String Interactions in a Gravitational Wave Background

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Plan of the talk

Gravitational waves in string theory

The structure of the H_4 WZW model

1) Representations

2) Spectral flow

3) OPE

Three-point couplings

Four-point correlators

Two different approaches

1) Free field realization

2) Penrose limit of $U(1) \times SU(2)_k$.

String amplitudes

Flat space limit

Gravitational waves as string backgrounds

Plane-fronted gravitational waves provide a large class of classical vacua for string theory Guven, Amati and Klimcik, Horowitz and Steif, Tseytlin

$$ds^2 = -dudv + dx^i dx_i + F(u, x^i) du^2 \ .$$

Explicit realizations in terms of a WZW model based on a non-semi-simple group. Nappi and Witten

Related by contraction to standard WZW models. Olive, Rabinovici and Schwimmer

The Nappi-Witten wave is the gravitational wave generated by a constant value of T_{uu} in the Einstein equation. It corresponds to the simple choice

$$F(u, x^i) = -\mu^2 x^i x_i \ .$$

Example of a non-compact curved CFT

Very few cases studied in detail: AdS_3 and its orbifolds. The H_4 model is much simpler but retains several essential features:

1) states belong to infinite dimensional representations

2) highest-weight representations appear together with their spectral flowed images

$AdS_3 \times S^3 \to H_6$

String correlators in H_6 and observables in the dual holographic theory.

Comparison between the R R and NS NS models.

The Nappi-Witten model

Consider a WZW model based on the non-semi-simple H_4 Lie algebra

$$[P^+, P^-] = -2i\mu K \ , \qquad [J, P^\pm] = \mp i\mu P^\pm \ .$$

The Killing form is degenerate but there is another non-degenerate invariant form and the stress-energy tensor is still bilinear in the currents

$$T = \frac{1}{4}\left(P^+P^- + P^-P^+\right) + JK + \frac{\mu^2}{2}K^2 \ .$$

This model describes a four-dimensional homogeneous space of Lorentzian signature and central charge $c = 4$

$$ds^2 = -2dudv - \frac{\mu^2 r^2}{4}du^2 + dr^2 + r^2 d\varphi^2 \ , \qquad B_{\varphi u} = \frac{\mu r^2}{2} \ .$$

The CFT on the string world-sheet can be quantized in a covariant way. Both the $p \neq 0$ and the $p = 0$ sector will be under control. The latter is crucial, after spectral flow, in order to describe the long string states.

Three different approaches

- Current algebra techniques: Knizhnik-Zamolodchikov equations.
- Free field resolution: twist field correlators.
- Contraction of the $U(1) \times SU(2)_k$ operator algebra: Penrose limit of the CFT.

Structure of the H_4 WZW model

The H_4 algebra has three types of unitary representations

	K	J	C
$V^+_{p,\hat{\jmath}}$	p	$\{\hat{\jmath}+n\}_{n\in\mathbb{N}}$	$-2p\hat{\jmath}+p$
$V^-_{p,\hat{\jmath}}$	$-p$	$\{\hat{\jmath}-n\}_{n\in\mathbb{N}}$	$2p\hat{\jmath}+p$
$V^0_{s,\hat{\jmath}}$	0	$\{\hat{\jmath}+n\}_{n\in\mathbb{Z}}$	s^2

States in $V^\pm_{p,\hat{\jmath}}$ correspond to particles describing a periodic orbit in the transverse plane. States in $V^0_{s,\hat{\jmath}}$ correspond to particles moving freely in the transverse plane.

