On correlation functions in the perturbed minimal models $\mathcal{M}_{2,2n+1}$

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1. Introduction

We concentrate on the minimal models of CFT $\mathcal{M}_{p,p'}$ (with coprime integers $1 < p < p'$) perturbed by the field Φ_{13}. These massive field theories having an infinite number of conservation laws are integrable.
The conformal field theories $\mathcal{M}_{p,p'}$ underlying the massive models under consideration are characterized by the central charge of the Virasoro algebra

$$c = 1 - 6\frac{(p' - p)^2}{pp'}\,.$$

There are primary fields $\Phi_{l,k}$ ($l = 1, \ldots, p-1$ and $k = 1, \ldots, p'-1$) in the model. The conformal dimensions $(\Delta_{l,k}, \Delta_{l,k})$ of these fields are determined by the Kac formula:

$$\Delta_{l,k} = \frac{(p'l - pk)^2 - (p' - p)^2}{4pp'}\,.$$

Our notations $I = \Phi_{11}$, $\Psi = \Phi_{12}$, $\Phi = \Phi_{13}$

$$\xi = \frac{p}{p' - p}$$

The scaling model we study can be formally defined through the action

$$\mathcal{A}_{\mathcal{S}_{p,p'}} = \mathcal{A}_{\mathcal{M}_{p,p'}} + g \int \Phi \; d^2x \;,$$

"Mass–coupling-constant relation":

$$\pi g = - \frac{(\xi+1)^2}{(\xi-1)(2\xi-1)}.$$

$$\left(\gamma(\frac{3\xi}{\xi+1})\gamma(\frac{\xi}{\xi+1})\right)^{\frac{1}{2}} m^{\frac{4}{\xi+1}},$$

where m is related to the mass of the lightest particle m_1 by

$$m = m_1 \left(\frac{\pi\gamma(1-\frac{\xi}{2})\gamma(\frac{1+\xi}{2})}{8\sin\pi\xi} \right)^{\frac{1}{2}}$$

and we set

$$\gamma(x) \;=\; \frac{\Gamma(x)}{\Gamma(1-x)}.$$

The correlation function of the spin fields, we have

$$\langle \Psi(x)\Psi(0)\rangle = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!}\cdot$$

$$\int \langle \Psi(x)\Psi(0)\Phi(y_1)\cdots\Phi(y_n)\rangle_{\mathrm{CFT}} d^2y_1 \cdots$$

It is assumed that if the renormalized fields $\{A_a,\ a = 0, 1, \ldots\}$ that are eigenvectors of the dilatation operator, i.e., the fields with a definite scaling dimensions, are chosen as basis elements, then the structure functions in the corresponding operator-product expansion

$$A_m(x)A_a(0) = \sum_b C^{A_b}_{A_m A_a}(r)A_b(0)\,,$$

are analytic functions of the coupling constant g.

The Lukyanov and Zamolodchikov formula for the VEVs in the first ground state:

$$\langle\Phi_{1k}\rangle = (-1)^{k-1} m^{2\Delta_{1k}} Q(1-\xi(k-1))\,.$$

$$Q(\eta) = \exp\int_0^\infty \frac{dt}{t}\Big(-\frac{\eta^2-1}{2\xi(\xi+1)}e^{-2(\xi+1)t}$$

$$+\frac{\cosh(2t)\sinh((\eta-1)t)\sinh((\eta+1)t)}{2\cosh(t)\sinh(\xi t)\sinh((\xi+1)t)}\Big)\,.$$

The one-point VEVs for the first nontrivial descendent operators

$$\langle L_{-2}\bar{L}_{-2}\Phi_{1k}\rangle =$$

$$-(1+\xi)^4\mathcal{W}(1-\xi(k-1))m^4\langle\Phi_{1k}\rangle\,,$$

where the function $\mathcal{W}(\eta)$ is

$$\mathcal{W}(\eta) := \frac{1}{\xi^2(\xi+1)^2}\gamma(\frac{1+\eta+\xi}{2})\cdot$$

$$\gamma(\frac{\eta-\xi}{2})\gamma(\frac{1-\eta+\xi}{2})\gamma(-\frac{\eta+\xi}{2})\,.$$

2. Short–distance expansion

We analyze the short-distance expansion based on the operator algebra

$$\langle \Psi(x)\Psi(0)\rangle = C^{I}_{\Psi\Psi}(r)\langle I\rangle + C^{\Phi}_{\Psi\Psi}(r)\langle \Phi(0)\rangle$$

$$+C^{\Phi_{15}}_{\Psi\Psi}(r)\langle \Phi_{15}\rangle + C^{L_{-2}\bar{L}_{-2}I}_{\Psi\Psi}(r)\langle L_{-2}\bar{L}_{-2}I\rangle$$

$$+C^{L_{-2}\bar{L}_{-2}\Phi}_{\Psi\Psi}(r)\langle L_{-2}\bar{L}_{-2}\Phi(0)\rangle + \cdots$$

