) \\
&+ \mathrm{Im} \int_{-\infty}^{\infty} dx\, G(\theta - x - i\eta)\, L_+(x + i\eta)
\end{aligned}
$$

G, χ : soliton-soliton scattering

$L_+(\theta) = \ln\left[1 + (-1)^\delta\, e^{iZ(\theta)}\right]$

Energy:

$$
\begin{aligned}
E &= M \sum_{\alpha=1}^{H} \cosh(h_\alpha) \\
&- \frac{M}{2} \mathrm{Im} \int_{-\infty}^{\infty} dx\, \sinh(x + i\eta)\, L_+(x + i\eta)
\end{aligned}
$$

connecting TBA and DdV (Kuniba)

Bethe Ansatz solution $\Longrightarrow$ Y-system for all states

$$\mathcal{F}(\theta) = (-1)^{\delta}\, e^{iZ\left(\frac{\pi}{2}\theta\right)}$$

$$\boxed{1 + Y_1(\tfrac{\pi}{2}\theta) = \{1 + \mathcal{F}(\theta + i)\}\left\{1 + \frac{1}{\mathcal{F}(\theta - i)}\right\}}$$

$$\begin{aligned} \text{Excited state TBA :} \qquad & \text{(same) Y} - \text{system} \\ & \text{same B. C.} \\ & \text{anal. ?} \end{aligned}$$

$R \to \infty$:

$$Z(\theta) \approx MR\sinh\theta + \sum_{\alpha=1}^{H} \chi(\theta - h_\alpha)$$

$$\Downarrow$$

anal. properties of Y-system functions

- $H = 0$ (ground state) no zeroes

- $H = 1$ double zeros ate $\theta = 0$ for even a

- $H = 2$

$Y_1(\theta)$: $\pm x_2$
$Y_2(\theta)$: $\pm x_1, \pm x_3$
$\vdots$
$Y_{p-2}(\theta)$: $\pm x_{p-3}, \pm x_{p-1}$
$Y_{p-1}(\theta)$: $\pm x_{p-2}$
$Y_p(\theta) = Y_{p+1}(\theta)$: $\pm x_{p-1}$

Summary (SG)

$$\left.\begin{array}{l} \text{(same) Y} - \text{system} \\ \text{same B. C.} \\ \text{roots from } R \to \infty \end{array}\right\} \Longrightarrow \text{excited state TBA integral eqs.}$$

Sausage model and SS-model

Assumption:

$$\left.\begin{array}{l}\text{(same) Y}-\text{system}\\ \text{same B. C.}\\ \text{roots from } R\to\infty\end{array}\right\} \Longrightarrow 1-\text{particle TBA eqs.}$$

TBA: $$E_1 = M - \frac{M}{2\pi}\int\limits_{-\infty}^{\infty} d\theta\ \cosh\theta\ \ln\left[1+Y_1(\theta)\right]$$

Lüscher: $$E_1 \approx M - \frac{M}{2\pi}\int\limits_{-\infty}^{\infty} d\theta\ \cosh\theta\ e^{-MR\cosh\theta} S\left(\theta+\frac{i\pi}{2}\right)$$

$$\Downarrow$$

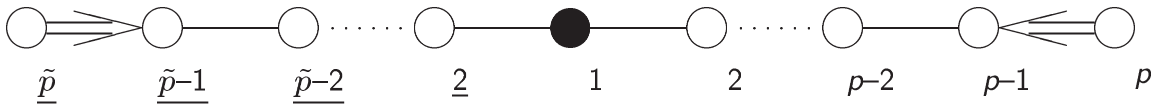
$$\boxed{Y_1(\theta) \approx e^{-MR\cosh\theta} S\left(\theta+\frac{i\pi}{2}\right)}$$

Sausage model 1-particle problem

same Y-system

two pairs of real roots for all $Y_a(\theta)$

SS-model 1-particle problem



(almost) same Y-system

double zeroes at $\theta = 0$ for all $Y_a(\theta)$

l	TBA	PT	MC	Lüscher
0.001	0.442126	0.442097		
0.003	0.496231	0.496176		
0.01	0.573883	0.573766		
0.03	0.670869	0.670606		
0.1	0.8269255	0.826130		0.4733
1/8	0.8648334	0.863821	0.863(2)	0.5265
1/4	1.0110649	1.00868	1.011(2)	0.7368
1/2	1.2255987	1.21867	1.228(2)	1.0395
1	1.5821717		1.584(4)	1.4941
2	2.3087261		2.309(10)	2.2909
4	4.0611160		4.132(10)	4.0607

Table 1: Results for the finite volume mass gap in the $O(4)$ nonlinear σ-model.

