The form factor program

scaling Z_N-Ising model + affine A_{N-1}-Toda model

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Florence, September 2003

- **The "Bootstrap Program"**
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The "Bootstrap Program"

Construct a quantum field theory explicitly by:

1. **Calculate the 2-particle S-matrix** using
   (a) general Properties: unitarity, crossing etc.
   (b) symmetry properties
   (c) "Yang-Baxter Equation"
   (d) "bound state bootstrap"
   (e) 'maximal analyticity'

2. **Calculate the "form factors"**

$$\langle\, 0 \,|\, \phi(x) \,|\, p_1, \ldots, p_n \,\rangle^{in} = e^{-ix(p_1+\cdots+p_n)}\, f^{\phi}\,(p_1, \ldots, p_n)$$

   using
   (a) the S-matrix
   (b) LSZ-assumptions
   (c) 'maximal analyticity'

3. **Calculate the "Wightman functions"**

$$\langle\, 0 \,|\, \phi(x)\phi(y) \,|\, 0 \,\rangle = \sum_n \int \langle\, 0 \,|\, \phi(x) \,|\, n \,\rangle^{in}\; {}^{in}\langle\, n \,|\, \phi(y) \,|\, 0 \,\rangle$$

The bootstrap program classifies integrable quantum field theoretic models

The models

Motivation:

Classical statistical lattice model in 2-dim

$$Z = \sum_{\{\sigma\}} \exp\left(-\frac{1}{kT}\sum_{\langle ij\rangle} E(\sigma_i, \sigma_j)\right)$$

$$\sigma_i \in \left\{1, \omega, \ldots, \omega^{N-1}\right\}, \; \omega = e^{2\pi i/N}$$

scaling limit $\to$ massive Z_N-Ising quantum field theory

Classical A_{N-1}-Toda field equations

$$\Box \vec{\alpha}_j \vec{\varphi} + \frac{\alpha}{\beta}\left(2e^{\beta\vec{\alpha}_j\vec{\varphi}} - e^{\beta\vec{\alpha}_{j+1}\vec{\varphi}} - e^{\beta\vec{\alpha}_{j-1}\vec{\varphi}}\right) = 0$$

$\vec{\varphi} = \left(\varphi^1, \ldots, \varphi^{N-1}\right)$ fields, $\vec{\alpha}_j$ = simple positive roots.

Both quantum models Z_N and A_{N-1} have the same particle spectrum:

$a = 1, \ldots, N-1$ such that $\bar{a} = N - a$

bound states: $(a, b) \to c = a + b$

in particular $\underbrace{(1, \ldots, 1)}_{N-1} \to (N-1) = \bar{1}$

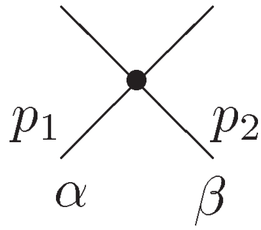
Integrability

$$\text{means}: \quad \partial_\mu J^\mu_L(t,x) = 0 \quad (L = \pm 1, \pm 3, \dots)$$

$$\text{Consequence}: \quad S^{(n)}(p_1, \dots, p_n) = \prod_{i<j} S_{ij}(p_i, p_j)$$

2-particle S-matrix

$S_{\alpha\beta}(\theta_{12}) =$



$$, \qquad p_i = m_i \begin{pmatrix} \cosh\theta_i \\ \sinh\theta_i \end{pmatrix}$$

The S-matrix for two type 1 particles:

Z_N-Ising model [Köberle Swieca 79]

$$S^Z_{11}(\theta) = \frac{\sinh\frac{1}{2}(\theta + \frac{2\pi i}{N})}{\sinh\frac{1}{2}(\theta - \frac{2\pi i}{N})}$$

A_{N-1}-Toda model [Arinshtein Fateev Zamolodchikov 79]

$$S^T_{11}(\theta) = \frac{\sinh\frac{1}{2}(\theta + \frac{2\pi i}{N})}{\sinh\frac{1}{2}(\theta - \frac{2\pi i}{N})} \frac{\sinh\frac{1}{2}(\theta - (2-B)\frac{\pi i}{N})}{\sinh\frac{1}{2}(\theta + (2-B)\frac{\pi i}{N})} \frac{\sinh\frac{1}{2}(\theta - B\frac{\pi i}{N})}{\sinh\frac{1}{2}(\theta + B\frac{\pi i}{N})}$$

