CFTs of SLEs

Conformal Field Theories .... Stochastic Loewner Evolutions

with Michel Bauer.

I CFTs and SLEs: outlook.

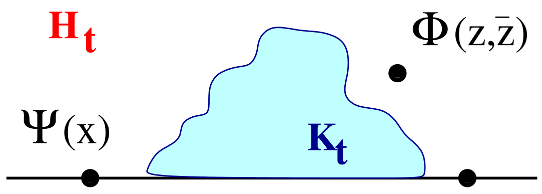
II Testing SLEs with CFTs.

III From the Virasoro algebra to SLE (martingales).

IV Crossing probabilities (SLEs = CFTs).

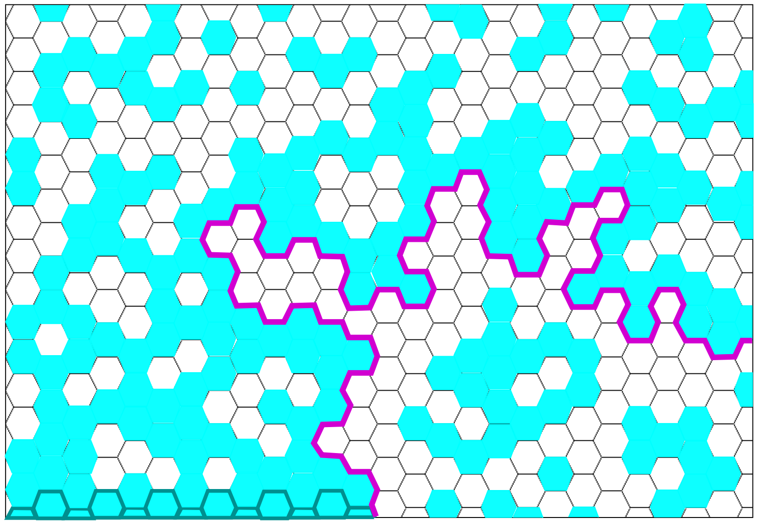
V Conclusions.

C.F.T.
in random domains.



CFTs and SLEs: outlook.

Conformal field theory.

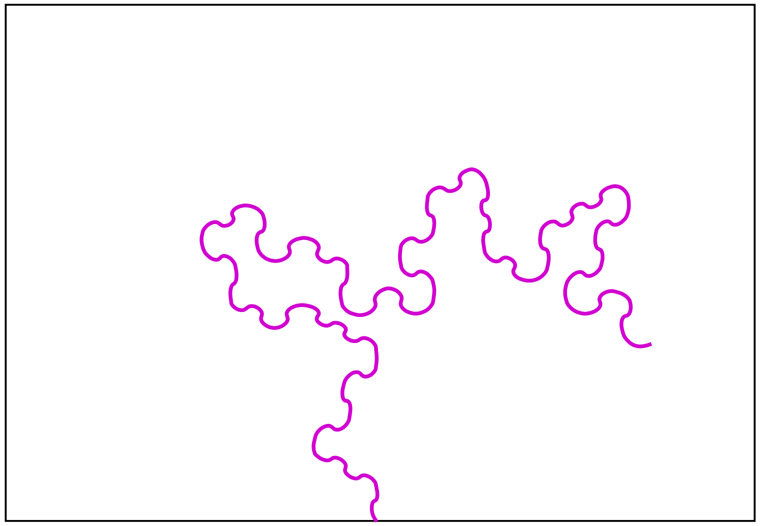


– Critical clusters. [c]

– local Boltzmann weights.

– $\mathrm{SLE}_{\kappa=6}$=$\mathrm{percolation}_{c=0}$

Stochastic Loewner evolution.



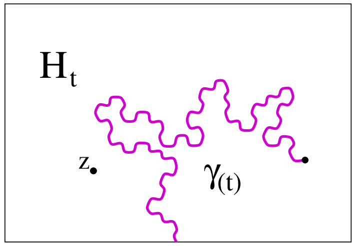
– Fractal curves. [κ]

– Conformally covariant.