The expansion for the structure functions

$$C^{K}_{\Psi\Psi}(r) = C^{K}_{\Psi\Psi}\ r^{2\Delta_K - 4\Delta_\Psi} + C^{K(1)}_{\Psi\Psi}(r) + \cdots$$

$$C^{K(1)}_{\Psi\Psi}(r) := \lim_{R\to\infty}\Big[$$

$$-g\int_{|y|<R}\langle \tilde{A}^{K}(\infty)\Phi(y)\Psi(x)\Psi(0)\rangle_{\mathrm{CFT}} d^2y$$

$$+\pi g\sum_{A}\frac{C^{A}_{\Psi\Psi}C^{K}_{\Phi A}}{\Delta_K - \Delta_A - \Delta_\Phi + 1}\times$$

$$R^{2(\Delta_K - \Delta_A - \Delta_\Phi + 1)} r^{2\Delta_A - 4\Delta_\Psi}\Big].$$

In our case

$$C^{I(1)}_{\Psi\Psi}(r) = -g\int' \langle\Phi(y)\Psi(x)\Psi(0)\rangle_{\mathrm{CFT}} d^2y\,,$$

$$C^{\Phi(1)}_{\Psi\Psi}(r) = -g\int' \langle\Phi(\infty)\Phi(y)\Psi(x)\Psi(0)\rangle_{\mathrm{CFT}} d^2y\,,$$

$$C^{\Phi_{15}(1)}_{\Psi\Psi}(r) = -g\int' \langle\Phi_{15}(\infty)\Phi(y)\Psi(x)\Psi(0)\rangle_{\mathrm{CFT}} d^2y\ .$$

For example

$$\langle\Psi(0)\Psi(z)\Phi(1)\Phi(\infty)\rangle_{\mathrm{CFT}} = C^{\Phi}_{\Psi\Psi}C^{\Phi}_{\Phi\Phi}\cdot$$

$$\left|z^{\frac{\xi}{4(\xi+1)}}\ (1-z)^{\frac{\xi}{2(\xi+1)}}\ {}_2F_1\Big(\frac{\xi}{\xi+1}, \frac{3\xi-1}{\xi+1}, \frac{2\xi}{\xi+1}|z\Big)\right|^2$$

$$+\ \left|z^{\frac{2-\xi}{4(\xi+1)}}\ (1-z)^{\frac{\xi}{2(\xi+1)}}\ {}_2F_1\Big(\frac{2}{\xi+1},\ \frac{2\xi}{\xi+1},\ \frac{2}{\xi+1}|z\Big)\right|^2 .$$

The integrals

$$J_1(p,q) = \int d^2y\ |y|^{2p}|y-1|^{2q}\,,$$

$$J_2(a,b,d,e,c) =$$

$$\int \mathrm{d}^2x \int \mathrm{d}^2y\ |x|^{2a}|1-x|^{2b}|y|^{2d}|1-y|^{2e}|x-y|^{2c}\,.$$

$$J_1(p,q) = \frac{\pi\gamma(p+1)\gamma(q+1)}{\gamma(p+q+2)}\,.$$

$$J_2\left(\frac{\beta-\alpha-1}{2}, \delta-\beta-1, \alpha-1, -\frac{\delta}{2}, \frac{\beta-\alpha-1}{2}\right)$$

$$= 2^{2\alpha-2}\pi^2\gamma\left(\frac{\alpha}{2}\right)\gamma\left(\frac{\beta}{2}\right)\gamma\left(\frac{1-\delta}{2}\right)\gamma\left(\frac{\delta-\beta}{2}\right)\cdot$$

$$\gamma\left(\frac{\alpha-\delta+1}{2}\right)\gamma\left(\frac{\beta-\alpha+1}{2}\right)\gamma\left(\frac{\delta-\alpha-\beta+1}{2}\right)\,.$$

