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Quantised String theory = M-Theory

Low energy limit: D=11 Supergravity

(graviton g, gravitino ψ, 3-Form A)

$$L = \sqrt{-g}R - \frac{1}{2}F \wedge *F - \frac{1}{6}A \wedge F \wedge F$$

$F = dA$

$R_{AB} = \frac{1}{12}\left(F_A^{CDE}F_{BCDE} - \frac{1}{12}g_{AB}F^{CDEF}F_{CDEF}\right)$

$d * F + \frac{1}{2}F \wedge F = 0$

Groundstate: $M^{11} \to M^4$

Supersymmetry

$\Longrightarrow$ compactification on manifold of **special holonomy**

Riemannian Manifold (M^n, g), simply connected,

Levi-Civita connection: $\nabla^{\mathrm{LC}} g = 0$

parallel transport around γ_p

holonomy group: $H = \{ \ P_\gamma : T_pM \to T_pM \ \} \ \subseteq \mathrm{SO(n)}$

subgroup of GL(n) preserving $g|_p$

Lie algebra: $\mathfrak{h} \ \subseteq \ \mathfrak{so}(n) \ \cong \ \wedge^2 T^*M$

Curvature

$$R \ \in \ \wedge^2 T^*M \ \otimes \ \wedge^2 T^*M \quad \cong \quad \mathfrak{so}(n) \ \otimes \ \wedge^2 T^*M$$

$$\mathfrak{h} \ \otimes \ \wedge^2 T^*M$$

restricted Hol $\quad \Longrightarrow \quad$ linear conditions on R

Berger's Holonomy Classification:

(M, g), simply connected and irreducible, is either

isometric to a symmetric space K/Hol or

Holonomy group H is one of:

SO(n)	
U(n)	Kähler
SU(n)	Calabi – Yau
Sp(n)	hyperKähler
$\mathrm{Sp(n)} \cdot \mathrm{Sp(1)}$	quaternionic Kähler
Spin(7)	Spin(7) – Manifolds
G_2	G_2–Manifolds

Special geometric structure Ω on M:

$$\nabla^{\mathrm{LC}}\Omega = 0$$

Hol $= \mathrm{Sp(n)},\ \mathrm{Sp(n)}\cdot\mathrm{Sp(1)},\ \mathrm{Spin(7)},\ \mathrm{G}_2 \quad \leftrightarrow \quad \Omega$ a 4-form

Kähler: Ω the Kähler 2-form

Manifolds are Einstein: $\mathrm{Ric} = \lambda g$

D-branes $\Longleftrightarrow$ Gauge fields

Yang-Mills curvature

$$F \in \wedge^2 T^*M \otimes \mathfrak{g} \quad , \quad \wedge^2 T^*M \cong \mathfrak{so}(TM)$$

$$F \in \rho(\mathfrak{h}) \otimes \mathfrak{g} \qquad\qquad \mathfrak{h}$$

$M = M^4$

$$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$$

$$\longleftrightarrow \qquad F = F_+ + F_-$$

$$\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = F_{\mu\nu} \quad \Longleftrightarrow \quad F = F[(\mathbf{3},1)]$$

$M = M^n$ with special Holonomy $H \subset \mathrm{SO(n)}$

$\exists$ covariantly constant 4-form T: $\nabla^{\mathrm{LC}} T = 0$

$$\tfrac{1}{2} T_{MNPQ} F^{PQ} = \lambda F_{MN} \quad \Longrightarrow \quad D^M F_{MN} = 0$$

[also on more general manifolds if $(\partial^M T_{MNPQ}) F^{PQ} = 0$]

$$\wedge^2 T^* M \cong \mathfrak{so}(n) \; = \; \bigoplus_i \mathfrak{a}(\lambda_i)$$

$$\longleftrightarrow \qquad F \; = \; \sum_i F[\mathfrak{a}(\lambda_i)]$$

$$\tfrac{1}{2} T_{MNPQ} F^{PQ} = \lambda F_{MN} \quad \Longleftrightarrow \quad F = F[\mathfrak{a}(\lambda)]$$

