Boundary Correlators in Integrable Models of 2D Quantum Gravity

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Two approaches to 2d quantum gravity

- **Continuous approach:**

matter field CFT coupled to Liouville field theory

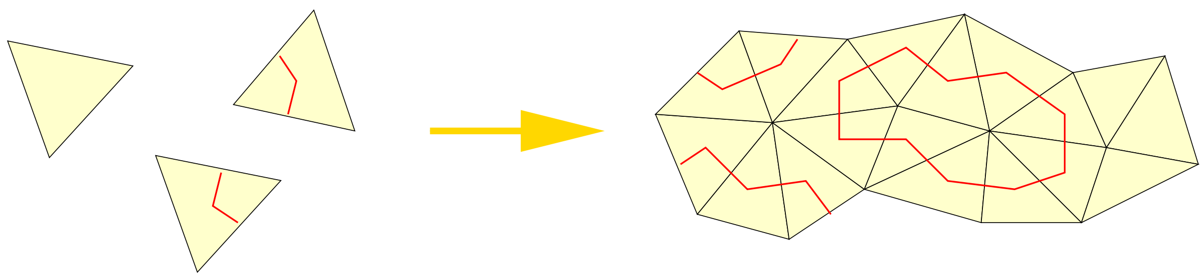
$$\left\langle e^{\sum \lambda_i \Phi_i} \right\rangle_{\mathrm{QG}} = \int \mathcal{D} g_{ab} \left\langle e^{\sum \lambda_i \Phi_i} \right\rangle_{QFT}$$

∗ Methods: Perturbative calculations, CFT methods, quantum groups, ...

- **Discrete approach:**

Statistical models on random planar graphs

Large-N matrix models: engines to produce planar graphs



∗ Methods: Loop equations, integrable hierarchies, ...

BPZ description of Liouville theory (DOZZ-proposal):

- Bulk 3p function [Dorn-Otto, Zamo-Zamo, Teschner] (1994)
- Boundary 2p function [Fateev-Zamo-Zamo] (2000)
- Bulk-boundary 2p function [Hosomichi] (2001)
- Boundary 3p function [Ponsot-Teschner] (2002)

The boundary 2p function have been obtained as solution of a **finite-difference equation**.

Are these results reproducible in the discrete approach?

Our statement:

1. All fundamental boundary correlation functions in Liouville theory satisfy similar finite-difference equations.

2. These equations can be derived in the discrete approach*

*) Up to a normalization

1. Liouville description of 2d QG on a disc

- Upper half plane (UHP) geometry: $\Rightarrow$ factorization: matter$\times$Liouville

$$\mathcal{A}_{\rm mat} + \mathcal{A}_{\rm Liouv} = \int_{UHP} \left[\frac{1}{4\pi} \left((\nabla\chi)^2 + (\nabla\phi)^2 \right) + \mu\, e^{2b\phi} \right] + \int_{\mathbb{R}} \mu_B\, e^{b\phi}$$

at infinity: $\phi(z,\bar z) \sim -Q \log|x|^2, \quad \chi(z,\bar z) \sim -e_0 \log|x|^2.$

- Weyl invariance : $c_{\rm mat} + c_{\rm Liouv} + c_{\rm ghosts} = 0$

$$\Rightarrow (1-6e_0^2) + (1+6Q^2) - 26 = 0$$

Solution: $\boxed{Q = \frac{1}{b} + b, \quad e_0 = \frac{1}{b} - b}$

$$b \in \mathbb{R} \quad \Rightarrow \quad c_{\rm mat} \le 1, \quad (b \le 1)$$

- Boundary conditions:

Matter field: Dirichlet $(\partial_{||}\chi = 0)$ or Neumann $(\partial_\perp \chi = 0)$

Liouville field: Neumann, labeled by μ_B $(\partial_\perp \phi = 4\pi\mu_B\, e^{b\phi})$

Boundary fields in Quantum Gravity (Boundary KPZ fields):

- matter fields dressed by Liouville vertex operators;
- characterized by the left/right boundary conditions [Cardy][1984]

— *matter* component:

