Solution of reflection equations related to (super) Yangians

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- D. Arnaudon, J. Avan, N. C., L. Frappat and É. Ragoucy, *R matrix presentation for (super)-Yangians $Y(g)$*, J. Math. Phys. **44** (2003) 302 and `math.QA/0111325`.

- D. Arnaudon, J. Avan, N. C., A. Doikou, L. Frappat and É. Ragoucy, *Classification of reflection matrices related to (super) Yangians and application to open spin chain models*, to appear in Nucl. Phys. B and `math.QA/0304150`.

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1 Introduction

$$\text{Model XXX}: \quad H = \sum_{i=1}^{N} H_{i,i+1}$$
$$\left(H_{i,i+1} = \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z\right)$$

PROPERTIES

Symmetry is $sl(2)$

Periodic boundary
$H_{N,N+1} = H_{N,1}$

GENERALISATIONS

Spin chain with
$osp(m|n)$ symmetry

Open spin chain
$H = \sum_{i=1}^{N-1} H_{i,i+1} + B_1 + B_N$

METHODS

Find a R-matrix which commutes with the generators of $osp(m|n)$

Find B_1 and B_N so that H be integrable

2 Yangian of $osp(m|n)$

M. Lüscher (1976) proved that an integrable Hamiltonian of a closed spin chain can be constructed from the transfer matrix:

$$t(\lambda) = tr_0(R_{01}(\lambda)\cdots R_{0N}(\lambda))$$

by:

$$H = \frac{d}{d\lambda}\Big(ln(t(\lambda))\Big)\Big|_{\lambda=0}$$

R-matrices $R_{0i}(\lambda)$ must satisfy the *Yang–Baxter equation*.

Goal

Find a closed spin chain with $osp(m|n)$ symmetry.

$$\Updownarrow$$

Find an R-matrix so that $[R_{12}(\lambda), U_1 + U_2] = 0$
where U is any generator of $osp(m|n)$.

Solution

The R-matrix of the Yangian of $osp(m|n)$.

2.1 Some definitions

- $\mathbb{Z}_2$-gradation $[i]$

$$(-1)^{[i]} = \begin{cases} \theta_0 & , \ 1 \le i \le m \\ -\theta_0 & , \ m+1 \le i \le m+n \end{cases} \qquad (\theta_0 = \pm 1)$$

- Signs θ_i

$$\theta_i = \begin{cases} +1 & , \qquad 1 \le i \le m + \frac{n}{2} \\ -1 & , \qquad m + \frac{n}{2} + 1 \le i \le m+n \end{cases}$$

- Conjugate index $\bar{\imath}$

$$\bar{\imath} = \begin{cases} m+1-i & , \qquad 1 \le i \le m \\ 2m+n+1-i & , \qquad m+1 \le i \le m+n \end{cases}$$

- Generalized transposition

$$\begin{aligned} A^t &= \sum_{ij} \left(A^{ij}\right)^t \, E_{ij} \\ &= \sum_{ij} (-1)^{[i][j]+[j]} \theta_i \theta_j \, A^{ij} \, E_{\bar{\jmath}\bar{\imath}} \end{aligned}$$

2.2 R-matrix

$$R(\lambda) = \lambda(\lambda + i\kappa)\mathbb{I} + i(\lambda + i\kappa)P - i\lambda Q$$

- $P = \sum_{i,j=1}^{m+n} (-1)^{[j]} \;\; E_{ij} \otimes E_{ji}$

  is the super permutation operator i.e

  $$P X_{12} P = X_{21} \qquad \big(X \in gl(m|n) \otimes gl(m|n)\big)$$

- $Q = P^{t_1} = \sum_{i,j=1}^{m+n} (-1)^{[i][j]} \;\; \theta_i \theta_j \;\; E_{\overline{j}\overline{i}} \otimes E_{ji}$

  is a projector.

- $2\,\kappa = (m - n - 2)\theta_0 \quad \in \mathbb{C}$

  depends only of the corresponding super algebra.

Properties satisfied by the R-matrix

- YBE

$$R_{12}(\lambda)\, R_{13}(\lambda+\mu)\, R_{23}(\mu) = R_{23}(\mu)\, R_{13}(\lambda+\mu)\, R_{12}(\lambda+\mu)$$

- Crossing symmetry

$$R_{12}^{t_1}(-\lambda - i\kappa) = R_{12}(\lambda)$$

- Unitarity

$$R_{12}(\lambda)\, R_{12}(-\lambda) = (\lambda^2+\kappa^2)(\lambda^2+1)\mathbb{I}$$

- $R_{12}(\lambda=0) \propto P_{12}$
- $[R_{12}(\lambda), U_1 + U_2] = 0$; U is any generator of $osp(m|n)$.