Example: H = Spin(7) $\subset$ SO(8)

$$\wedge^2 T^* M \;=\; \mathfrak{spin}_7 \;\oplus\; \mathbb{R}^7$$

$$\mathfrak{a}(\lambda{=}1) \qquad \mathfrak{a}(\lambda{=}-3)$$

The $\lambda{=}1$ equation

$$\tfrac{1}{2} T_{MNPQ} F^{PQ} = F_{MN} \quad \Longleftrightarrow \quad F[\mathfrak{a}(\lambda{=}-3)] = 0$$

$T_{8abc} = \psi_{abc}$

$T_{abcd} = \frac{1}{6} \epsilon_{abcdefg} \psi_{efg}$

$[e_a, e_b] = \psi_{abc} e_c$

$\longrightarrow \quad F_{8a} - \psi_{abc} F_{bc} = 0$

[Corrigan–Fairlie–Devchand–Nuyts (1983)]

[Capria–Salamon (1988), Baulieu–Kanno–Singer (1998)
Donaldson–Thomas (1998), Tian (2000),....]

for pseudo-Riemannian manifolds with possible torsion:

[Alekseevsky–Cortés–Devchand (math.DG/0209124)]

Superspace generalisation:

[Devchand–Nuyts (hep-th/0109072)]

conformal space $\quad y^A \quad , \quad A = 1, \ldots, 10$

$y^A y_A = 0, \qquad$ scale invariance

3-form in $d = 10 \quad F_{ABC} := y_A F_{BC} + y_B F_{CA} + y_C F_{AB}$

selfduality: $\quad \frac{1}{6}\Omega_{ABCDEF}F^{DEF} = \lambda F_{ABC}$

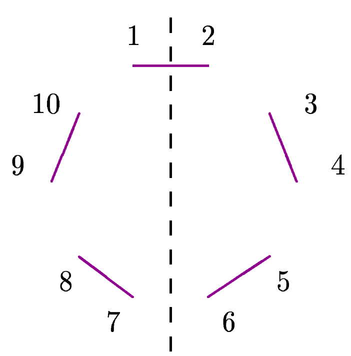
trivial choice

$$\Omega^0_{MNPQ\,9\,0} = T_{MNPQ}$$

non-zero components of Spin(7)-invariant 4-form

$$\begin{aligned} & T_{1234} = T_{1256} = T_{1278} = T_{1357} = -T_{1368} \\ = & -T_{1458} = -T_{1467} = -T_{2358} = -T_{2367} = -T_{2457} \\ = & T_{2468} = T_{3456} = T_{3478} = T_{5678} = 1 \end{aligned}$$

$$\sigma : (1, 2, 3, 4 \ldots, 9\,, 10\,) \longrightarrow (3, 4, \ldots, 9\,, 10\,, 1, 2)$$



$$\Omega^1_{\sigma(m)\,\sigma(n)\,\sigma(p)\,\sigma(q)\,1\,2} = T_{mnpq}$$

satisfies compatibility condition

$$\Omega^1_{mnpq\,9\,0} = \Omega^0_{mnpq\,1\,2}$$

Similarly

$$\Omega^2_{\sigma^2(m)\,\sigma^2(n)\,\sigma^2(p)\,\sigma^2(q)\,3\,4} = T_{mnpq}$$

also compatible with Ω^0 and Ω^1:

$$\begin{aligned} \Omega^0_{mnpq\,3\,4} &= \Omega^2_{mnpq\,1\,2} \\ \Omega^1_{mnpq\,3\,4} &= \Omega^2_{mnpq\,9\,0} \end{aligned}$$

$\exists$ five compatible 6-forms:

$$\Omega^N_{\sigma^N(m)\,\sigma^N(n)\,\sigma^N(p)\,\sigma^N(q)\,\sigma^N(9)\,\sigma^N(0)} = T_{mnpq} \quad , \quad N = 0, \ldots, 4$$

SU(4)×U(1)-invariant 6-form:

$$\Omega_{\,\sigma^N(m)\,\sigma^N(n)\,\sigma^N(p)\,\sigma^N(q)\,\sigma^N(9)\,\sigma^N(0)} = T_{mnpq} \quad , \quad N = 0, \ldots, 4$$

$\exists$ 4 independent SU(4)×U(1)-invariant 6-forms in $D = 10$

[Devchand–Nuyts–Weingart (work in progress)]