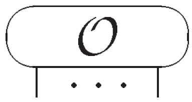
$$\text{where} \qquad B = \frac{2\beta^2}{4\pi + \beta^2}$$

Form factors

Definition: Let $\mathcal{O}(x)$ be a local operator

$$\langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_1, \ldots, p_n \,\rangle^{in}_{\alpha_1 \ldots \alpha_n} = f^{\mathcal{O}}_{\alpha_1 \ldots \alpha_n}(\theta_1, \ldots, \theta_n)$$

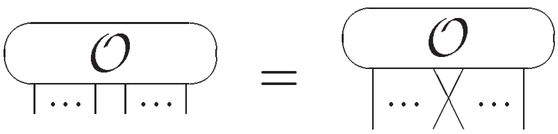
$$=$$



LSZ-assumptions
\+ 'maximal analyticity' $\Big\}$ $\Longrightarrow$ Properties of form factors

(i) Watson's equation

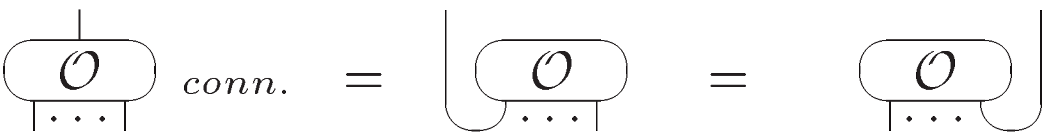
$$f^{\mathcal{O}}_{\ldots ij \ldots}(\ldots, \theta_i, \theta_j, \ldots) = f^{\mathcal{O}}_{\ldots ji \ldots}(\ldots, \theta_j, \theta_i, \ldots)\, S_{ij}(\theta_i - \theta_j)$$



(ii) Crossing

$${}_{\bar{\alpha}_1}\langle\, p_1 \,|\, \mathcal{O}(0) \,|\, \ldots, p_n \,\rangle^{in, conn.}_{\ldots \alpha_n} =$$

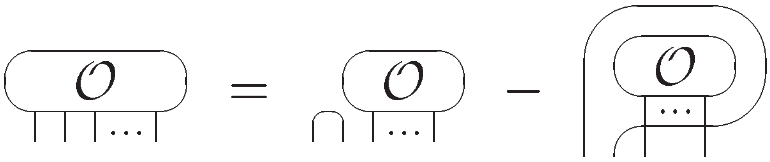
$$f^{\mathcal{O}}_{\alpha_1 \ldots \alpha_n}(\theta_1 + i\pi, \ldots, \theta_n) = f^{\mathcal{O}}_{\ldots \alpha_n \alpha_1}(\ldots, \theta_n, \theta_1 - i\pi)$$



(iii) Annihilation recursion relation

$$\frac{1}{2i}\underset{\theta_{12}=i\pi}{\mathrm{Res}}\, f^{\mathcal{O}}_{1\dots n}(\theta_1,\dots) = \mathbf{C}_{12}\, f^{\mathcal{O}}_{3\dots n}(\theta_3,\dots)\,(\mathbf{1} - S_{2n}\dots S_{23})$$

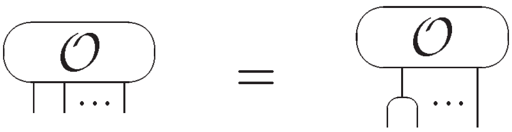
$$\frac{1}{2i}\underset{\theta_{12}=i\pi}{\mathrm{Res}}$$



(iv) Bound state form factors

$$\frac{1}{\sqrt{2}}\underset{\theta_{12}=ia}{\mathrm{Res}}\, f^{\mathcal{O}}_{123\dots n}(\underline{\theta}) \quad = \quad f^{\mathcal{O}}_{(12)3\dots n}(\theta_{(12)},\underline{\theta}')\,\Gamma^{(12)}_{12}$$

$$\frac{1}{\sqrt{2}}\underset{\theta_{12}=ia}{\mathrm{Res}}$$



(v) Lorentz invariance

$$f^{\mathcal{O}}_{1\dots n}(\theta_1+u,\dots,\theta_n+u) = e^{su}\, f^{\mathcal{O}}_{1\dots n}(\theta_1,\dots,\theta_n)$$

where s is the "spin" of $\mathcal{O}$

[Karowski Weisz 1978] [Berg Karowski Weisz 1979]

[Smirnov 1992]

[Babujian Fring Karowski Zapletal 1998]

Solutions of the form factor eqs. (i) – (v)

