Integrable massless sigma models with singular metric and RG flows.

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2D non-linear sigma models

$$A[G] = \frac{1}{2}\int G_{ij}(X)\, \partial_\mu X^i \partial_\mu X^j \, d^2x$$

$x_{1,2} \in R_2$; X^i, $i=1,\dots d$; $X^i \in M$

RG flow in 2D SM (Ricci flow)

$$\frac{dG_{ij}}{dt} = -\frac{1}{2\pi} R_{ij} + \cdots .$$

t is the log. scale, R_{ij} : Ricci tensor

$$t \to \begin{cases} -\infty & \text{U.V. asymptotics} \\ +\infty & \text{I.R. Long dist. behav.} \end{cases}$$

$d=2$ $\quad R_{ij} = \frac{\mathcal{R}}{2} G_{ij}$; $\quad G_{ij} = e^{\phi}\delta_{ij}$

$$\frac{\partial}{\partial t} e^{\phi} = \frac{1}{4\pi}\, \partial^2_{x_i} \phi$$

$$V(t) = \int \sqrt{G}\, d^2x = 2(t_0 - t)$$

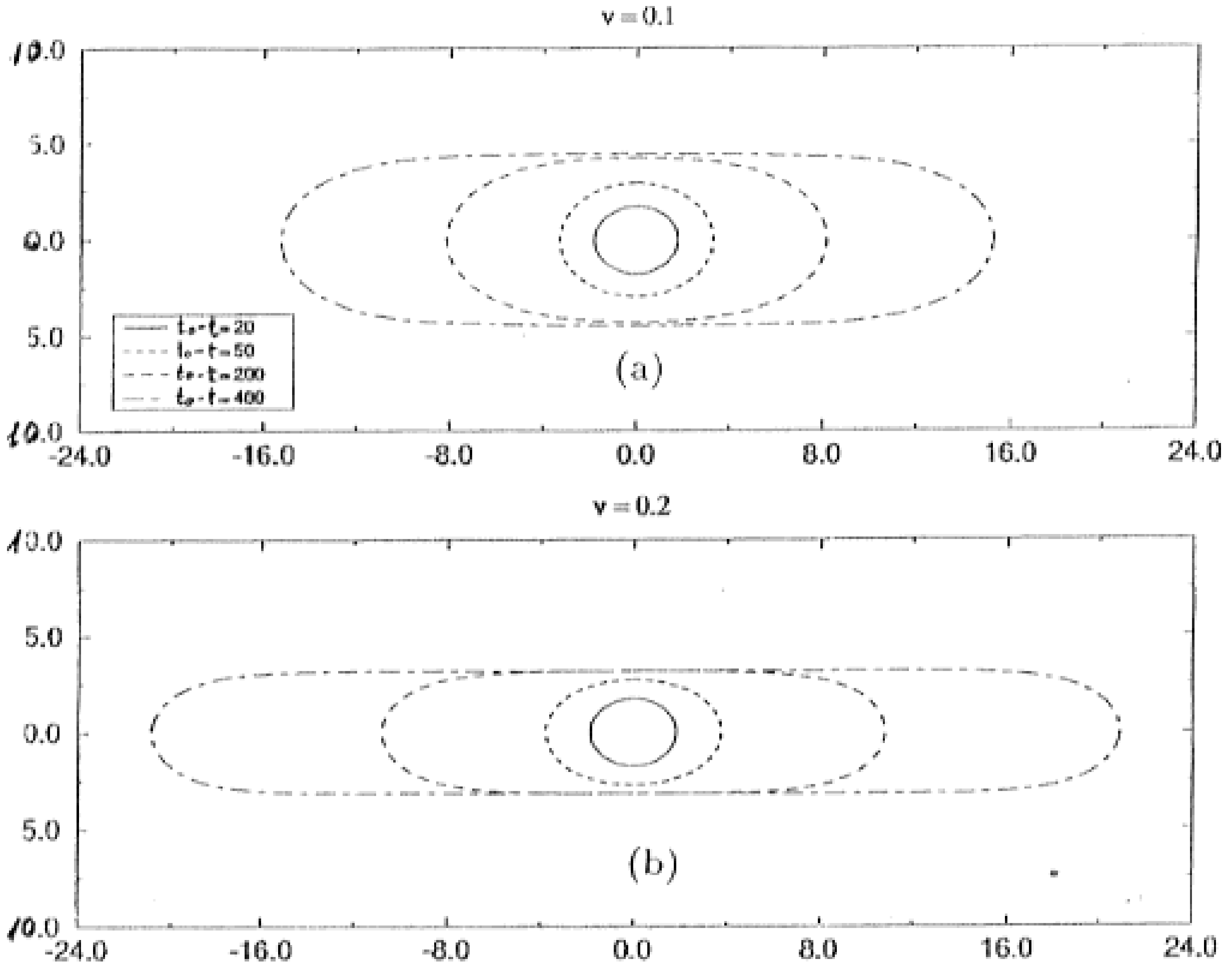

v = 0.1
10.0
5.0
0.0
5.0
10.0
t₀-t=20
t₀-t=50
t₀-t=200
t₀-t=400
(a)
-24.0
-16.0
-8.0
0.0
8.0
16.0
24.0
v = 0.2
10.0
5.0
0.0
5.0
10.0
(b)
-24.0
-16.0
-8.0
0.0
8.0
16.0
24.0

Fig.5.