– cf. O. Schramm.

Stochastic Loewner evolutions.

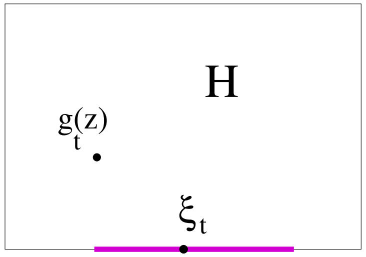
Uniformizing map $g_t(z)$ for the outer domain of a ‘fractal bubble’.



$\xrightarrow{g_t(z)}$

by Riemann

mapping theorem



• Loewner evolution for the uniformizing map.

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \qquad \begin{cases} g_{t=0}(z) = z, \quad g_t(z) = z + 2t/z + \cdots, \text{ at } \infty. \\ \text{up to time } \tau_z\text{: } g_{\tau_z}(z) = \xi_{\tau_z}. \end{cases}$$

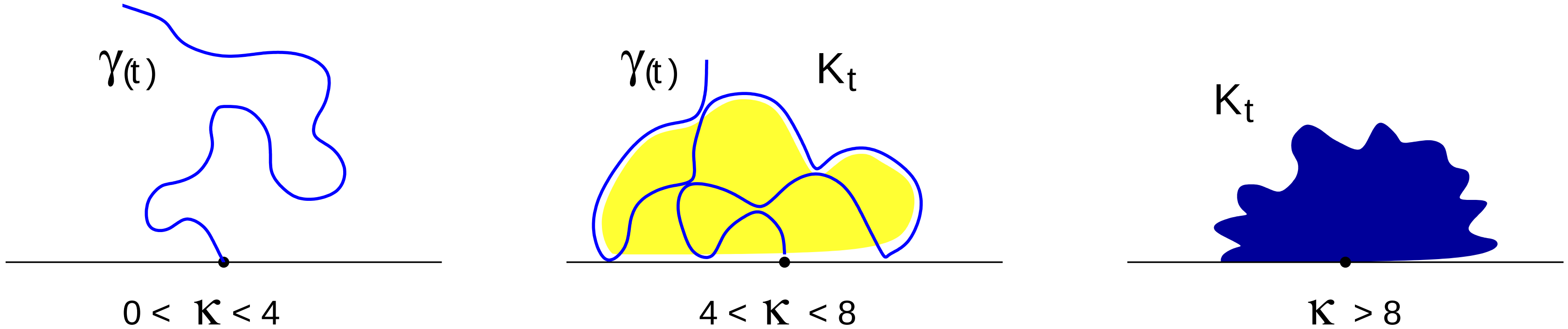
• Following Lawler, Schramm, Werner:

. SLEs = <u>Loewner evolution with a Brownian motion as source:</u>

. $\mathbf{E}[\xi_t \, \xi_s] = \kappa \min(t, s)$.

Different phases (Rhode-Schramm).

- SLEs hull: $K_t = \{z \in \mathbf{H};\ \tau_z < t\}$ with $\tau_z =$ swallowing time.



- SLEs trace: $\gamma(t) = g_t^{-1}(\xi_t)$.

SLE $\equiv$ growth process of K_t.

Coupling SLEs to CFTs.

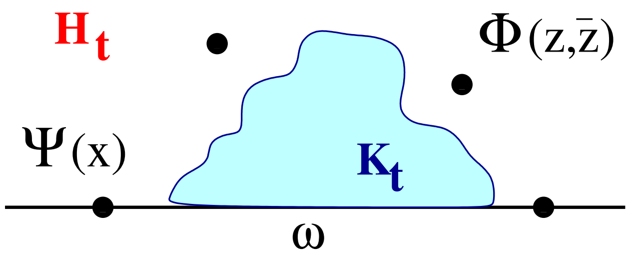
- Testing the SLE hulls with a boundary CFT.