Applying the mass–coupling-constant relation, we have

$$C^{I}_{\Psi\Psi}(r) = r^{\frac{2-\xi}{\xi+1}}\Bigg\{1-(mr)^{\frac{4}{\xi+1}}.$$

$$\frac{(\xi+1)^2}{(\xi-1)(2\xi-1)}\left(\frac{\gamma^2(\frac{\xi}{\xi+1})\gamma^5(\frac{2}{\xi+1})\gamma(\frac{3\xi}{\xi+1})}{\gamma(\frac{2\xi}{\xi+1})\gamma(\frac{2-\xi}{\xi+1})\gamma^2(\frac{4}{\xi+1})}\right)^{\frac{1}{2}}\Bigg\},$$

$$C^{\Phi}_{\Psi\Psi}(r) = -r^{\frac{\xi}{\xi+1}}\Bigg\{\left(\frac{\gamma(\frac{\xi}{\xi+1})\gamma(\frac{2}{\xi+1})}{\gamma(\frac{2\xi}{\xi+1})\gamma(\frac{2-\xi}{\xi+1})}\right)^{\frac{1}{2}}+(mr)^{\frac{4}{\xi+1}}.$$

$$\frac{\xi^2(1-\xi)^3}{4(\xi+1)^2(2\xi-1)}\left(\frac{\gamma^8(\frac{1-\xi}{1+\xi})\gamma^9(\frac{\xi}{\xi+1})\gamma(\frac{3\xi}{\xi+1})}{\gamma^2(\frac{2-2\xi}{\xi+1})}\right)^{\frac{1}{2}}\Bigg\},$$

$$C^{\Phi_{15}}_{\Psi\Psi}(r) = r^{\frac{7\xi-2}{\xi+1}}(mr)^{\frac{4}{\xi+1}}.$$

$$\frac{\xi^2(1-\xi)}{4(2\xi-1)(3\xi+1)^2}\left(\frac{\gamma^7(\frac{\xi}{\xi+1})\gamma^4(\frac{1-\xi}{\xi+1})}{\gamma(\frac{4\xi}{\xi+1})\gamma(\frac{2-2\xi}{\xi+1})\gamma(\frac{2-3\xi}{\xi+1})}\right)^{\frac{1}{2}}.$$

We also need the structure functions for the descendent fields in the main order:

$$C_{\Psi\Psi}^{L_{-2}\bar{L}_{-2}I}(r) = \frac{\xi^2}{4(\xi+3)^2} r^{\frac{3(\xi+2)}{\xi+1}} C_{\Psi\Psi}^{I}$$

$$C_{\Psi\Psi}^{L_{-2}\bar{L}_{-2}\Phi}(r) = \frac{\xi^2}{4(3\xi+1)^2} r^{\frac{5\xi+4)}{\xi+1}} C_{\Psi\Psi}^{\Phi}$$

The explicit first-order UV expansion in g for the correlation function

$$\langle \Psi(r)\Psi(0)\rangle = r^{\frac{2-\xi}{\xi+1}} \Bigg\{ A(r) - B(r)Q(1-2\xi)$$

$$+ D(r)Q(1-4\xi) \Bigg\}$$

The $A(r)$, $B(r)$, $D(r)$ have the forms:

$$A(r) = 1 - \frac{\xi^2}{4(\xi+3)^2}\frac{\gamma^2(\frac{\xi}{2})}{\gamma^2(\frac{1+\xi}{2})}(mr)^4 -$$

$$\frac{(\xi+1)^2}{(1-\xi)(1-2\xi)}\left(\frac{\gamma^2(\frac{\xi}{\xi+1})\gamma^5(\frac{2}{\xi+1})\gamma(\frac{3\xi}{\xi+1})}{\gamma(\frac{2\xi}{\xi+1})\gamma(\frac{2-\xi}{\xi+1})\gamma^2(\frac{4}{\xi+1})}\right)^{\frac{1}{2}}(mr)^{\frac{4}{\xi+1}}$$

$$B(r) = \left\{\left(\frac{\gamma(\frac{\xi}{\xi+1})\gamma(\frac{2}{\xi+1})}{\gamma(\frac{2\xi}{\xi+1})\gamma(\frac{2-\xi}{\xi+1})}\right)^{\frac{1}{2}}\left\{1 + \frac{(\xi+1)^2}{(\xi-1)^2(3\xi+1)^2}\cdot\right.\right.$$

$$\left.\frac{\gamma(\frac{3\xi}{2})\gamma(\frac{1-3\xi}{2})}{\gamma(\frac{\xi}{2})\gamma(\frac{1-\xi}{2})}(mr)^4\right\} + \frac{\xi^2(1-\xi)^3}{4(\xi+1)^2(1-2\xi)}\cdot$$