Sausage model solution

SM with a metric

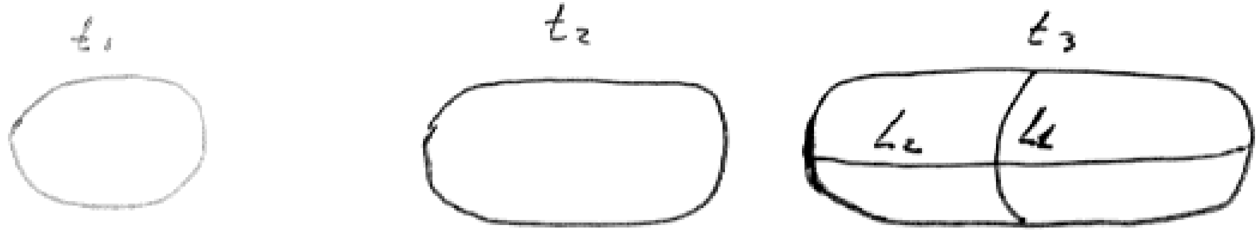
$$A_s = \int \frac{(\partial_\mu X)^2 + (\partial_\mu Y)^2}{a(t) + b(t)\cosh 2X}\, d^2x$$

$$\frac{da}{dt} = \frac{1}{2\pi} b^2 \;; \quad \frac{db}{dt} = \frac{1}{2\pi} ab$$

$\nu^2 = a^2 - b^2$ parametrizes the theory

$$a(t) = \nu \coth u \qquad\qquad u = \frac{\nu(t_0 - t)}{4\pi}$$
$$b(t) = \nu / \sinh u$$

$\nu \to 0 \qquad O(3)$ SM.



$$L_1 \sim 2\pi\sqrt{\frac{2}{\nu}} \;, \qquad L_2 \sim \sqrt{2\nu}\,(t_0 - t)$$

UV limit: two Wittens cigars

$$\frac{1}{\nu}\left(dr^2 + \tanh^2 r\,(d\theta)^2\right) \qquad SL(2,R)/U(1) \text{ CFT}$$

RG analysis, Fact. Scat. Th., TBA

Singular solution

$Y \to Y + i\pi/4$; $u \to u + i\pi/4$

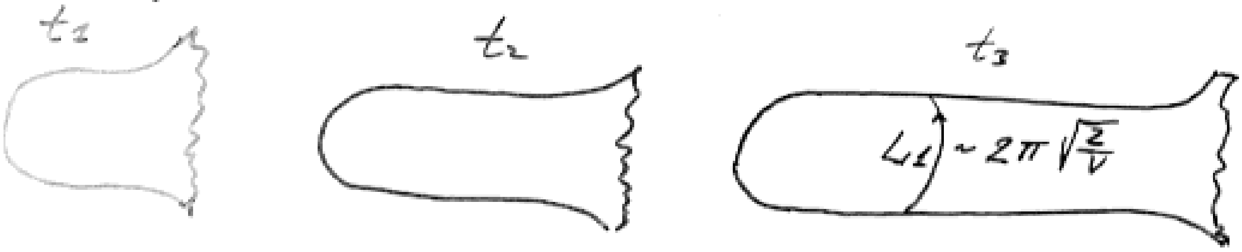
$$A_{SSM} = \int \frac{(\partial_\mu X)^2 + (\partial_\mu Y)^2}{C(t) + d(t)\sinh 2Y}\, d^2x$$

$C(t) = \nu \tanh u$ $\qquad \nu^2 = c^2 + d^2$

$d(t) = \nu/\cosh u$ $\qquad u = \nu(t_0 - t)/4\pi$

singularity at $Y = -u$; $Y > -u$

$\Delta Y \sim \nu \ll 1$ curvat. is not small but all geodesic distances are finite



UV limit

$\frac{1}{\nu}\left(dr^2 + \tanh^2 r\,(d\theta)^2\right)$ $\qquad \frac{1}{\nu}\left(dr^2 + \coth^2 r\,(d\tilde\theta)^2\right)$

$\theta \to \theta + 2\pi$ $\qquad \tilde\theta \to \theta + \nu/2$

Two dual descriptions of $SL(2,R)/U(1)$ CFT

$2\pi/(\nu/2) = \frac{4\pi}{\nu} = N$ $\qquad \nu = 4\pi/N$

In the IR limit: $t \to \infty$

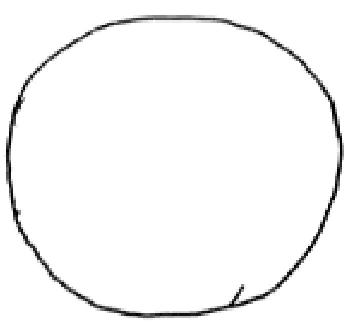
$$u = \frac{V}{4\pi}(t_0 - t) \to -\infty$$

$$A \to \frac{1}{V}\int \frac{(\partial_\mu Y)^2 + (\partial_\mu X)^2}{\exp(2Y+2u)-1}\, d^2x$$

For $V = \frac{4\pi}{N}$ the sigma mod. with this metric can be derived from $SU(2)$ level N WZW model by gauging $U(1)$ symmetry. Resulting SM describes $SU(2)_N/U(1)$ coset model $\to$ Z_N-paraferm. CFT

$$c = 2 - \frac{6}{N+2}$$

$$\Delta_{j,m} = \frac{j(j+1)}{N+2} - \frac{m^2}{4N}\ ; \quad j = \frac{|m|}{2}, \frac{|m|}{2}+1, \leq N/2$$



R $\qquad (t - t_0) = \log R\Lambda_0$

$E_i(R)$ – finite size spectrum

$$E_0(R) = -\frac{\pi c(R)}{6R} \qquad \text{TBA}$$

$E_i(R)$ – excited states

One loop approximation

$$C(R) = 2 - e_0(R) \qquad (d - e_0(R))$$

$$E_i(R) = E_0(R) + \frac{\pi}{6R}(e_i - e_0)$$

$$\hat{h}\,\Psi^{(i)} = e_i(R)\,\Psi^{(i)}$$

$$\hat{h} = -\frac{1}{2}\nabla_t^2 + \frac{1}{8}\mathcal{R}_t \qquad t - t_0 = \log R\Lambda_0$$

$$(\Psi_1, \Psi_2) = \int \Psi_1^* \Psi_2 \sqrt{G}\, d^d X$$

In the CFT limit

$$(e_i - e_0)/24 = \Delta_i \qquad \text{spectr. of an. dim.}$$

Operator $\hat{h}/\nu$ depends only on

$$u = -\nu/4\pi \log R\Lambda_0 = -\frac{1}{N}\log R\Lambda_0$$

$$e_i(R) = \nu/4\pi\; \mathcal{X}_i(u) = \frac{1}{N}\mathcal{X}_i(u)$$

$$\nabla_t^2 = \exp(\Phi(y,u))(\partial_x^2 + \partial_y^2)$$

$$\Psi^{(i)} = \Psi_m^{(i)}(y)\, e^{imx} \qquad m \in \mathbb{Z}$$

$$\left[-\partial_y^2 + m^2 + \frac{\sinh 2y \sinh 2u}{(\sinh 2y + \sinh 2u)^2}\right] \Psi_m^{(i)} =$$

$$= \frac{1}{6} \frac{\varkappa_m^{(i)} \cosh 2u}{\sinh 2y + \sinh 2u} \Psi_m^{(i)}$$