. in $\mathbf{H_t} \equiv \mathbf{H} \setminus K_t$: complement of the SLE hull

$$\langle \cdots \Psi_\delta(x) \cdots \Phi_h(z,\bar{z}) \cdots \rangle_{\omega;\mathbf{H_t}}$$

- Main property...

. For an appropriate choice of CFT and boundary conditions:



is conserved in mean

$\cdots\cdots$ How to deal with finite (random) deformations in (boundary) CFT ?

(Random) conformal maps in CFT.

- CFT in a infinitesimally deformed geometry.

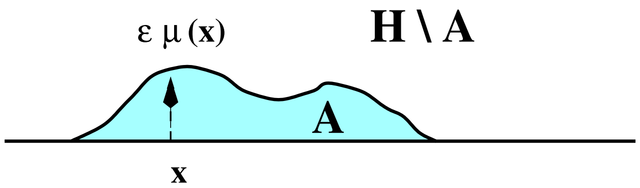
Uniforming map of $\mathbf{H} \setminus \mathbf{A}$ onto $\mathbf{H}$:

$g_A(z) = z + \varepsilon \int \frac{\mu_A(x)}{z-x} dx + \cdots$

Represented in CFT by insertion of:

$$[1 - \varepsilon \int \mu_A(x) T(x) dx + \cdots]$$

For a tiny hull A $\cdots$



Infinitesimal conformal deformations are implemented in CFT via insertions of the stress tensor $T(z) = \sum_n L_n z^{-n}$ with L_n the Virasoro generators.

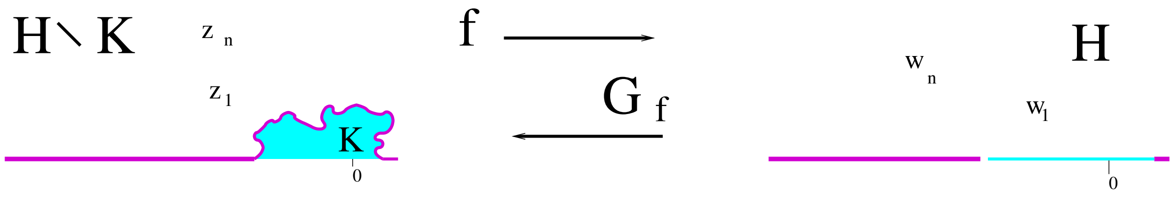
$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m;0}$.

(Random) conformal maps in CFT.

- Finite conformal transformations in (boundary) CFT:

Associate an operator G_f to any conformal map f (regular at infinity):

$f = z + \sum_{m<0} f_m z^{m+1} \in N_-$ $=$ Group of germs of conformal maps at ∞.

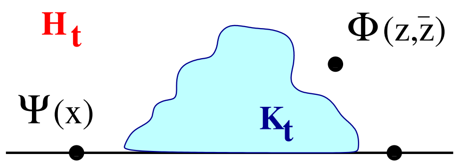


Homomorphism: $f \in N_-$ ... to ... $G_f \in \overline{\mathcal{U}(\mathrm{vir}_-)}$, with $G_{f\circ g} = G_g\, G_f$,

$$\begin{aligned} \langle \cdots \Psi_\delta(x) \cdots \rangle_{\mathbf{H}\setminus\mathbf{K}} &= \langle \cdots |f'(x)|^\delta \Psi_\delta(f(x)) \cdots \rangle_{\mathbf{H}} \\ &\equiv \langle G_f^{-1} \cdots \Psi_\delta(x) \cdots G_f \rangle_{\mathbf{H}} \end{aligned}$$

G_f intertwines the CFT in $H \setminus K$ and in H.

Testing SLE with CFT.