Since we are dealing with infinite dimensional representations it is useful to realize the H_4 algebra in terms of differential operators acting on auxiliary variables x and $\bar{x}$. For the $V^+_{p,\hat{\jmath}}$ representation they are given by

$$P^+_0 = \sqrt{2}p\ x\ , \quad P^-_0 = \sqrt{2}\ \partial_x\ , \quad J_0 = i(\hat{\jmath}+x\partial_x)\ , \quad K_0 = ip\ .$$

Semiclassical expression for the vertex operators

$$\Phi^+_{p,\hat{\jmath}} = e^{2ipv+iju-\frac{\mu p}{2}z\tilde{z}+i\mu pzxe^{\frac{i\mu u}{2}}+i\mu p\tilde{z}\bar{x}e^{\frac{i\mu u}{2}}+\mu px\bar{x}e^{i\mu u}}\ .$$

Highest-weight representations are constructed by acting with the modes J^a_{-n} on states $|R_i\rangle$ which form an unitary representations of the global H_4 and satisfy

$$J^a_n|R_i\rangle = 0\ , \qquad n>0\ .$$

When $p>1$, the modes $P^\pm_{-1}$ create negative norm states that can not be removed from the physical string spectrum. Bound on the eigenvalues of K in the highest weight representations: $|p|<1$.

We can collect all the primary fields corresponding to the states in a given H_4 representation in a single field

$$\Phi^{+}_{p,\hat{\jmath}}(z,x) = \sum_{n=0}^{\infty} R^{+}_{p,\hat{\jmath};n}(z) \frac{(x\sqrt{p})^n}{\sqrt{n!}} \ ,$$

and write the OPE with the currents in the simple form

$$P^{+}(z)\Phi^{+}_{p,\hat{\jmath}}(w,x) = \sqrt{2px}\frac{\Phi^{+}_{p,\hat{\jmath}}(w,x)}{z-w} \ , \quad P^{-}(z)\Phi^{+}_{p,\hat{\jmath}}(w,x) = \sqrt{2}\partial_x \frac{\Phi^{+}_{p,\hat{\jmath}}(w,x)}{z-w} \ .$$

States carrying $p > 1$ are not in the highest-weight representations but can be obtained using the spectral flow.

Consider the algebra $\tilde{H}_{4,w}$

$$\tilde{P}^{\pm}_{n} = P^{\pm}_{n\mp w} \ , \quad \tilde{K}_n = K_n - iw\delta_{n,0} \ , \quad \tilde{J}_n = J_n \ , \quad \tilde{L}_n = L_n - iwJ_n \ .$$

The spectral-flowed representations of the H_4 algebra are representations that are highest-weight with respect to the $\tilde{H}_{4,w}$ algebra. We will denote them by $\Omega_w(\Phi^{+}_{p,\hat{\jmath}})$.

From a geometric point of view, spectral flow generates new solutions of the world-sheet σ-model. Maldacena and Ooguri

In our case its action is given by

$$g(\tau,\sigma) \to e^{w(\tau+\sigma)J} g(\tau,\sigma) e^{w(\tau-\sigma)J} \ ,$$

and then generates the following coordinate transformation Kiritsis and Pioline

$$u \to u + w\tau \ , \quad v \to v \ , \quad \rho e^{i\varphi} \to \rho e^{i(\varphi - w\sigma)} \ .$$

States in the spectral flowed continuous representations are the so called long strings. They have $p \in \mathbb{Z}$ but move freely in the transverse plane: the coupling to the metric and to the antisymmetric tensor compensate exactly.

The action of the original modes on the ground states $|\psi\rangle$ of a spectral flowed representations is given by

$$P_n^+|\psi\rangle = 0 \ , \quad n \geq -w \qquad P_n^-|\psi\rangle = 0 \ , \quad n > w \ .$$

The spectrum of L_0 is then generically unbounded from below.

Two exceptions: $\Omega_{-1}(\Phi^+_{p,\hat{\jmath}}) = \Phi^-_{1-p,\hat{\jmath}}$ and $\Omega_1(\Phi^-_{p,\hat{\jmath}}) = \Phi^+_{1-p,\hat{\jmath}}$.