$\Psi_m = 0 : y < -u$; $\Psi_m \sim \sqrt{y+u}$, $y \to -u$

Elliptic substitution:

$$k^2 = \frac{1}{1+e^{-4u}} ; \quad k^2 \begin{cases} \to 0, \text{ IR}, & u \to -\infty \\ \to 1, \text{ UV}, & u \to \infty \end{cases}$$

$$\exp(y-u) = \frac{dn(z|k^2)}{k\, sn(z|k^2)} ; \quad \underset{y=\infty}{0} \leq z \leq \underset{y=-u}{\mathbb{K}}$$

$\psi(z) \sim \sqrt{-z+\mathbb{K}}$

$$\left(-\frac{d^2}{dz^2} - \frac{dn^2(2z)}{sn^2(2z)} + \frac{m^2 cn^2(z)}{sn^2(z) dn^2(z)}\right)\psi_m^{(i)} = \frac{\varkappa_m^{(i)}}{6} \psi_m^{(i)}$$

IR, $k^2 \to 0$, $\mathbb{K} \to \pi/2$. Trigon. lim.

$$\psi_{m,j} = \sqrt{\sin 2z}\, \sin^m(z)\, P_n^{(0,m)}(\cos 2z) ; \quad j = \frac{|m|}{2} + n$$

$$\varkappa_0 = N e_0 = N(2-c) = 6 - k^4 - \frac{k^6}{2} - \dots$$

$$N(e_{j,m} - e_0) = 24\left(j(j+1) - \frac{m^2}{4}\right) - \frac{12\left[j(j+1) - \frac{m^2}{4}\right]^2}{j(j+1)} k^2 + \dots$$

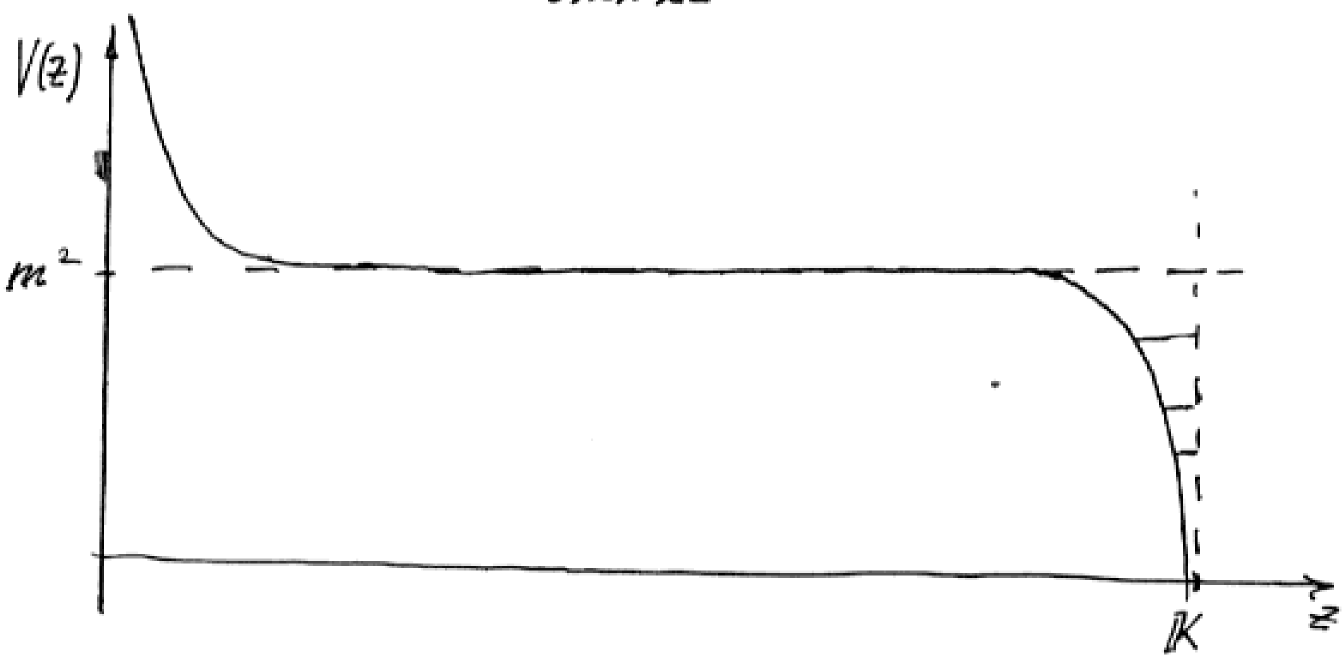
Spectr. dim. of Z_N paraf. CFT

UV asymptot. $u \to \infty$; $k^2 \to 1$

$$\mathbb{K}(k^2) \simeq u + 2\log 2 \to \infty$$

$$\left(-\frac{d^2}{dz^2} + V(z)\right)\psi_{m,j} = \frac{\varkappa_{m,j}}{6}\psi_{m,j}$$

$$V(z) = \begin{cases} -\dfrac{1}{\sinh^2 2z} + m^2\coth^2 z\,, & 0 < z \ll \mathbb{K} \\[2ex] -\dfrac{1}{\sinh^2 2z} + m^2\tanh^2 z\,; & 0 < \mathbb{K} - z \ll \mathbb{K} \end{cases}$$