- Translation in the target space:

$$f_t(z) \equiv g_t(z) - \xi_t$$

implemented in CFT by $G_{f_t} \equiv G_t$

$$G_t^{-1} dG_t = dt(-2L_{-2} + \tfrac{\kappa}{2} L_{-1}^2) + L_{-1} d\xi_t$$

A Markov process in $\overline{\mathcal{U}(\mathrm{vir}_-)}$, intertwining CFT in $\mathbf{H_t}$ and $\mathbf{H}$.

- Main property:

Let $|\omega\rangle$ be the h.w.v. of the irreducible Virasoro representation with weight $h^{\kappa}_{1,2} = (6-\kappa)/2\kappa$ central charge $c_\kappa = (3\kappa - 8)(6-\kappa)/2\kappa$

Then,

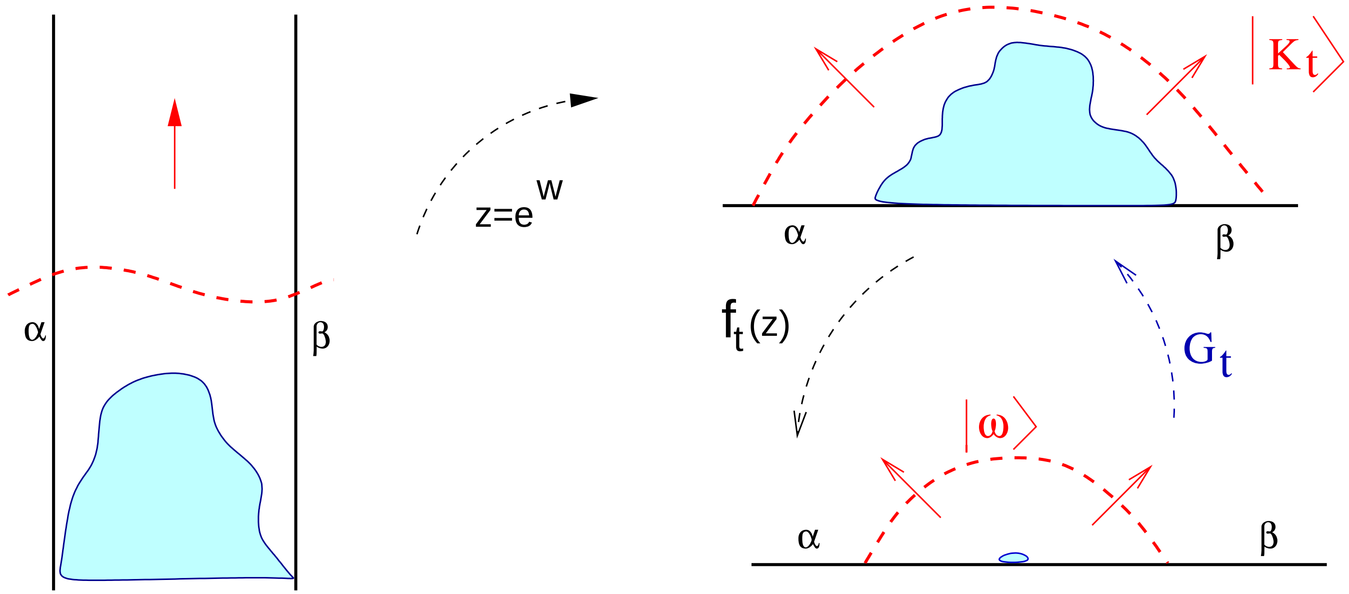
$G_t|\omega\rangle$ is conserved in mean

$$\mathbf{E}[\, G_t|\omega\rangle \, |\{G_u\}_{u\le s}] = G_s|\omega\rangle$$

.... $G_t|\omega\rangle$ is a (universal) martingale.

Geometrical interpretation.