The spectrum of the H_4 model is then given by

- $\Omega_{\pm w}(\Phi^{\pm}_{p,\hat{\jmath}})$ with $p<1$ and $w \in \mathbb{N}$,
- $\Omega_w(\Phi^0_{s,\hat{\jmath}})$ with $\hat{\jmath} \in [-1/2, 1/2)$ and $w \in \mathbb{Z}$.

We can now discuss the correlators between the primary fields of the H_4 algebra.

Two-point functions

$$\langle \Phi^+_{p_1,\hat{\jmath}_1} \Phi^-_{p_2,\hat{\jmath}_2} \rangle = \delta(p_1-p_2)\delta(\hat{\jmath}_1+\hat{\jmath}_2) \frac{e^{-p_1(x_1x_2+\bar{x}_1\bar{x}_2)}}{|z_{12}|^{4h}} \ .$$

Three-point functions

$$\langle \Phi^a_{q_1} \Phi^b_{q_2} \Phi^c_{q_3} \rangle = \frac{\mathcal{C}_{abc}(q_1,q_2,q_3) D_{abc}(x_1,x_2,x_3)}{|z_{12}|^{2(h_1+h_2-h_3)} |z_{13}|^{2(h_1+h_3-h_2)} |z_{23}|^{2(h_2+h_3-h_1)}} \ .$$

Completely fixed by the conformal and H_4 Ward identities, up to the structure constants $\mathcal{C}_{abc}$.

As an example we consider the OPE

$$[\Phi^+_{p_1,\hat{\jmath}_1}] \otimes [\Phi^+_{p_2,\hat{\jmath}_2}] = \sum_{n=0}^{\infty} [\Phi^+_{p_1+p_2,\hat{\jmath}_1+\hat{\jmath}_2+n}] \ .$$

The corresponding D function is

$$D_{++-} = \left| e^{-x_3(p_1x_1+p_2x_2)} (x_2-x_1)^{-L} \right|^2 \delta(p_3-p_1-p_2) \delta_{\mathbb{N}}(-L) \ ,$$

where $L = \hat{\jmath}_1 + \hat{\jmath}_2 + \hat{\jmath}_3$.

The functions D are generating functions for the H_4 Clebsch-Gordan coefficients.

Three-point couplings

There are three classes of three-point couplings

- $\langle V^+V^+V^-\rangle$ couplings.

$$\mathcal{C}_{++-} = \frac{1}{\Gamma(1-\hat{\jmath}_3-\hat{\jmath}_1-\hat{\jmath}_2)}\left[\frac{\gamma(p_3)}{\gamma(p_1)\gamma(p_2)}\right]^{\frac{1}{2}-\hat{\jmath}_3-\hat{\jmath}_1-\hat{\jmath}_2}$$

  Here

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} ,$$

  $p_3 = p_1 + p_2$ and $\hat{\jmath}_3 = -\hat{\jmath}_1 - \hat{\jmath}_2 - n$, $n \in \mathbb{N}$.

  Other couplings of the same type are given by similar expression. For instance $\langle V^+V^-V^-\rangle$ with $p_3 = p_1 - p_2 > 0$ and $\hat{\jmath}_3 = -\hat{\jmath}_1 - \hat{\jmath}_2 + n$, $n \in \mathbb{N}$ is

$$\mathcal{C}_{+--} = \frac{1}{\Gamma(1+\hat{\jmath}_3+\hat{\jmath}_1+\hat{\jmath}_2)}\left[\frac{\gamma(p_1)}{\gamma(p_2)\gamma(p_3)}\right]^{\frac{1}{2}+\hat{\jmath}_3+\hat{\jmath}_1+\hat{\jmath}_2}$$

- $\langle V^+V^-V^0\rangle$ couplings.

$$\mathcal{C}_{+-0} = e^{\frac{s^2}{2}[\psi(p)+\psi(1-p)-2\psi(1)]}$$

  where $\psi(x) = \frac{d\ln\Gamma(x)}{dx}$ is the digamma function.