$$\left.\left(\frac{\gamma^8(\frac{1-\xi}{1+\xi})\gamma^9(\frac{\xi}{\xi+1})\gamma(\frac{3\xi}{\xi+1})}{\gamma^2(\frac{2-2\xi}{\xi+1})}\right)^{\frac{1}{2}}(mr)^{\frac{4}{\xi+1}}\right\}(mr)^{\frac{2(\xi-1)}{\xi+1}}$$

$$D(r) = \frac{\xi^2(1-\xi)}{4(1-2\xi)(3\xi+1)^2}\cdot$$

$$\left(\frac{\gamma^7(\frac{\xi}{\xi+1})\gamma^4(\frac{1-\xi}{\xi+1})}{\gamma(\frac{4\xi}{\xi+1})\gamma(\frac{2-2\xi}{\xi+1})\gamma(\frac{2-3\xi}{\xi+1})}\right)^{\frac{1}{2}}(mr)^{\frac{8\xi}{\xi+1}}$$

3. Long–distance expansion

Form-factor expansion

$$\langle \Psi(x)\Psi(0)\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d\beta_1 \cdots d\beta_n}{(2\pi)^n} \times$$

$$\langle 0|\Psi(x)|\beta_1, \ldots, \beta_n\rangle\langle\beta_1, \ldots, \beta_n|\Psi(0)|0\rangle =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\{a_j\}} \int \frac{d\beta_1 \cdots d\beta_n}{(2\pi)^n} e^{-r\sum_j m_{a_j} \cosh\beta_j}$$

$$\times F_{a_n \cdots a_1}(\beta_n, \ldots, \beta_1) F_{a_1 \cdots a_n}(\beta_1, \ldots, \beta_n)$$

where the matrix elements of the local operators in the basis of the asymptotic states

$$F_{a_1 \cdots a_n}(\beta_1, \ldots, \beta_n) = \langle 0|\Psi(0)|\beta_1, \ldots, \beta_n\rangle_{a_1 \cdots a_n}$$

satisfying Smirnov's axioms.

The solutions of Smirnov's axioms for the primaries in perturbed $\mathcal{M}_{2,2n+1}$ models have already been discussed in the literature.

$$\langle 0|\Phi_{1k}|0\rangle = \langle \Phi_{1k}\rangle$$

$$\langle 0|\Phi_{1k}|\beta\rangle = iC\frac{\sin\Big((k-1)\pi\xi/2\Big)}{\sin(\pi\xi)}\ \langle \Phi_{1k}\rangle$$

$$\langle 0|\Phi_{1k}|\beta_2,\beta_1\rangle = i^2C^2\frac{\sin^2\Big((k-1)\pi\xi/2\Big)}{\sin^2(\pi\xi)}\ R(\beta_1-\beta_2)\ \langle \Phi_{1k}\rangle$$

$$\langle 0|\Phi_{1k}|\beta_3,\beta_2,\beta_1\rangle = i^3C^3\frac{\sin\Big((k-1)\pi\xi/2\Big)}{\sin(\pi\xi)}\prod_{s<j}R(\beta_s-\beta_j)$$

$$\times\left\{\frac{\sin^2\Big((k-1)\pi\xi/2\Big)}{\sin^2(\pi\xi)}+\frac{1}{\prod_{s<j}2\cosh(\frac{\beta_s-\beta_j}{2})}\right\}\ \langle \Phi_{1k}\rangle$$

The function $R(\beta)$ determining the rapidity-dependent part of the two-particle form factor is given explicitly as

$$R(\beta) = \exp\Big\{4\int_0^\infty \frac{dt}{t}\frac{\sinh t \sinh \xi t \sinh(\xi+1)t}{\sinh^2 2t}\cosh 2(1-\frac{i}{\pi}\beta)t\Big\}$$

Even the first two-term expression

$$\langle \Phi_{1k}(x)\Phi_{1k}(0)\rangle = \langle \Phi_{1k}\rangle^2 \Big\{1-$$

$$-\frac{2}{\pi}\frac{\sin^2(\frac{\pi}{2}(k-1)\xi)}{\sin(\frac{\pi}{2}\xi)}\ \exp\Big(-\int_0^{\pi\xi}\frac{dt}{\pi}\frac{t}{\sin t}\Big)K_0(m_1 r)\Big\} + \cdots ,$$

where $K_0(x)$ is a Macdonald function, turns out to be a good approximation for the correlation functions at long distances.