Discrete spectrum (bound states)

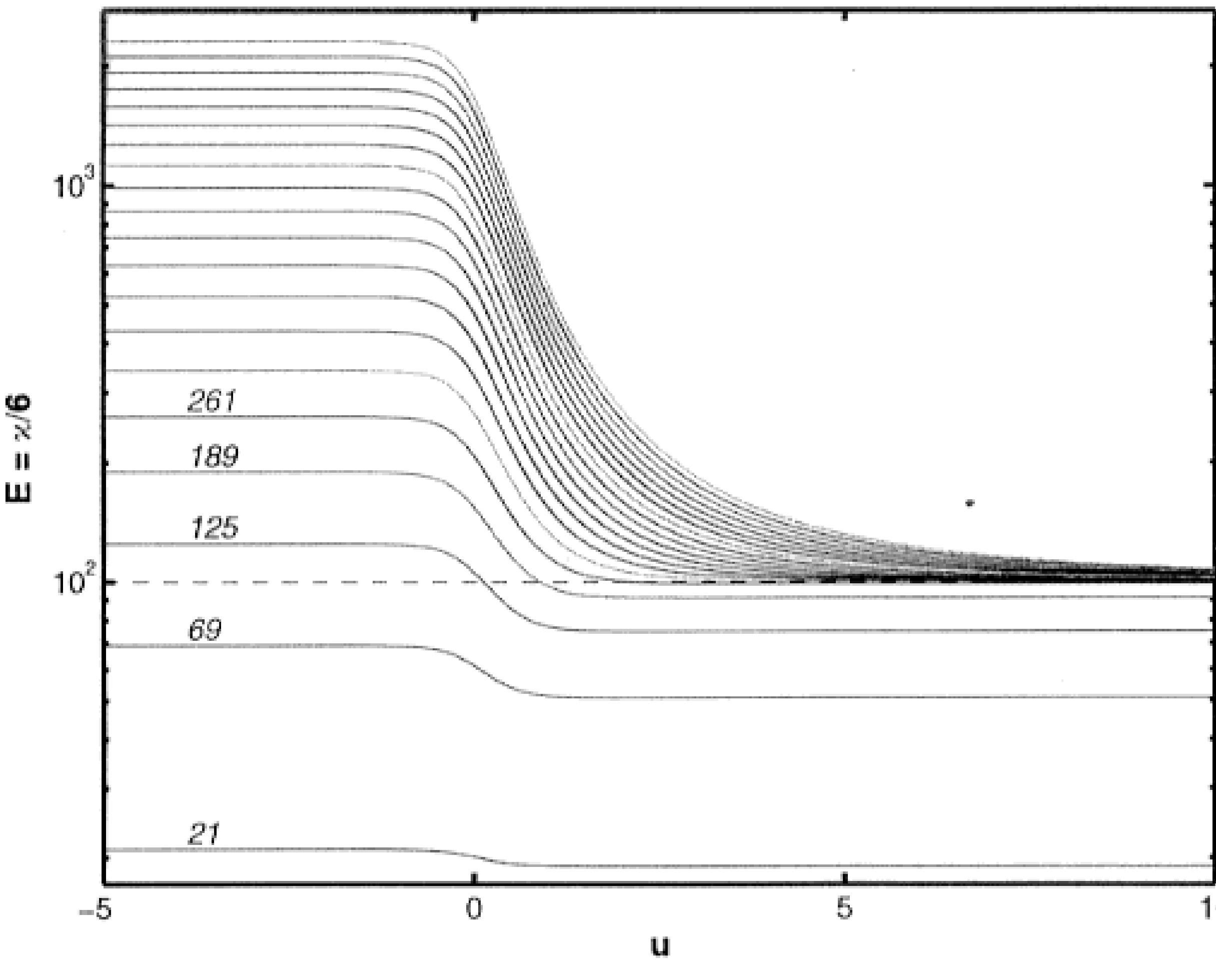
$$N e_{j,m} = 6\{m^2 - (2j-2m+1)^2\} \qquad \frac{|m|}{2} \le j < m-1$$

$$N e_{j,m} = 6m^2 + \frac{3\pi^2}{2}\frac{(j-m+1)}{(u+v_m)^2} + O(1/u^5) \qquad j \ge m-1$$