- $\langle V^0V^0V^0\rangle$ couplings. They are the same as in flat-space.

Four-point correlators and the KZ equation

The KZ equation follows from the existence of the null vector

$$\left[L_{-1}-\frac{1}{2}(P^-_{-1}P^+_0+P^+_{-1}P^-_0)-J_{-1}K_0-K_{-1}J_0-K_{-1}K_0\right]|V\rangle$$

in any highest-weight representation V of the affine algebra.

In our case we write this equation as a partial differential equation

$$\partial_{z_i}A=\sum_{j=1,j\neq i}^{4}\frac{1}{z_{ij}}\left[\frac{D_i^+D_j^-+D_j^-D_i^+}{2}+D_i^JD_j^K+D_j^KD_i^J+D_i^KD_j^K\right]A\ .$$

We fix the global conformal and H_4 symmetry

$$A(z_i,\bar{z}_i,x_i,\bar{x}_i)=\prod_{i<j}^{4}|z_{ij}|^{2\left(\frac{h}{3}-h_i-h_j\right)}K(x_i,\bar{x}_i)\mathcal{A}(z,\bar{z},x,\bar{x})\ .$$

where $h=\sum_{i=1}^4 h_i$ and z is the cross-ratio.

Using the operator algebra, we decompose each four-point function as a sum over intermediate representations of the affine algebra.

The functions that appear in this decomposition are called conformal blocks.

We can choose to decompose the four-point functions in different ways and all of them must agree due to the associativity of the operator algebra.

Four-point function $\rightarrow$ monodromy invariant combination of the conformal blocks

$\langle \Phi^{+}_{p_1,\hat{\jmath}_1}\Phi^{+}_{p_2,\hat{\jmath}_2}\Phi^{+}_{p_3,\hat{\jmath}_3}\Phi^{-}_{p_4,\hat{\jmath}_4}\rangle$ with $p_1+p_2+p_3=p_4$.

These correlators are the simplest ones, they factorize on a finite number of blocks.

$$\mathcal{A}(z,x)=\sum_{n=0}^{|L|}\mathcal{C}_{++-}\mathcal{C}_{+-+}|\mathcal{F}_n(z,x)|^2\ ,\quad L=\sum_{i=1}^{4}\hat{\jmath}_i\leq 0\ .$$

States with $p=0$ or $p\geq 1$ can not appear in the intermediate channels. They can be easily obtained as limits of $SU(2)_k$ correlators.

$\langle \Phi^{+}_{p_1,\hat{\jmath}_1}\Phi^{-}_{p_2,\hat{\jmath}_2}\Phi^{+}_{p_3,\hat{\jmath}_3}\Phi^{-}_{p_4,\hat{\jmath}_4}\rangle$ with $p_1+p_3=p_2+p_4$.

These correlators are the most interesting. When $p_1>p_2$ they factorize on an infinite number of intermediate $V^{+}_{p_1-p_2,\hat{\jmath}_1+\hat{\jmath}_2-n}$ representations

$$\mathcal{A}(z,x)=\sum_{n=0}^{\infty}\mathcal{C}_{+--}\mathcal{C}_{+-+}|\mathcal{F}_n(z,x)|^2\ .$$

When $p_1=p_2$ they factorize on the continuum of the V^0_s representations

$$\mathcal{A}(z,x)=\int_0^{\infty}dss\ \mathcal{C}^2_{+-0}|\mathcal{F}_s(z,x)|^2\ .$$

Finally when $p_1+p_3\geq 1$ we can see explicitly that the correlator factorizes on spectral flowed representations and therefore that these representations are necessary for the closure of the operator algebra of the model.

$\langle \Phi^{+}_{p_1,\hat{\jmath}_1}\Phi^{+}_{p_2,\hat{\jmath}_2}\Phi^{-}_{p_3,\hat{\jmath}_3}\Phi^{0}_{s,\hat{\jmath}_4}\rangle$ with $p_1+p_2=p_3$.