4. Numerical results

We take model $\mathcal{M}_{2,7}$ as an example with the central charge

$$c = -\frac{68}{7}\,.$$

There are three primaries: the unit field $I := \Phi_{11} = \Phi_{16}$, the spin field $\Psi := \Phi_{12} = \Phi_{1,5}$, and the energy field $\Phi := \Phi_{13} = \Phi_{14}$ with the respective conformal dimensions

$$\Delta_I \;=\; 0\,, \quad \Delta_\Psi = -\frac{2}{7}\,, \quad \Delta_\Phi = -\frac{3}{7}$$

The VEVs of operators:

$$m_1^{\frac{6}{7}}\langle\Phi_{13}\rangle = 2.269550689$$

$$m_1^{\frac{4}{7}}\langle\Phi_{15}\rangle = -2.325136\ i$$

for primary fields and

$$m_1^{\frac{8}{7}} C_{\Psi\Psi}^{L_{-2}\bar{L}_{-2}I}(r)\langle L_{-2}\bar{L}_{-2}I\rangle =$$

$$= -0.0005899419474(m_1 r)^{\frac{36}{7}}$$

$$m_1^{\frac{8}{7}} C_{\Psi\Psi}^{L_{-2}\bar{L}_{-2}\Phi}(r) \langle L_{-2}\bar{L}_{-2}\Phi(0)\rangle =$$

$$= 0.01123584810 (m_1 r)^{\frac{30}{7}}$$

for the first nontrivial descendants.
The mass coupling constant for $\xi = 2/5$ is

$$g = -0.04053795542378225 m_1^{20/7}$$

Collecting all data together, we obtain up to r^6:

$$m_1^{\frac{8}{7}} \langle \Psi(x)\Psi(0)\rangle \sim -5.83 (m_1 r)^{\frac{2}{7}} + (m_1 r)^{\frac{8}{7}}$$

$$-0.265 (m_1 r)^{22/7} + 0.233 (m_1 r)^{24/7} - 0.045 (m_1 r)^4$$

$$+0.011 (m_1 r)^{\frac{30}{7}} - 0.0006 (m_1 r)^{\frac{36}{7}} + O(r^6)$$

5. Conclusion

We have demonstrated that there is a region in the mass scale where the long- and short-distance expansions for the correlation functions of the spin operators match each other. We can therefore conclude that the combination of the two approaches allows reconstructing the correlation functions at all scales.

This also serves as a good consistency check for the correctness of both methods as well as the VEVs of local operators.

Integrals for the first corrections

The integrals from the correlation functions that appear at the first order correction can be represented in the form of double integrals over a plane because of the equation

$$\pi\frac{\gamma(a+1)\gamma(c+1)}{\gamma(a+c+2)}\int \mathrm{d}^2x|x|^{2(a+d+c+1)}|1-x|^{2e}\times$$

$$\left|{}_2F_1\left(\begin{array}{cc} a+1, & -b \\ a+c+2 & \end{array}\middle| x\right)\right|^2$$

$$+\pi\frac{\gamma(b+1)\gamma(a+c+1)}{\gamma(a+b+c+2)}\int \mathrm{d}^2x|x|^{2d}|1-x|^{2e}\times$$

$$\left|{}_2F_1\left(\begin{array}{cc} -a-b-c-1, & -c \\ -a-c & \end{array}\middle| x\right)\right|^2$$

$$=\int \mathrm{d}^2x\int \mathrm{d}^2y|x|^{2a}|1-x|^{2b}|y|^{2d}|1-y|^{2e}|x-y|^{2c}$$

Using the technique of analytical continuations and manipulation with the contours we obtain the following expression

$$I(a,b,d,e,c)=\sin\pi b K(a,b,d,e,c)\times$$

$$[\sin\pi e L(a,b,d,e,c)+\sin\pi(e+c)L(d,e,a,b,c)]$$

where

$$K = \int_0^1 dx \int_0^1 dy \, x^{a+d+c+1}(1-x)^b y^d (1-y)^c (1-xy)^e \,,$$

$$L = \int_0^1 dy \int_0^1 dx \, x^{-a-b-c-2} \times$$

$$(1-x)^c y^{-a-b-d-e-c-3}(1-y)^e(1-xy)^b \,.$$

Standard formulas for the integral representations of the hypegeometric functions give a form convenient for numerical studies:

$$K = B(d+1, c+1)B(a+d+c+2, b+1) \times$$

$$_3F_2\left(\begin{matrix} -e, & a+d+c+2, & d+1 \\ a+d+c+3+b, & d+2+c & \end{matrix} \middle| 1 \right),$$

$$L = B(-a-b-d-e-c-2, e+1)B(-a-b-c-1, c+1) \times$$

$$_3F_2\left(\begin{matrix} -b, & -a-b-d-e-c-2, & -a-b-c-1 \\ -a-b-d-c-1, & -a-b & \end{matrix} \middle| 1 \right)$$