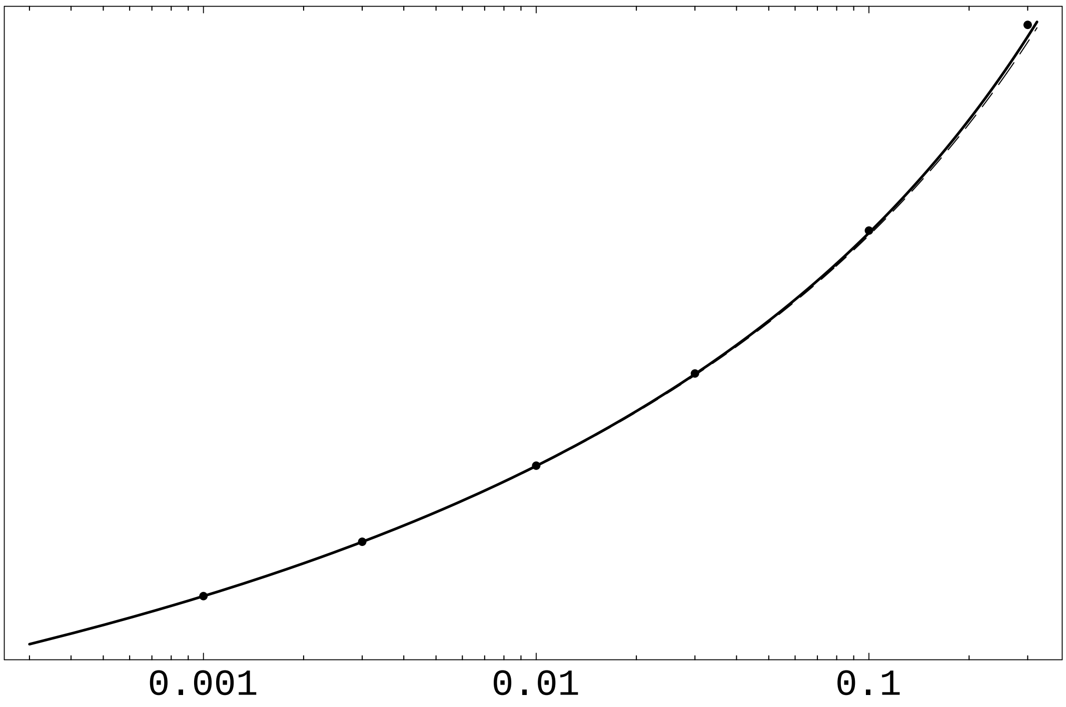


Figure 1: Finite volume mass gap of the $O(3)$ model. Comparison of numerical solution of the TBA integral equations (dots) to three-loop perturbation theory (solid line). The two-loop perturbative curve (dashed line) is also shown.

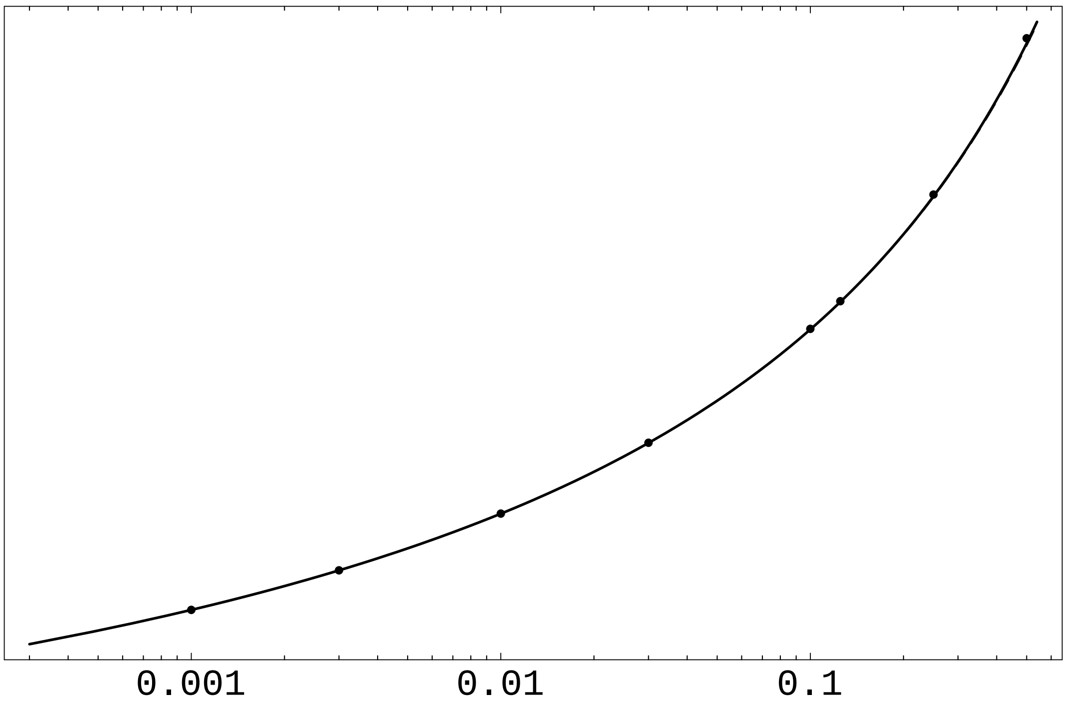


Figure 2: Finite volume mass gap of the $O(4)$ model. Comparison of numerical solution of the TBA integral equations (dots) to three-loop perturbation theory (solid line). The two-loop perturbative result is too close to the three-loop result and is not shown here.