$$N e_0 = \frac{3\pi^2}{2}\frac{1}{(u+2\log 2)^2} + O(1/u^5)$$

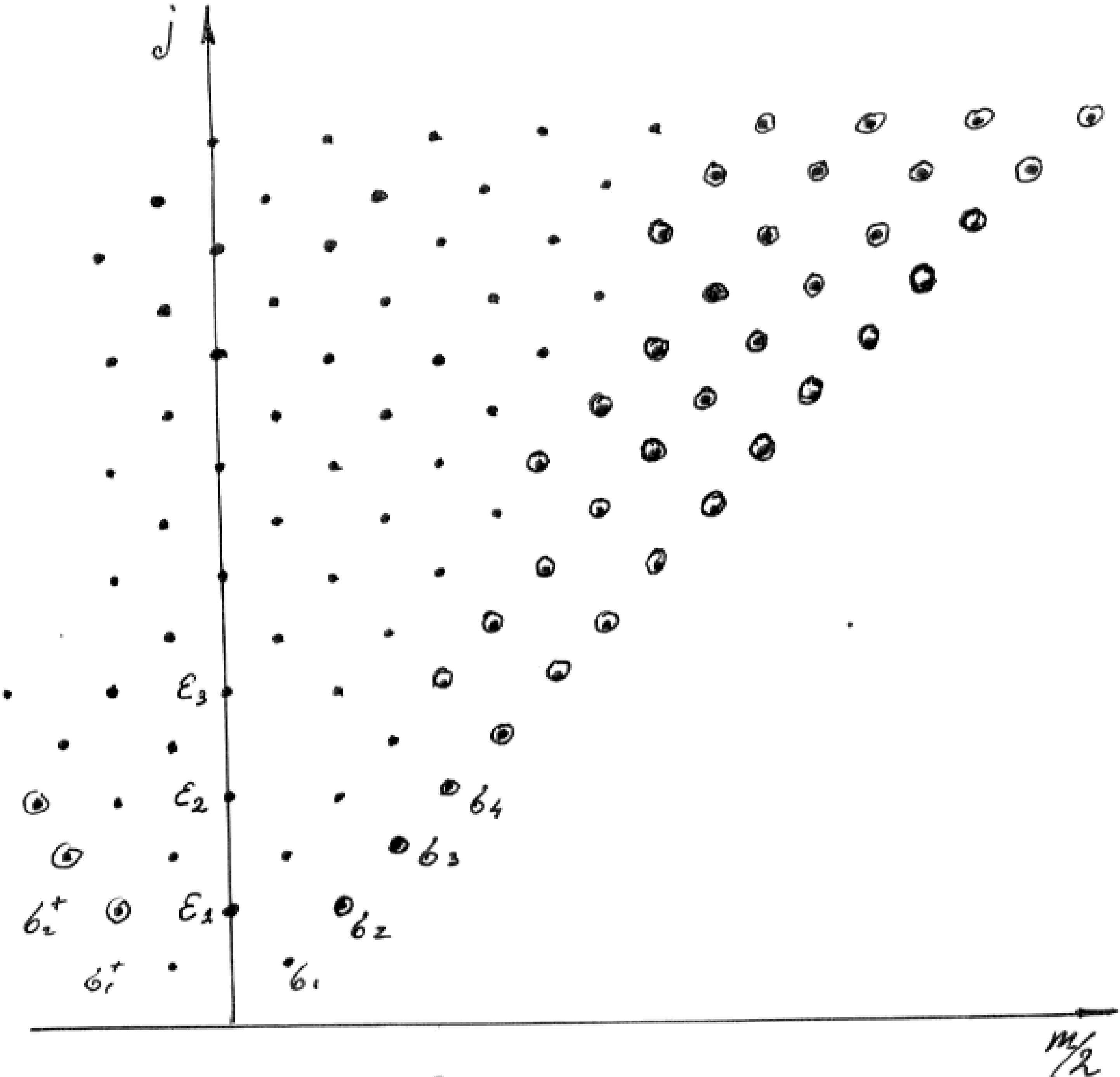
$$u = 1/N \log(1/R\Lambda_0)$$

RG flow of levels $m=10$



Parafermionic CFT states

$$\Delta_{j,m} = \frac{j(j+1)}{N+2} - \frac{m^2}{4N}$$



States ⊙ flow in UV to discr. levels

$$N\, e_{j,m/6} \to m^2 - (2j - 2m + 1)^2$$

$$\frac{|m|}{2} \le j < |m| - 1$$

Z_N - parafermionic CFT

$\{\Psi_1(z), \Psi_2(z) \dots \Psi_N(z)\}$ paraf. curr. alg.

$\{W^{(s)}(z)\}$ $s = 2, \dots, N$; $W^{(2)} = T(z)$

W_N or $W(A_{N-1})$ algebra is a symm.

The fields $\chi_{j,m}$ $\quad j = \frac{|m|}{2}, \frac{|m|}{2}+1, \dots \leq N/2$

$$\Delta_{j,m} = \frac{j(j+1)}{N+2} - \frac{m^2}{4N}$$

σ_j : order parameters $\chi_{j,j}$, $j = 1/2, 1, \dots \leq N/2$

ϵ_j : thermal operators $\chi_{j,0}$, $j = 1, 2, \dots \leq N/2$

Q. G. symmetry $SU(2,q)$ $\quad q = \exp\left(\frac{2\pi i}{N+2}\right)$

Integrable perturbation (relevant)

ϵ_1 , ϵ_2 ; $\quad (\Psi_1 \bar{\Psi}_1 + \Psi_1^+ \bar{\Psi}_1^+)$

IR integr. perturb. $\quad \Delta_p > 1$

$\Delta_p = 1 + \frac{2}{N} + O(1/N^2)$; $\quad W_{-1}^{(3)} \epsilon_1$: $\Delta = 1 + \frac{2}{N+2}$

$$\Phi_p \sim W_{-1}^{(3)} \bar{W}_{-1}^{(3)} \epsilon_1 \,, \quad \langle \Phi_p(x), \Phi_p(0) \rangle = \frac{1}{|X|^{4\Delta}}$$

$$A^{(N)} = A^{(N)}_{CFT} + \lambda \int \Phi_p(x)\, d^2x$$

$$\epsilon = p = \frac{M}{2} e^{\beta} \qquad \text{Right mov. part.}$$

$$\epsilon = -p = \frac{M}{2} e^{-\beta} \qquad \text{Left mov. part}$$

$$(\pi\lambda)^2 = \frac{(N-2)^2}{(N+2)^4} \frac{\Gamma(1/N+2)\,\Gamma(3/N+2)}{\Gamma\left(\frac{N+1}{N+2}\right)\Gamma\left(\frac{N-1}{N+2}\right)} \left(\frac{4N+2}{M}\right)^{8/N+2}$$

Expansion parameter in CPT

$$\lambda R^{-4/N+2} \sim (MR)^{-4/N+2} \rightarrow e^{4u} \text{ in one loop}$$

$$\delta e^{(1)}_{j,m} = \lambda\left(\frac{R}{2\pi}\right)^{-4/N+2} \langle \chi_{j,m} | \phi_p | \chi_{j,m} \rangle$$