$\langle \Phi^{+}_{p_1,\hat{\jmath}_1}\Phi^{-}_{p_2,\hat{\jmath}_2}\Phi^{0}_{s_3,\hat{\jmath}_3}\Phi^{0}_{s_4,\hat{\jmath}_4}\rangle$ with $p_1=p_2$.

$\langle\Phi^+_{p_1,\hat{j}_1}\Phi^+_{p_2,\hat{j}_2}\Phi^+_{p_3,\hat{j}_3}\Phi^-_{p_4,\hat{j}_4}\rangle$

The four-point function is

$$\mathcal{A}(z,\bar{z},x,\bar{x}) \sim \left(C_{12}|f(z,x)|^2 + C_{34}|g(z,x)|^2\right)^{|L|} ,$$

where $x = \frac{x_2-x_1}{x_3-x_1}$. The conformal blocks are

$$F_n(z,x) = f^n(z,x)(g(z,x))^{|L|-n} , \qquad n = 0, ..., |L| .$$

Here f and g are combination of hypergeometric functions, linear in x.

$\langle\Phi^+_{p_1,\hat{j}_1}\Phi^-_{p_2,\hat{j}_2}\Phi^+_{p_3,\hat{j}_3}\Phi^-_{p_4,\hat{j}_4}\rangle$

The four-point function is

$$\mathcal{A}(z,\bar{z},x,\bar{x}) \sim \frac{1}{S^{1+|L|}}\left|e^{xq(z)-xz(1-z)\partial \ln S}\right|^2 \left(\frac{u}{2}\right)^{-|L|} I_{|L|}(u) ,$$

where $x = (x_1-x_3)(x_2-x_4)$ and everything can be expressed in terms of hypergeometric functions. For instance $S = |f_1|^2 - r|f_2|^2$ and f_1, f_2 are two independent solutions of an hypergeometric equation.

The decomposition of the correlator in conformal blocks depends on the value of the p_i.

When $p_1 \neq p_2$ the conformal blocks are

$$F_n(z,x) \quad \sim \quad \frac{e^{xg_1}}{f_1^{1-L}} \quad L_n^{|L|}(x\gamma_\psi)\ \psi^n , \qquad n \in \mathbb{N} .$$

When $p_1 = p_2$ the conformal blocks have a similar structure

$$F_s(z,x) \quad \sim \quad \frac{e^{xg_1}}{c_1^{1-L}} \quad e^{\frac{s^2}{2}\rho}\ [xz(1-z)\partial\rho]^{\frac{L}{2}} J_{|L|}(v) .$$

It is interesting to study the factorization in the t channel.

When $p_1 + p_3 < 1$ the intermediate states are highest-weight representations.

When $p_1 + p_3 = 1$ the correlator factorizes on a continuum $\rightarrow$ $\Omega_1(\Phi^0_{s,\hat{\jmath}})$. The three-point coupling $\mathcal{C}_{p_1,p_3,1} \sim \mathcal{C}_{p_1,-p_1,0}$.

When $p_1 + p_3 > 1$ the correlator factorizes on $\Omega_1(\Phi^+_{p,\hat{\jmath}})$.

Fusion rules between spectral-flowed representations can be derived using

$$\Omega_{w_1}(\Phi_1) \otimes \Omega_{w_2}(\Phi_2) = \Omega_{w_1+w_2}(\Phi_1 \otimes \Phi_2) \ . \quad \text{Gaberdiel}$$

In our case this leads to

$$[\Phi^+_{p_1,\hat{\jmath}_1}] \otimes [\Phi^+_{p_3,\hat{\jmath}_3}] = \sum_{n=0}^{\infty} [\Omega_1(\Phi^+_{p_1+p_3-1,\hat{\jmath}_1+\hat{\jmath}_3-n})] \ .$$