- The hull boundary state $\equiv |K_t\rangle$ ······

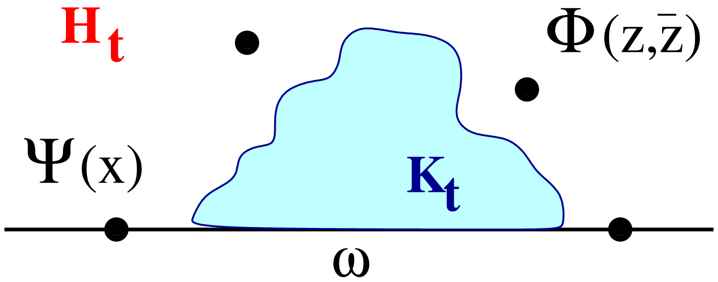


······ the hull boundary state $|K_t\rangle = G_t|\omega\rangle$.

Coupling SLEs to CFTs.

- Main property (with appropriate boundary condition):

$$\begin{aligned}\langle \Psi_\delta(x)\cdot\Phi_h(z,\bar z)\rangle_{\omega;\mathbf{H_t}} &\equiv \langle 0|\Psi_\delta(x)\cdot\Phi_h(z,\bar z)\cdot G_t|\omega\rangle\\ &= \langle 0|[f_t'(x)]^\delta\Psi_\delta(f_t(x))\cdot|f_t'(z)|^{2h}\Phi_h(f_t(z),\bar f_t(z))|\omega\rangle\end{aligned}$$



is conserved in mean

Crossing probabilities: SLEs = CFTs.

Crossing probabilities $\equiv$ Stopping time event probabilities (eg Cardy's formula).

- The martingale property may be extended to stopping time τ:

$$\mathbf{E}[G_\tau\,|\omega\rangle] = |\omega\rangle \qquad \text{or} \qquad \mathbf{E}[\langle v|\,G_\tau\,|\omega\rangle] = \langle v\,|\omega\rangle$$

Appropriate choice of $\langle v|$ depending on the problem.

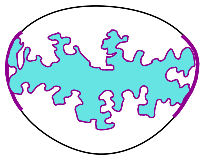
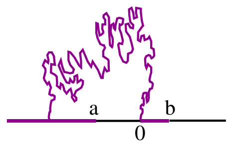
For event $\mathcal{E}$:

$$\langle v_{\mathcal{E}}|\,G_\tau\,|\omega\rangle = \mathbf{1}_{\mathcal{E}}$$

$\Longrightarrow$ CFT amplitudes = SLE probabilities.

$$\mathrm{Prob}[\mathcal{E}] \;\equiv\; \mathbf{E}[\mathbf{1}_{\mathcal{E}}] \;=\; \langle v_{\mathcal{E}}|\omega\rangle$$

Crossing probablilities: Cardy’s formula.

Prob ()$_{\text{CFT}}$ = Prob (

)$_{\text{SLE}}$

In SLE terms, L.S.W.,Smirnov: $\mathbf{P}[\gamma \text{ hits } (-\infty, a] \text{ first}] = \mathbf{P}[\tau_a < \tau_b]$

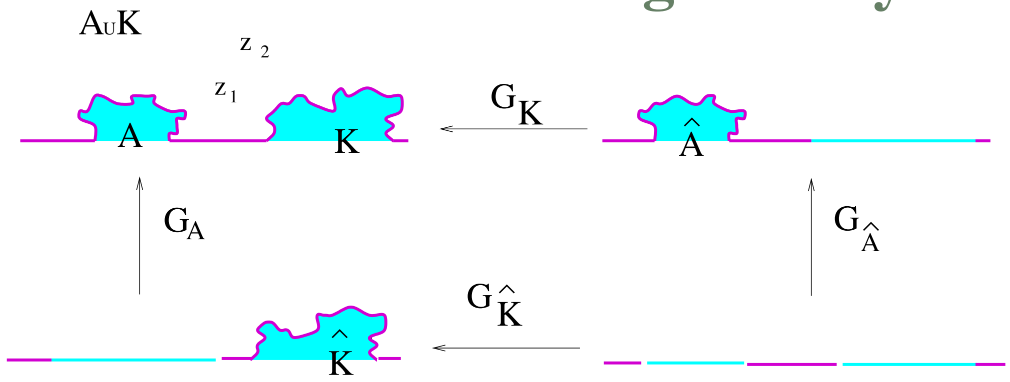
- For $\kappa > 4$, consider the CFT correlation: $u = \frac{a}{a-b}, \; u_\tau = \frac{f_\tau(a)}{f_\tau(a)-f_\tau(b)}$