Can be calculated exactly

$$\frac{(e_{j,m} - e_0)}{24} = \Delta_{j,m} + b_1(j,m,N)(MR)^{-4/N+2} + \dots$$

$$C(R) = 2 - \frac{6}{N+2} + d_2(N)(MR)^{-8/N+2} + d_3(MR)^{-8/N+2} + \dots$$

$$b_1(j,m,N) = -\frac{12\,(j(j+1) - m^2/4)^2}{N\, j(j+1)} (1 + O(1/N)) =$$

$$= b_1^{(\text{one loop})}(1 + O(1/N))$$

$$d_2(N) = \frac{1}{N}(1 + O(1/N)) = d_2^{(\text{one loop})}(1 + O(1/N))$$

$$d_3(N) = -\frac{3}{2N}(1 + O(1/N)) = d_3^{(\text{one loop})}(1 + O(1/N))$$

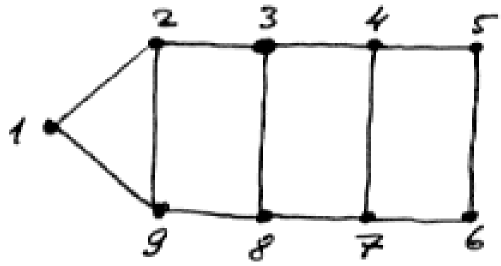
FST and TBA

$$\epsilon = p = \frac{M}{2} e^{\beta} \qquad \text{Right}$$

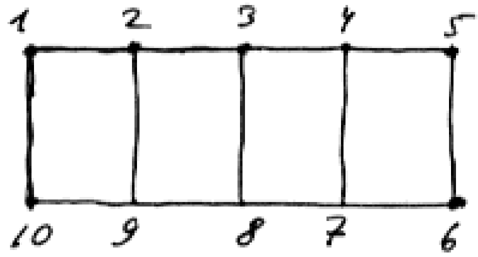
$$\epsilon = -p = \frac{M}{2} e^{-\beta} \qquad \text{Left}$$

$$S_{RR}(\beta) = S_{LL}(\beta) = S_{RL}(\beta) = S_{PF}(\beta)$$

Admissibility diagram for kinks



$N = 9$



$N = \frac{4\pi}{\nu} = 10$

$$K_{e,e_H}(\beta_1)\, K_{e_H,e_{H2}}(\beta_2)\rangle_{in} =$$

$$A_{e_H}(\beta_1 - \beta_2)\, | K_{e,e_H}(\beta_2)\, K_{e_H,e_{H2}}(\beta_1)\rangle_{out}$$

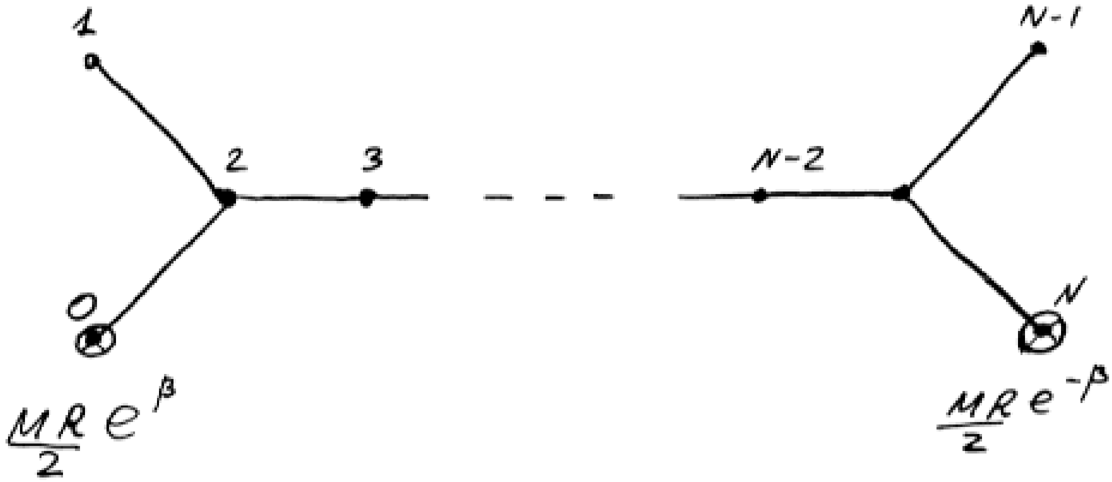
$$A_{e_H}(\beta) = \frac{\sinh\left[\frac{1}{N+2}(\theta - i\pi)\right]}{\sinh\left[\frac{1}{N+2}(\theta + i\pi)\right]}$$

. . . .

TBA equations $\qquad \nu = 4\pi/N$

$\epsilon_a(\beta) \quad a = 0, 1, \dots, N \; ; \quad L_a = 1 + \exp(-\epsilon_a(\beta))$

$$\epsilon_a(\beta) + \frac{1}{2\pi}\int \frac{I_{ab}\, L_b(\beta')}{\cosh(\beta-\beta')}\, d\beta' = \frac{MR}{2}\left(\delta_{a,0}\, e^{\beta} + \delta_{a,N}\, e^{-\beta}\right)$$



$$C(R) = \frac{3}{2\pi^2}\int d\beta \left[e^{\beta} L_0(\beta) + e^{-\beta} L_N(\beta)\right] MR$$