From the factorization of the correlator one can see that

1) the intermediate states have the correct dimension for being identified with $\Omega_1(\Phi^+_{p,\hat{\jmath}})$

2) arbitrary negative powers of the cross-ratio can appear

3) $\mathcal{C}_{++-}(p_1, p_3, p_1 + p_3) \sim \mathcal{C}_{+--}(p_1, 1 - p_3, 1 - p_1 - p_3)$

The three-point coupling can be written as

$$\langle \Phi^+_{p_1,\hat{\jmath}_1} \Omega_1(\Phi^-_{1-p_3,\hat{\jmath}_3}) \Omega_{-1}(\Phi^-_{1-p_1-p_3,\hat{\jmath}}) \rangle \ .$$

This result can be verified also using the OPE with the degenerate representations $\Phi^{\pm}_{p=1}$.

Fusion and braiding

We can factorize the four-point functions around $z = 0$, 1 and ∞ $\rightarrow$ three sets of conformal blocks.

Linear transformations between the different basis: braiding and fusion matrices.

Previous discussions for non-compact models (Liouville, $SL_2(\mathbb{R})$) Teschner, Ponsot

The change of basis between blocks corresponding to $V^{\pm}$ states can be written as $(u = 1 - z)$

$$F_n(z, x) = \sum_{m=0}^{\infty} c_{nm}^L G_m(u, x) \ ,$$

where

$$c_{nm}^L = \frac{\Gamma(m + n + |L| + 1)}{m! \Gamma(m + |L| + 1)} \frac{1}{r_1^{n+m+|L|+1}} \left(\frac{r_2}{p_1 - p_2}\right)^n [(p_3 - p_2)s_1]^m F(-m, -n, -m-n-|L|; \theta)$$

and

$$\theta = \frac{\sin(\pi p_4)\sin(\pi p_2)}{\sin(\pi p_1)\sin(\pi p_3)} \ .$$

Similarly we can change basis from the blocks corresponding to the V^0 representations to the blocks corresponding to the $V^{\pm}$ representations

$$F_n(u, x) = \int_0^{\infty} ds \ c_n^L(s) F_s(z, x) \ .$$

Relations with the quantum Heisenberg group.

Penrose limit

The Nappi-Witten gravitational wave can be obtained as the Penrose limit of an $U(1) \times SU(2)_k$ WZW model. Starting from

$$ds^2 = k[d\gamma^2 + \sin^2\gamma d\beta^2 + \cos^2\gamma d\alpha^2 - dt^2] \ ,$$

we set

$$\frac{\mu u}{2} = t \quad , \quad \frac{v}{\mu} = \frac{k}{2}(t-\alpha) \quad , \quad r = \sqrt{k}\gamma \ ,$$

and take the limit $k \to \infty$. From the point of view of the CFT this limit is a contraction of the original current algebra. The new currents are related to the $U(1) \times SU(2)_k$ currents by

$$K = \frac{2i}{k}J^0 \ , \quad J = i(J^0 - J^3) \ , \quad P^{\pm} = \sqrt{\frac{2}{k}}J^{\pm} \ .$$

In the limit $T_{U(1)\times SU(2)_k} \to T_{H_4}$.

Two different classes of states are relevant in the limit

- States around the top or the bottom of an $SU(2)$ representation:

  $$l = \frac{kp}{2} \pm \hat{\jmath} \ , \qquad m = \pm\frac{kp}{2} \mp \hat{\jmath} \mp n \ , \qquad q = \pm\frac{k}{2}p \ ,$$

  with $0 < p < 1$. They form in the limit the $V^{\pm}_{p,\hat{\jmath}}$ representations. Moreover

  $$h = -\frac{q^2}{k} + \frac{l(l+1)}{k+2} \to \mp p\hat{\jmath} + \frac{p}{2}(1-p) \ .$$

- States in the middle of an $SU(2)$ representation:

  $$q, m \sim O(1) \ , \qquad l = \sqrt{\frac{k}{2}}s \ .$$

  They form in the limit the $V^0_{s,\hat{\jmath}}$ representations and $h \to \frac{s^2}{2}$.