Hamiltonian

In this case, the closed spin chain hamiltonian is

$$H \propto \sum_{i=1}^{N} \left(P_{i,i+1} - \frac{\theta_0}{\kappa}\, Q_{i,i+1} \right)$$

$$P_{N,N+1} = P_{N,1} \quad \text{and} \quad Q_{N,N+1} = Q_{N,1}$$

Remark

The known cases of $so(m)$ and $sp(n)$ are particular cases by taking $n=0$ or $m=0$, respectively.

2.3 Algebra $\mathcal{U}(R)$

Goal

Construct the FRT realisation of $Y(osp(m|n))$.

Definition

$\mathcal{U}(R)$ is the complex associative super-algebra generated by

1 and $T_{ij}^{(k)}$, for $1 \le i, j \le m+n$ and $k \in \mathbb{Z}_{>0}$

subject to the defining relations:

$$R_{12}(\lambda-\mu)\, T_1(\lambda)\, T_2(\mu) = T_2(\mu)\, T_1(\lambda)\, R_{12}(\lambda-\mu)$$

where

$$T(\lambda) \;=\; 1 + \sum_{i,j=1}^{m+n} \sum_{k \in \mathbb{Z}_{>0}} \frac{T_{ij}^{(k)}}{\lambda^k} \otimes\, E_{ij}$$

$$\in\, \mathcal{U}(R)\left[\left[\lambda^{-1}\right]\right] \otimes gl(m|n)$$

Hopf structure

$\mathcal{U}(R)$ has a Hopf structure

$$\begin{array}{ll} \textit{coproduct} & \Delta(T_{ij}(\lambda)) = \sum_{k=1}^{m+n} T_{ik}(\lambda) \otimes T_{kj}(\lambda) \\ \textit{antipode} & S(T(\lambda)) = T^{-1}(\lambda) \\ \textit{counit} & \varepsilon(T(\lambda)) = \mathbb{I} \end{array}$$

2.4 Yangian $Y(osp(m|n))$

Center of $\mathcal{U}(R)$

$C(\lambda) = T^t(\lambda - \kappa)\, T(\lambda)$ lies in the center of $\mathcal{U}(R)$ and

$$\Delta(C(\lambda)) = C(\lambda) \otimes C(\lambda).$$

$\mathcal{I}$, the two-sided ideal generated by $C(\lambda) = \mathbb{I}$, is also a coideal.

$$\Downarrow$$

$$\mathcal{U}(R)/\mathcal{I} \text{ is a Hopf algebra.}$$

Theorem

The quotient $\mathcal{U}(R)/\mathcal{I}$ is a quantization of the algebra $osp(m|n) \otimes \mathbb{C}[\lambda]$ endowed with its standard bialgebra structure.

Definition

The Yangian of $osp(m|n)$ is the quotient $\mathcal{U}(R)/\mathcal{I}$.

Remark

This definition is consistent with the cases $so(m)$ and $sp(n)$.

2.5 Open problems about the Yangian

- Find the FRT realisation for the Yangian of the exceptional Lie algebra E_6, E_7, F_4 and G_2.
  For $Y(E_8)$, see K.Koepsell et al. `hep-th/9903111`.

- Classify the representations of $Y(osp(m|n))$

- Determine the form of the other bases of $Y(osp(m|n))$

- Find the universal R-matrix of $DY(osp(m|n))$

Remark

The last two points have been solved for $Y(osp(1|2))$.
See D.Arnaudon et al. `math.QA/0209167`.

3 Classification of the reflection matrices

Introduce non trivial boundary effects

$$H = \sum_{i=1}^{N-1} \left(P_{i,i+1} - \frac{\theta_0}{\kappa}\, Q_{i,i+1} \right) + B_1 + B_N$$