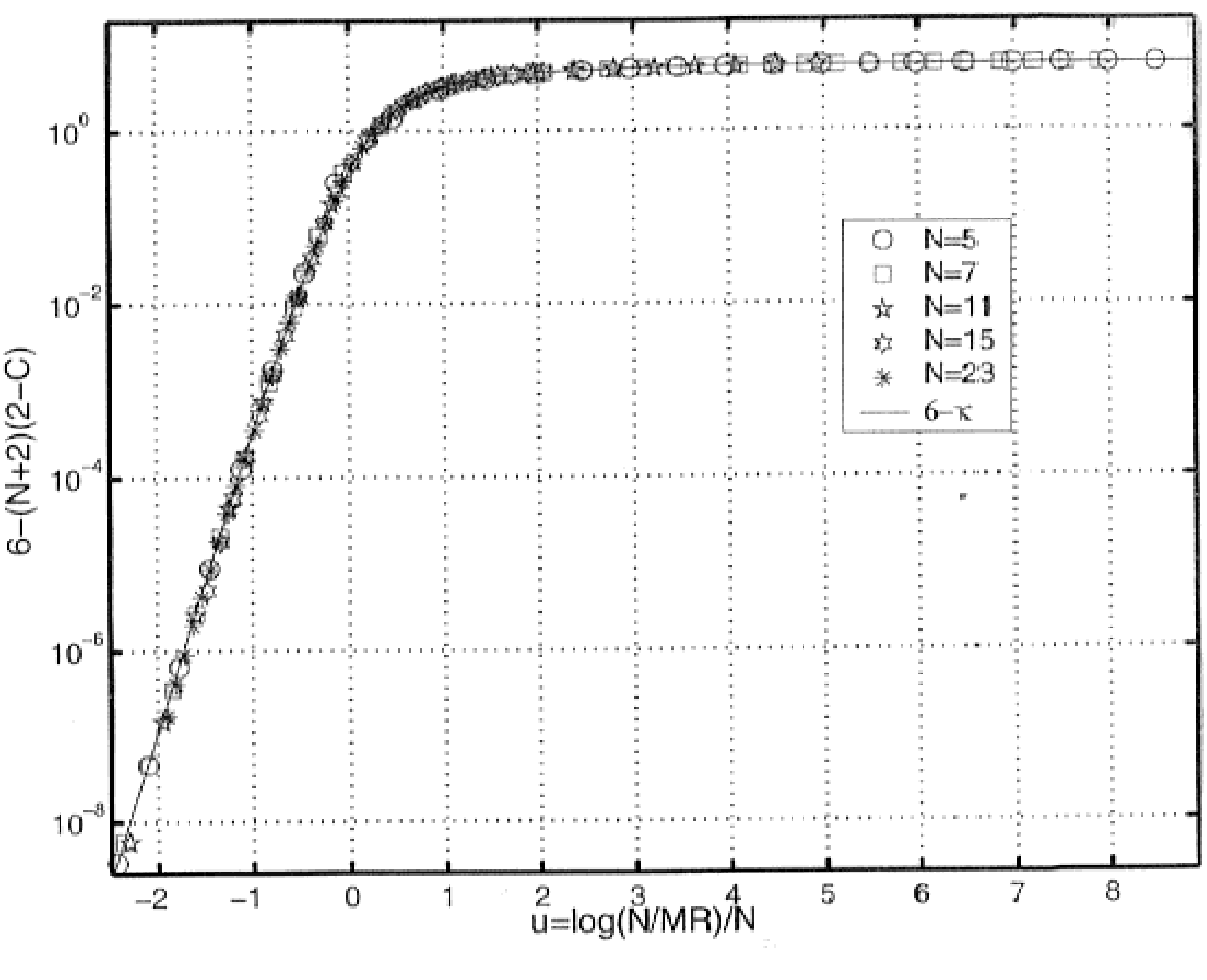
UV $\qquad C_{uv} = C(0) = 2$

$$C(R,N) = 2 - \frac{3\pi^2 (N-2)}{2\left[\log\left(\frac{8\pi}{RM}\right) + (N-2)\log 4\right]^2} + O\left(\frac{1}{\log^5\left(\frac{1}{RM}\right)}\right)$$