2-particle form factors

Watson's equations + crossing

$$
\begin{aligned}
F(\theta) &= F(-\theta)\, S(\theta) \\
F(i\pi - \theta) &= F(i\pi + \theta)
\end{aligned}
$$

"maximal analyticity" $\Rightarrow$ unique solution

The minimal Z_N-Ising form factor [Karowski 79]

$$
F^Z(\theta) = \exp \int_0^\infty \frac{dt}{t} \frac{2 \sinh t \frac{N-1}{N} \cosh t \frac{1}{N}}{\sinh^2 t} \left(1 - \cosh t \left(1 - \frac{\theta}{i\pi}\right)\right)
$$

The minimal A_{N-1}-Toda form factor [Destri de Vega 90]

$$
F^T(\theta) = \exp \int_0^\infty \frac{dt}{t} \frac{-4 \sinh t \frac{N-1}{N} \sinh t \frac{B}{2N} \sinh t \frac{2-B}{2N}}{\sinh^2 t} \times \cosh t \left(1 - \frac{\theta}{i\pi}\right)
$$

The General solution for $f^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\theta_1,\ldots,\theta_n)$

($\alpha_i =$ type of particle)

Babujian Fring Karowski Zapletal Nucl. Phys. B538 [FS] (1999) 535-586
Babujian Karowski, Nucl. Phys. B620 [FS] (2002) 407-455

General form factor formula

$$f^{\mathcal{O}}_{\alpha_1\ldots\alpha_n}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \, h(\underline{\theta},\underline{z})\, p^{\mathcal{O}}(\underline{\theta},\underline{z})\, \Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z})$$

satisfies the properties (i) – (v).

$$\begin{cases} \text{integration contour } \mathcal{C}_{\underline{\theta}} \\ h(\underline{\theta},\underline{z}) = \text{ scalar function depending on the S-matrix} \\ \text{“Bethe ansatz” state } \Psi_{\alpha_1\ldots\alpha_n}(\underline{\theta},\underline{z}) \\ \text{p-function } p^{\mathcal{O}}(\underline{\theta},\underline{z}) \text{ depending on the operator } \mathcal{O}(x) \end{cases}$$

A similar formula has been proposed by Smirnov, World Scientific 1992

Two examples $\longrightarrow$

Z_N-Ising and A_{N-1}-Toda form factors

For models without backward scattering

$$f_n^{\mathcal{O}}(\underline{\theta}) = K_n^{\mathcal{O}}(\underline{\theta}) \prod_{1 \le i < j \le n} F(\theta_{ij})$$

Form factors for n-particles of type 1:
$K_n^{\mathcal{O}}(\underline{\theta})$ = linear combination of the fundamental blocks:

$$\begin{aligned} I_{n\underline{m}}(\underline{\theta}, p_n^{\mathcal{O}}) &= \frac{1}{m_1! \dots m_{N-1}!} \left(\prod_{k=1}^{N-1} \prod_{j=1}^{m_k} \int_{\mathcal{C}_{\underline{\theta}}} dz_{kj} \right) \\ &\times \prod_{k=1}^{N-1} \left(\prod_{j=1}^{m_k} \prod_{i=1}^{n} \phi(z_{kj} - \theta_i) \prod_{1 \le i < j}^{m_k} \tau(z_{ki} - z_{kj}) \right) \\ &\times \prod_{1 \le k < l}^{N-1} \prod_{i=1}^{m_k} \prod_{j=1}^{m_l} \kappa(z_{ki} - z_{lj}) p_{n\underline{m}}^{\mathcal{O}}(\underline{\theta}, \underline{z}) \end{aligned}$$

with

$$\kappa(z) = \frac{1}{\phi(-z)}, \quad \tau(z) = \frac{1}{\phi(z)\phi(-z)}$$

The p-functions $p_{n\underline{m}}^{\mathcal{O}}(\underline{\theta}, \underline{z})$ depend on the operator $\mathcal{O}(x)$ whereas the other functions are determined by the S-matrix only.

$$\underbrace{(1,\ldots,1)}_{N-1} \to (N-1) = \bar{1} = \text{anti-particle of } 1$$

and the form factor recursion relations (iii) + (iv) $\Longrightarrow$

$$\prod_{k=1}^{N-1} \phi\left(\theta + k\frac{2\pi i}{N}\right) \prod_{k=0}^{N-1} F\left(\theta + k\frac{2\pi i}{N}\right) = 1$$