$$\Downarrow$$

Describe the reflection process on the boundaries

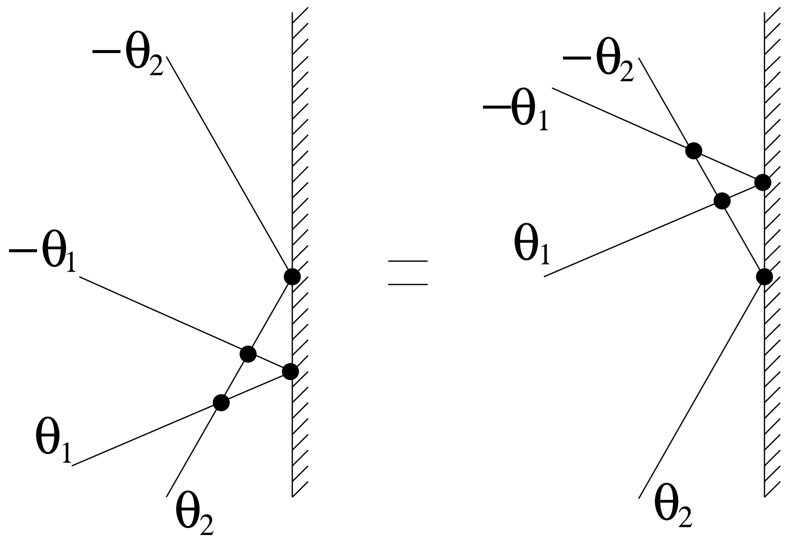
$$\Downarrow$$

Reflection matrices solutions of the reflection equations.

3.1 Reflection equations

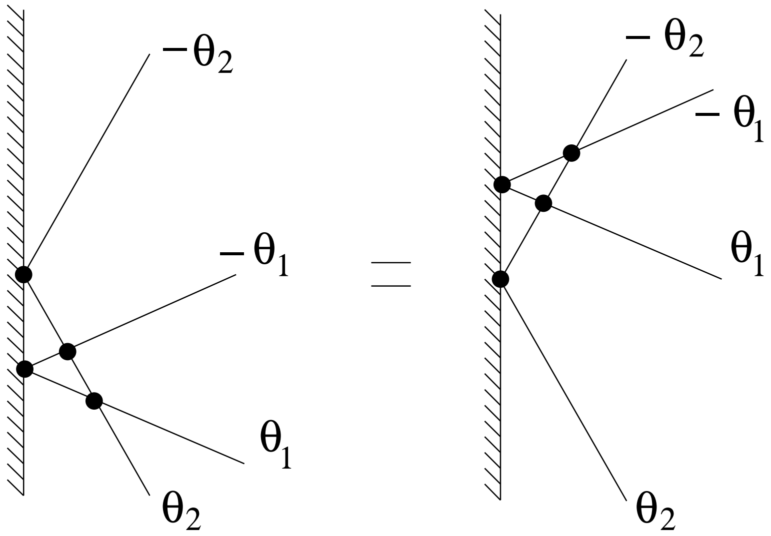
Reflection on the right boundary

$$R_{12}(\theta_1-\theta_2)\,K_1(\theta_1)\,R_{12}(\theta_1+\theta_2)\,K_2(\theta_2)$$
$$=K_2(\theta_2)\,R_{12}(\theta_1+\theta_2)\,K_1(\theta_1)\,R_{12}(\theta_1-\theta_2)\,.$$



Reflection on the left boundary

$$R_{12}(-\theta_1+\theta_2)\,K_1^t(\theta_1)\,R_{12}(-\theta_1-\theta_2-2i\kappa)\,K_2^t(\theta_2)$$
$$=K_2^t(\theta_2)\,R_{12}(-\theta_1-\theta_2-2i\kappa)\,K_1^t(\theta_1)\,R_{12}(-\theta_1+\theta_2)\,.$$



3.2 Hamiltonian

E.K. Sklyanin (1987) showed that, in the case of open spin chain, the transfer matrix becomes :

$$t(\theta) = tr_0\Big(K_0^+(\theta)R_{0N}(\theta)\cdots R_{01}(\theta)\times \\ K_0^-(\theta)R_{01}^{-1}(-\theta)\cdots R_{0N}^{-1}(-\theta)\Big)$$

and the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2trK^+(0)} \quad \frac{d}{d\theta}\Big(t(\theta) - tr\left(K^+(\theta)\right)\Big)\Big|_{\theta=0} \\ &= \sum_{k=1}^{N-1} H_{k,k+1} + B_1 + B_N \end{aligned}$$

is integrable.

$H_{k,k+1}$ are the same operators as for the closed spin chain.

The boundary terms, B_1 and B_N, act in the space 1 and N

$$\begin{aligned} B_1 &= \frac{1}{2}\frac{d}{d\theta}\Big(K_1^-(\theta)\Big)\Big|_{\theta=0} \\ B_N &= \frac{tr_0\left(K_0^+(0)H_{N0}\right)}{tr\left(K^+(0)\right)} \end{aligned}$$

3.3 Classification of the matrices $K^{-}(\theta)$

Goal

Classify the reflection matrices in order to classify the boundary terms for the open spin chain.

Achievements

We classify the diagonal, antidiagonal and mixed solutions.