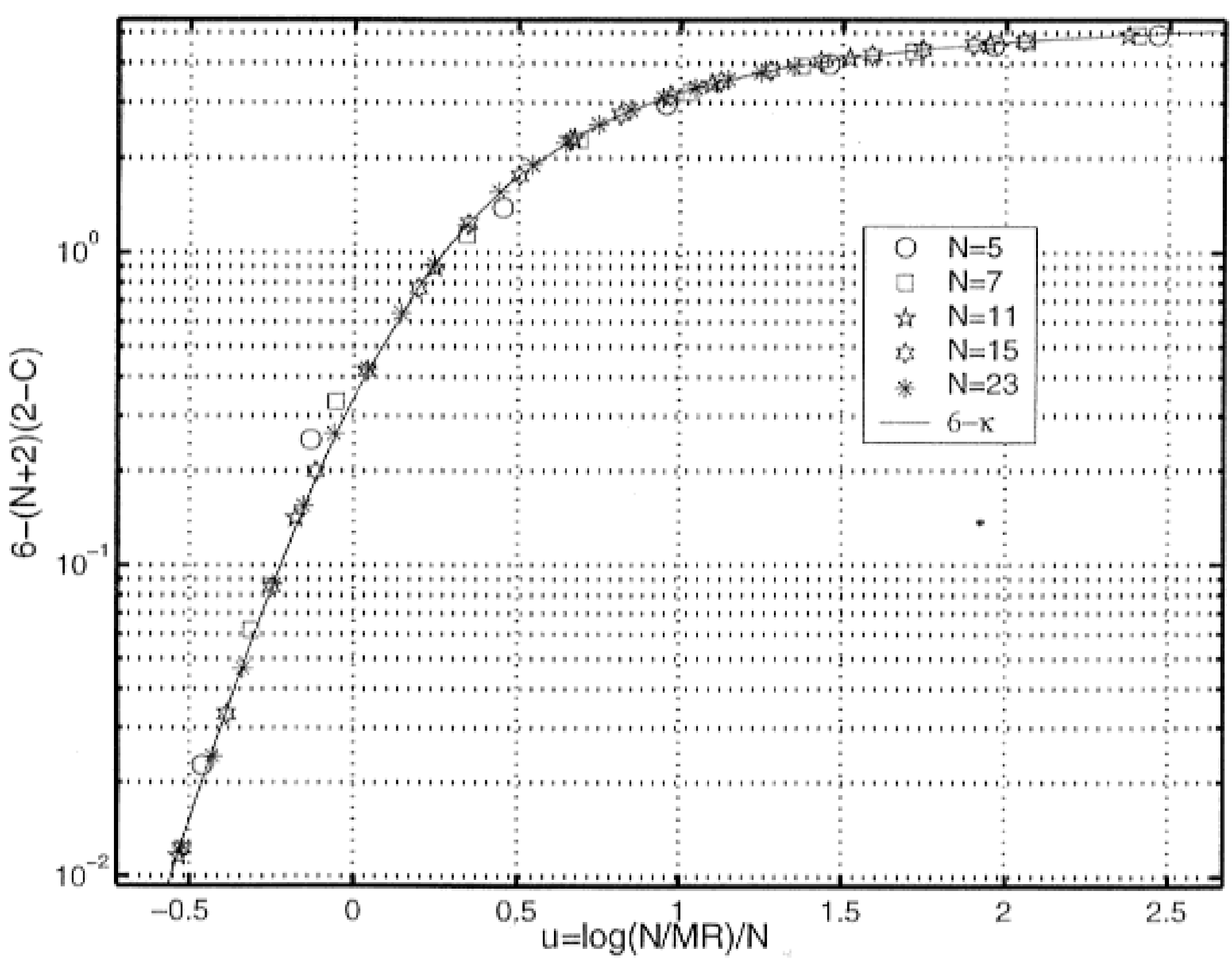
IR asymptot. $\qquad MR \gg 1$

$$C(R,N) = 2 - \frac{6}{N+2} - d_2 (MR)^{-8/N+2} - d_3 (MR)^{-\frac{12}{N+2}} + \cdots$$

Numerical values of $d_2(N)$ and $d_3(N)$ coincide with exact coefficients derived from IR perturbed CFT.



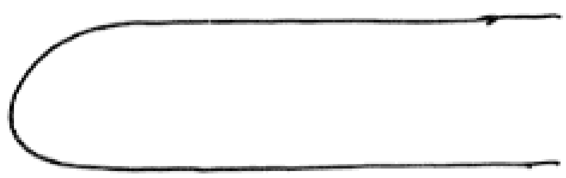
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
6−(N+2)(2−C)
−2 −1 0 1 2 3 4 5 6 7 8
u=log(N/MR)/N
N=5
N=7
N=11
N=15
N=23
6−κ



N=5
N=7
N=11
N=15
N=23
6−κ
6−(N+2)(2−C)
u=log(N/MR)/N
10^0
10^{-1}
10^{-2}
−0.5
0
0.5
1
1.5
2
2.5

Dual description

$$\frac{1}{\nu}\left(dr^2 + \tanh^2 r\, (d\theta)^2 \right)$$



$\frac{SL(2,R)}{U(1)}$ CFT

Dual descript. : sine-Liouville model

$$A_{SL} = \int \left[\frac{(\partial_\mu \varphi)^2}{16\pi} + \frac{(\partial \phi)^2}{16\pi} + \mu e^{\alpha\varphi} \cos\beta\phi \right] d^2x$$

$$\beta^2 - \alpha^2 = \frac{1}{2} \; ; \quad \beta^2 = \frac{\pi}{\nu} \, , \qquad \varphi \sim -\log r$$

The same spectrum and corr. func.

Sausage model dual repres.

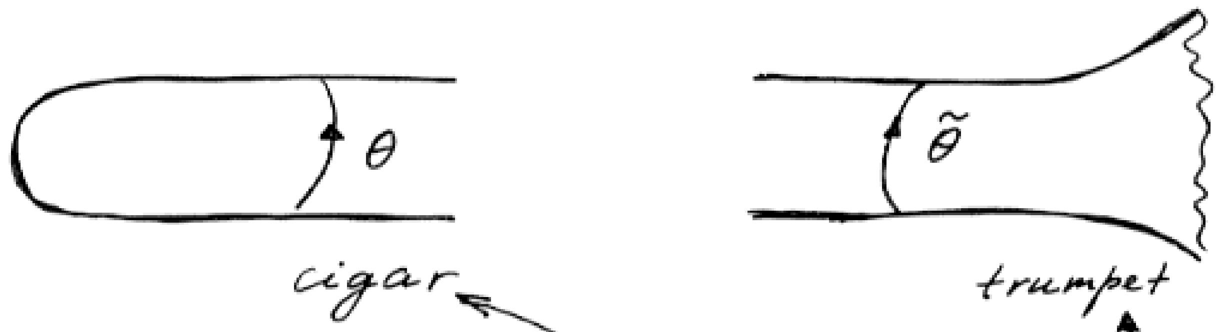
$$A_{SSG} = \int \left[\frac{(\partial_\mu \varphi)^2}{16\pi} + \frac{(\partial_\mu \phi)^2}{16\pi} + \mu \left(e^{\alpha\varphi} \cos\beta\phi + e^{-\alpha\varphi} \cos\beta\phi \right) \right] d^2x$$

Integrability cond $\beta^2 - \alpha^2 = 1/2$

$$\beta^2 = \frac{\pi}{\nu} \; ; \quad \nu \text{ - arbitrary}$$

Dual representation for sing. SM.

two dual descriptions of $SL(2,R)/U(1)$ CFT



$$A_{SSM} = \int \left[\frac{(\partial_\mu \varphi)^2}{16\pi} + \frac{(\partial_\mu \phi)^2}{16\pi} + \mu \left(e^{\alpha\varphi} \cos\beta\phi + e^{-\alpha\varphi} \cos\beta\tilde\phi \right) \right] d^2x$$

$$\partial_\mu \phi = \epsilon_{\mu\nu} \partial_\nu \tilde\phi$$

$$\alpha^2 - \beta^2 = 1/2$$

Two terms are mutually local and the theory has sense (well defined) only

if $\beta^2 = \frac{\pi}{\nu} = N/4$ or

$$\nu = \frac{4\pi}{N}$$

discrete spectrum:

$$e_{j,m}/6 = \frac{m^2}{N} - \frac{(2j-2m+1)^2}{N-2}, \quad \frac{|m|}{2} \le j < m-1$$

$$e_{j,m}/6 = m^2 + \frac{\pi^2 (N-2)(j-m+1)^2}{4 Z_m^2} + O(1/Z_m^5) \quad ; j \ge m-1$$

$$Z_m = \log\left(\frac{8\pi(N-2)}{RM}\right) + (N-2)\left(\Gamma'(1) - \Gamma'\left(\frac{m+1}{2}\right)\right)$$