Solutions:

$$\phi^Z(z) = \frac{1}{\sinh\frac{1}{2}z \sinh\frac{1}{2}(z - 2\pi i/N)}$$

$$\phi^T(z) = \phi^Z(z) \sinh\tfrac{1}{2}\left(z - \frac{i\pi}{N}B\right) \sinh\tfrac{1}{2}\left(z - \frac{i\pi}{N}(2-B)\right)$$

they satisfy the 'Jost-function property'

$$\frac{\phi(\theta)}{\phi(-\theta)} = S(\theta)$$

Examples of operators and their p-functions
The Z_N-Ising model

For $Q, \tilde{Q} = 0, \dots, N-1$:

$$\text{operators} \quad \psi_{Q,\tilde{Q}}(x) \quad \text{with} \begin{cases} \text{charge} & Q \\ \text{spin} & Q\tilde{Q}/N \\ \text{statistics} & e^{2\pi i \tilde{Q}/N} \end{cases}$$

$$\leftrightarrow \qquad p_{n\underline{m}}^{Q\tilde{Q}}(\underline{\theta}, \underline{z}) = \exp\left(\frac{\tilde{Q}}{N}\sum_{i=1}^{n}\theta_i - \sum_{k=1}^{\tilde{Q}}\sum_{j=1}^{m_k} z_{kj}\right)$$

where $n = Nm + Q$, $m_k = m + 1$ for $k < Q$, $m_k = m$ else

In particular:

$\tilde{Q} = 0$	: order parameter $\sigma_Q(x)$
$Q = 0$	: disorder parameter $\mu_{\tilde{Q}}(x)$
$Q = \tilde{Q}$	para-fermi fields $\psi_Q(x)$

with spacelike commutation rules:

$$\sigma_Q(x)\sigma_{Q'}(y) = \sigma_{Q'}(y)\sigma_Q(x)$$

$$\mu_{\tilde{Q}}(x)\mu_{\tilde{Q}'}(y) = \mu_{\tilde{Q}'}(y)\mu_{\tilde{Q}}(x)$$

$$\sigma_Q(x)\mu_{\tilde{Q}}(y) = \mu_{\tilde{Q}}(y)\sigma_Q(x)\begin{cases} 1 & \text{for} \quad y^1 > x^1 \\ e^{2\pi i Q\tilde{Q}/N} & \text{for} \quad y^1 < x^1 \end{cases}$$

$$\psi_Q(x)\psi_Q(y) = \psi_Q(y)\psi_Q(x)e^{2\pi i Q^2/N\epsilon(y^1 - x^1)}$$

which agrees with conformal Z_N para-fermi field theory
[Fateev Zamolodchikov 1985]

The form factors agree for $N = 3$ with [Kirilov Smirnov 1988]

The A_{N-1} Toda model

form factors for exponential of the fields

$$\mathcal{O}(x) = :\exp\vec{\gamma}\vec{\varphi}:(x)\,.$$

$$K_n^{\mathcal{O}}(\underline{\theta}) = N_n \sum_{m_1=0}^{n} \cdots \sum_{m_{N-1}=0}^{n} \prod_{k=1}^{N-1} q_k^{m_k} I_{n\underline{m}}^{T}(\underline{\theta}, 1)$$

satisfies the 'clustering' property for $\operatorname{Re}\theta_1 \to \infty$

$$f_n(\theta_1,\theta_2,\ldots,\theta_n) \to f_1(\theta_1)\, f_{n-1}(\theta_2,\ldots,\theta_n)$$

which is typical for exponential of the fields of bose fields

In particular

$$:\exp\gamma\vec{\alpha}_j\vec{\varphi}:(x) \quad \leftrightarrow \quad q_k = \omega^{-k}\omega^{-\hat{\gamma}\left(\delta_{kj}-\delta_{kj+1}\right)}$$

The form factors of the bound states $|\,p\,\rangle_b$ are

$$\langle\, 0\,|\, :\exp\gamma\vec{\alpha}_j\vec{\varphi}:(0)\,|\,p\,\rangle_b = \frac{\beta}{\pi B}\sqrt{N Z^b}\,\chi_b(\gamma,j)$$

$$\begin{aligned}\chi_b(\gamma,j) &= \prod_{k=1}^{N-1} q_k^{-b/N} \sum_{0\le k_1<\cdots<k_b<N} q_{k_1}\cdots q_{k_b} \\ &= -4\omega^{-bj}\frac{\sin\frac{b\pi}{N}}{\sin\frac{\pi}{N}}\sin\left(\frac{\pi}{N}\frac{B\gamma}{2\beta}\right)\sin\frac{\pi}{N}\left(1-\frac{B\gamma}{2\beta}\right)\end{aligned}$$

which agrees with [Lukyanov 1997]