Lemma

If $K(\theta)$ is solution of the reflection equation and U is a matrix of the orthosymplectic super group, then $K^t(\theta)$, $UK(\theta)U^t$ and $UK^t(\theta)U^t$ are also solutions.

Diagonal solutions

We look for the solution of the following type:

$$K(u) = diag\Big(k_1(u), \cdots, k_m(u); k_{m+1}(u), \cdots, k_{m+n}(u)\Big)$$

There are 4 families of diagonal solutions

- D1: Solution only valid for m even.

$$diag\Big(\underbrace{1,...,1}_{\frac{m}{2}},\underbrace{\alpha(c),...,\alpha(c)}_{\frac{m}{2}};\ \underbrace{1,...,1}_{\frac{n}{2}},\underbrace{\alpha(c),...,\alpha(c)}_{\frac{n}{2}}\Big)$$

- D2: Solution valid for $m > 1$.

$$diag\Big(\underbrace{\alpha(c_1),1,...,1,\alpha(c_m)}_{m};\ 1,...,1\Big)$$

where: $(\kappa - \theta_0)c_1c_m + c_1 + c_m = 0$

- D3:

$$diag\Big(\underbrace{1,...,1}_{m_1},\underbrace{\alpha(c),...,\alpha(c)}_{m-2m_1},\underbrace{1,...,1}_{m_1};$$
$$\underbrace{1,...,1}_{n_1},\underbrace{\alpha(c),...,\alpha(c)}_{n-2n_1},\underbrace{1,...,1}_{n_1}\Big)$$

where: $c = \frac{2}{\kappa-\theta_0(2m_1-2n_1-1)}$.

- D4: Solution valid only for $so(4)$

$$diag\big(1\ ,\ \alpha(c_2)\ ,\ \alpha(c_3)\ ,\ \alpha(c_2)\ \alpha(c_3)\big)$$

where:

$$\alpha(c) = \frac{1 + c\theta}{1 - c\theta}$$

Antidiagonal solutions

The solutions exist only for the algebras $so(2m)$ and $sp(2m)$, are constant and have the following form:

$$\begin{pmatrix} & & & & & \ell_1 \\ & 0 & & & \cdot^{\cdot^{\cdot}} & \\ & & & \ell_m & & \\ & & \frac{1}{\ell_m} & & & \\ & \cdot^{\cdot^{\cdot}} & & & 0 & \\ \frac{1}{\ell_1} & & & & & \end{pmatrix}$$

Mixed solutions

We look for solutions with terms only in the diagonal and the antidiagonal

$$\left(\begin{array}{ccccc|ccccc}
k_1(u) & 0 & \cdots & 0 & \ell_1(u) & & & & & \\
0 & \ddots & & \iddots & 0 & & & & & \\
\vdots & 0 & \ddots & 0 & \vdots & & & 0 & & \\
0 & \iddots & & \ddots & 0 & & & & & \\
\ell_m(u) & 0 & \cdots & 0 & k_m(u) & & & & & \\
\hline
& & & & & k_{m+1}(u) & 0 & \cdots & 0 & \ell_{m+1}(u) \\
& & & & & 0 & \ddots & & \iddots & 0 \\
& & 0 & & & \vdots & & \ddots & & \vdots \\
& & & & & 0 & \iddots & & \ddots & 0 \\
& & & & & \ell_{m+n}(u) & 0 & \cdots & 0 & k_{m+n}(u)
\end{array}\right)$$

Lemma

These solutions

- are constant
- are valid only for m even
- fall into 2 classes: $\begin{cases} \ell_1, \cdots, \ell_m = 0 \\ or \\ \ell_{m+1}, \cdots, \ell_{m+n} = 0 \end{cases}$

Example of $osp(4|2)$

One finds 2 solutions :

$$\left(\begin{array}{cccc|cc}
1 & & & & & \\
 & 1 & & & & \\
 & & -1 & & & \\
 & & & -1 & & \\
\hline
 & & & & k_5 & \ell_5 \\
 & & & & \ell_6 & -k_5
\end{array}\right)$$

where $k_5^2 + \ell_5 \ell_6 = 1$.

$$\left(\begin{array}{cccc|cc}
1 & & & 0 & & \\
 & 0 & \ell_2 & & & \\
 & \ell_2^{-1} & 0 & & & \\
0 & & & 1 & & \\
\hline
 & & & & 1 & \\
 & & & & & 1
\end{array}\right)$$

3.4 Conclusion

- Complete the classification of the reflection matrices

- Diagonalize the Hamiltonian
  $A.Doikou$ will present the derivation of the Bethe equations and their solutions in the limit :

$$N \longrightarrow +\infty$$