We can go a bit further and derive the H_4 three-point couplings from the $U(1) \times SU(2)_k$ three-point couplings, first computed by Fateev and Zamolodchikov.

$SU(2)$ primaries

$$\Phi^l(z,x) = \sum_{m,\bar{m}=-l}^{l} c^l_{m,\bar{m}} \; x^{l+m} \bar{x}^{l+\bar{m}} \; \Phi^l_{m,\bar{m}} \; .$$

They are related to the H_4 primaries by

$$\Phi^-_{p,\hat{\jmath}}(z,x) = \lim_{k\to\infty} \Phi^l \left(z, \sqrt{\frac{p}{2l}} x \right) \; ,$$

and by similar expressions for the other operators. Scaling the fields and the quantum numbers in the three-point couplings

$$\langle \prod_{i=1}^{3} \Phi^{l_i}(z_i,x_i) \rangle = C(l_1,l_2,l_3) \prod_{i<j}^{3} \frac{(x_{ij}\bar{x}_{ij})^{l_{ij}}}{(z_{ij}\bar{z}_{ij})^{h_{ij}}} \; ,$$

we obtain the H_4 three-point couplings up to factors $\sim \sqrt{k}$, due to the continuum δ-function imposing momentum conservation.

Limit of the $SU(2)$ KZ equation $\to$ H_4 KZ equation.

If we scale the quantum numbers in an $SU(2)$ four-point correlator in such a way as to obtain an H_4 $\langle + + + - \rangle$ correlator, the number of conformal blocks remains finite in the limit and equal to $|L| + 1$, as expected.

In the Dotsenko-Fateev integral representations of the correlator the various hypergeometric integrals become independent since $|D(t)|^{\frac{4}{k+2}} \to 1$ where $D(t) = \prod_{i<j}(t_i - t_j)$.

Spectral flowed states are generated by acting on the highest-weight states of an affine representation with an affine Weyl translation and dressing the state with a suitable $U(1)$ charge.

Free field resolution

The H_4 current algebra has a free field realization introduced by Kiritsis and Kounnas.

The currents can be written as

$$J = \partial v \ , \qquad K = \partial u \ , \qquad P^{+} = i e^{-iu} \partial y \ , \qquad P^{-} = i e^{iu} \partial \tilde{y} \ ,$$

where u, v, y and $\tilde{y}$ are free fields.

Using this representation we can establish a relation between H_4 primary vertex operators and orbifold twist fields $H_p^{\mp}(z)$.

In terms of free fields, the ground state of a $V^{\pm}$ representation is given by

$$R^{\pm}_{p,\hat{\jmath};0} = e^{i\hat{\jmath}u \pm ipv} H_p^{\mp} \ .$$

Acting with $P_0^{\mp}$ we obtain the other states $R^{\pm}_{p,\hat{\jmath};n}$ and from the OPEs

$$\partial y(z) H_p^{-}(w) \sim (z-w)^{-p} H_p^{-(1)}(w) \ , \qquad \partial y(z) H_p^{+}(w) \sim (z-w)^{-1+p} H_p^{+(1)}(w) \ ,$$

we can see that the higher components in an H_4 representation involve excited twist fields $H_p^{\pm(n)}$.

Therefore we can evaluate the H_4 correlators using twist field correlators, following Dixon, Friedan, Martinec and Shenker.

H_4 four-point correlators $\rightarrow$ generating functions for correlators between arbitrary twist fields.

Semiclassical wave functions $\rightarrow$ representations of the twist fields in terms of the wave functions of a two-dimensional harmonic oscillator.

Spectral flow by w units in this setting corresponds to multiplication by e^{iwv}.