$$\begin{aligned} F(u_\tau) &\equiv \langle \omega | \Psi_{\delta=0}(a) \Psi_{\delta=0}(b) G_\tau | \omega \rangle \\ &= \mathbf{1}_{\{\tau_a < \tau_b\}} F(0) + \mathbf{1}_{\{\tau_a > \tau_b\}} F(1) \end{aligned}$$

- Apply the martingale property to $F(u_\tau)$:

$$\mathbf{E}[F(u_\tau)] = F(u) \Longrightarrow \mathbf{P}[\tau_a < \tau_b] = \frac{F(u) - F(1)}{F(0) - F(1)}$$

SLE in a deformed geometry.



$f_K \equiv f_t$ and $f_{\hat{K}}$ fix $\infty \cdots$ $\qquad$ $\cdots$ f_A and $f_{\hat{A}}$ fix 0.

$G_K \equiv G_t$ and $G_{\hat{K}} \in \overline{\mathcal{U}(\mathrm{vir}_-)}$ $\qquad$ G_A and $G_{\hat{A}} \in \overline{\mathcal{U}(\mathrm{vir}_+)}$

- "Virasoro Wick theorem": $\qquad G_A^{-1}\, G_K = Z(A,K)\; G_{\hat{K}}\, G_{\hat{A}}^{-1}$

- The SLE partition function martingale:

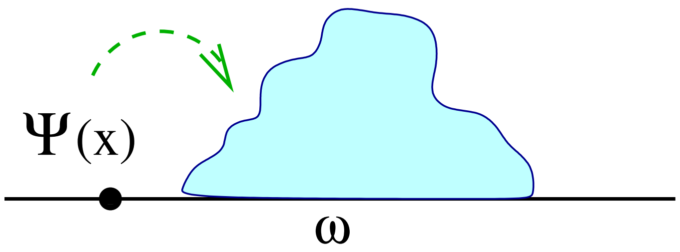
$$M_A \equiv \langle \omega | G_A^{-1}\, G_t | \omega \rangle = Z(A, K_t)\, f'_{\hat{A}_t}(0)^{h_\kappa}$$

$\cdots\cdots$ Probability that the $SLE_{\kappa=8/3}$ trace does not touch A.

A relation with 2D gravity.

CFTs coupled to SLE_κ hulls as CFT's on random domains $\mathbf{H_t}$
or in $\mathbf{H}$ with a random metric as in 2D gravity: cf. Duplantier

$$\mathbf{E}[\,\langle \Psi_\delta(x)\cdot\Phi_h(z,\bar z)\rangle_{\omega;\mathbf{H_t}}\,]$$



$x \to 0$ mimics the approach to the hull:

$$\mathbf{E}[\,\langle \cdots \Psi_\delta(x)\rangle_{\omega;\mathbf{H_t}}\,] \simeq_{x\to 0} x^{\Delta}$$

Fusion rules determine Δ KPZ formula ($\kappa < 4$).

$$\Delta\left(\Delta - \frac{\kappa-4}{\kappa}\right) = \frac{4}{\kappa}\,\delta \qquad \text{identical to} \qquad \Delta(\Delta - \gamma_{\mathrm{str.}}) = (1-\gamma_{\mathrm{str.}})\,\delta$$

Conclusions.

• Conformal transformations in CFT...

...Random

• Testing SLE:

Martingales and null vectors.

Other (random) applications ??...

• $\langle C.F.T. \rangle = \mathbf{P}[S.L.E.]$

Reconstruction ??..

A new perspective to field theory ??....