The wave function renormalization constants Z^a are defined as

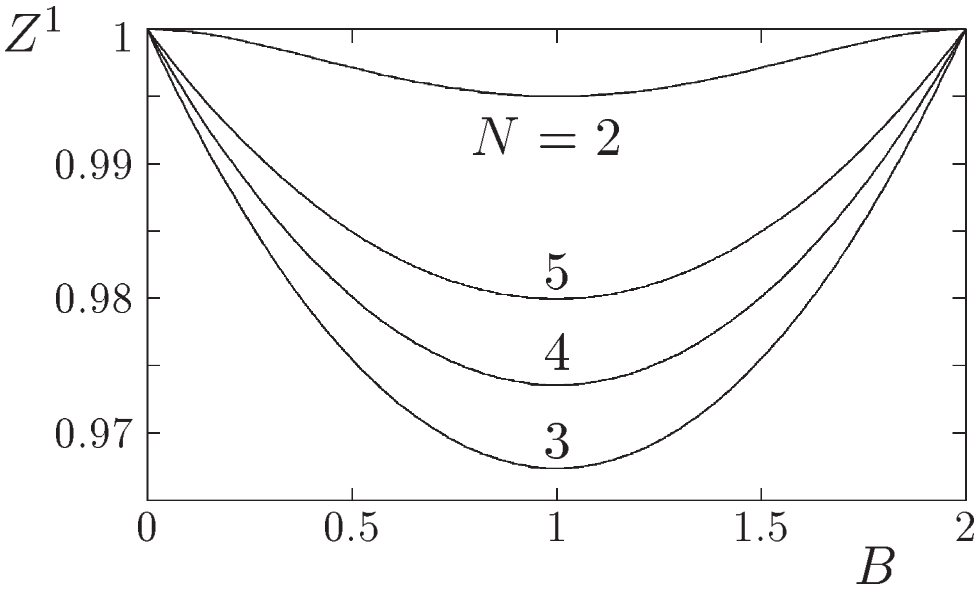
$$\langle\, 0 \,|\, \varphi^a(0) \,|\, p\, \rangle_b = \delta_{ab}\sqrt{Z^a}$$

We calculate

$$Z^a = \frac{\pi B(2-B)}{4\sin\frac{\pi}{2}B}\exp\int_0^\infty \frac{dt}{t}$$
$$\times \frac{2\sinh\frac{1}{2}\frac{t}{N}B\sinh\frac{1}{2}t\frac{2-B}{N}}{\sinh^2 t\sinh t\frac{1}{N}}\left(\sinh^2 t\frac{a}{N} + \sinh^2 t\frac{\bar{a}}{N}\right)$$

which agrees with [Destri de Vega 1990]

For example:



The wave function renormalization constants $Z^1(B, N)$

Quantum field operator equations

The following operator equations are equations of all their matrix elements:

$$\Box\vec{\alpha}_j\vec{\varphi} + \frac{\alpha}{\beta} : \left(2e^{\beta\vec{\alpha}_j\vec{\varphi}} - e^{\beta\vec{\alpha}_{j+1}\vec{\varphi}} - e^{\beta\vec{\alpha}_{j-1}\vec{\varphi}}\right) := 0$$

where the renormalized and the "bare" mass are related by

$$m^2 = 4\,\alpha\,\sin^2\frac{\pi}{N}\underbrace{\frac{2N\sin\frac{\pi}{2N}B\sin\frac{\pi}{2N}(2-B)}{\pi B\sin\frac{\pi}{N}}}$$

quantum correction ↑

∞ set of conservation laws

$$\partial_\mu J_L^\mu(t,x) = 0 \quad (L = \pm1, \pm3, \dots)$$

p-functions (for both models Z_N-Ising and A_{N-1}-Toda):

$$p_{n\underline{m}}^{J_L^\pm}(\underline{\theta}, \underline{z}) = \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{k=1}^{N-1}\sum_{j=1}^{m_k} e^{Lz_{kj}}$$

such that the charges $Q_L = \int dx J_L^0(x)$ satisfy

$$\left(Q_L - \sum_{i=1}^{n} e^{L\theta_i}\right) |\, p_1, \dots, p_n\,\rangle^{in} = 0$$