String amplitudes

Consider the Nappi-Witten gravitational wave as part of a critical string theory background $\mathcal{C} = \mathcal{C}_{H_4} \times \mathcal{C}_{int} \times \mathcal{C}_{gh}$. Simplest choice $\mathcal{C}_{int} = \mathbb{R}^{22}$. Vertex operators: $e^{i\vec{p}\vec{X}}$, $h = \frac{\vec{p}^2}{2}$.

Four-point string amplitudes $\sim$ integrated CFT four-point correlators

$$\mathcal{A}_{string} = \int d^2z|z|^{2\sigma_{12}-\frac{4}{3}}|1-z|^{2\sigma_{14}-\frac{4}{3}} K(x_i, \bar{x}_i)\mathcal{A}(z, \bar{z}; x, \bar{x}) \ .$$

In flat space

$$\mathcal{A}_{string} \sim \int d^2z|z|^{\frac{\alpha'}{2}(p_1+p_2)^2-4}|1-z|^{\frac{\alpha'}{2}(p_2+p_3)^2-4} \ ,$$

and the amplitude has a pole whenever

$$\alpha'(p_1+p_2)^2 = 4(1-N) \ .$$

In our case for given external quantum numbers there are several intermediate states, in various representations of the current algebra. For instance, in the case of a $\langle + - + - \rangle$ correlator

$$h_{int}(p_1 - p_2, \hat{\jmath}_1 + \hat{\jmath}_2 - n) = 1 - N \ , \qquad n \in \mathbb{N} \ .$$

When the intermediate state belongs to a $\Omega_w(V^0_{s,\hat{\jmath}})$ representation the amplitude develops a logarithmic branch cut. The integral on the cross-ratio can be approximated by

$$\mathcal{A}_{string} \sim \int_0^{\sigma} dr \frac{1}{r^{3-2h_{int}} \ln r} \ ,$$

and in the limit $h_{int} \to 1$ the amplitude behaves as

$$\mathcal{A}_{string} \sim \ln(h_{int} - 1) \ .$$

When $w = 0$ only the tachyon can be on-shell but for $w \neq 0$ (long strings in the intermediate channel) a branch cut appears for each level.

Flat space limit

The value of the parameter μ that appear in the metric

$$ds^2 = -2dudv - \frac{\mu^2 r^2}{4} du^2 + dx_T^2 \ ,$$

can be changed by performing a boost $u \to \lambda u$, $v \to \frac{v}{\lambda}$. There are two interesting limits to consider, $\mu \to 0$ and $\mu \to \infty$.

In both cases one recovers string theory in flat space, even though the states that survive in the two limits are very different.

- Short strings $\Omega_{\pm w}(\Phi^{\pm}_{p,\hat{\jmath}})$

$$h = \mp \left(p + \frac{w}{\mu} \right) \hat{\jmath} + \frac{\mu p}{2} (1 - \mu p) \ , \quad p \in (0, 1/\mu) \ , \ w \in \mathbb{N} \ .$$

- Long strings $\Omega_w(\Phi^0_{s,\hat{\jmath}})$

$$h = -\frac{w \hat{\jmath}}{\mu} + \frac{s^2}{2} \ , \quad \hat{\jmath} \in [-\mu/2, \mu/2) \ , \ w \in \mathbb{Z} \ .$$

The limit $\mu \to 0$ can be considered as a further contraction of the H_4 algebra. The highest weight representations reconstruct the flat space spectrum: the potential flattens and the confined states describe larger and larger orbits until they become free.

In the limit $\mu \to \infty$ the long string states, which do not feel the potential, remain free, $p = w/\mu$ becomes a continuous variable and $\hat{\jmath} \in \mathbb{R}$. All operators with $\mu p \notin \mathbb{Z}$ decouple: they are so strongly trapped by the potential that disappear from the spectrum.

Compare with the small and large radius limit of a compactified boson.