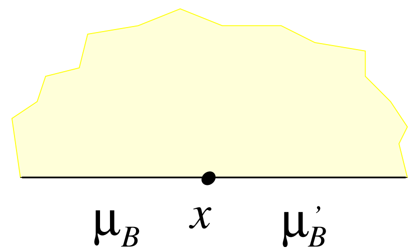
– electric or vertex operators (N/N)

– magnetic operators (D/D)

– twist operators (N/D or D/N)

— *Liouville* component:

separates 2 pieces of boundary with μ_B and μ'_B:



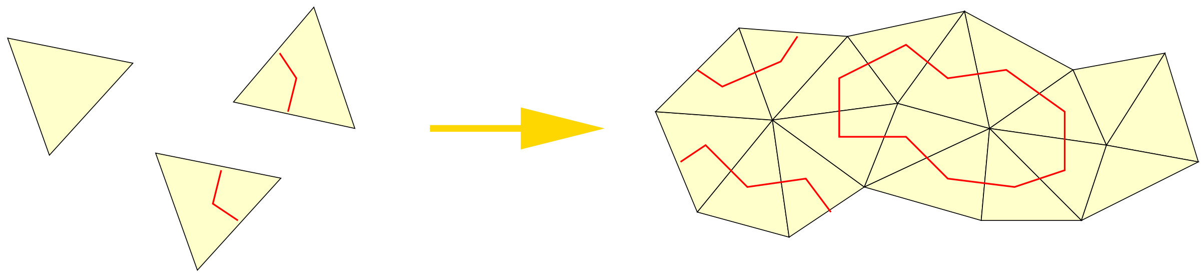
- Example: The electric KPZ operator:

$$\mathbf{B}_e(x) = e^{ie\chi(x)} \times \ e^{\beta\phi(x)}$$

$$\Delta_B = e(e-e_0) + \beta(Q-\beta) = 1 \quad \Rightarrow \beta_\pm(e) \quad (\beta_- \leq \beta_+)$$

2. Discrete approach: Loop gas on a random graph

World sheet $\to$ random planar graph



Matter field $\to$ gas of nonintersecting loops on the random graph

Fugacity: $\quad n = -2\cos\pi g, \qquad g$ – 'Coulomb-gas coupling

- Loop gas partition function [I.K.[89]]:

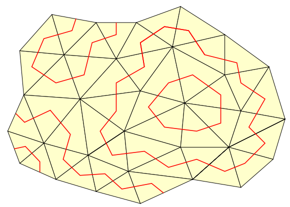
$$\mathcal{Z}(\mu, M, z) = \sum_{\text{triangulations}} \sum_{\text{loops}} n^{[\#\ \text{loops}]} \; e^{-\mu A - M L_{\text{loops}} - \mu_B L_{\text{bound}}}$$

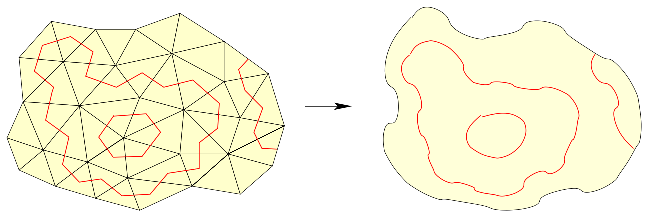
A: area of the world sheet, $\quad\mu$: cosmological constant

L_{loops}: total length of the loops, $\quad M$: loop tension,

L_{bound}: length of the boundary, μ_B: boundary cosmological constant.

- Critical phases: matter CFT with $c = 1 - 6\frac{(g-1)^2}{g}$





dense phase:	dilute phase:
$0 < g < 1, \quad g = b^2$	$1 < g < 2\,, \quad g = 1/b^2$
$\mu \sim M^{2g}$	$\mu \sim M^2$

- loop configurations $\leftrightarrow$ local SOS height variable

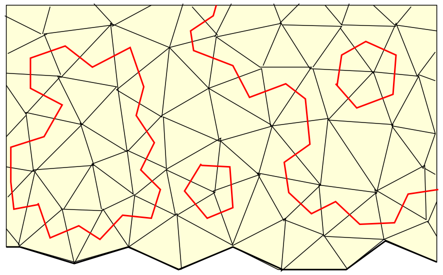
discretizing the gaussian field χ

- The dilute and dense phases are related by

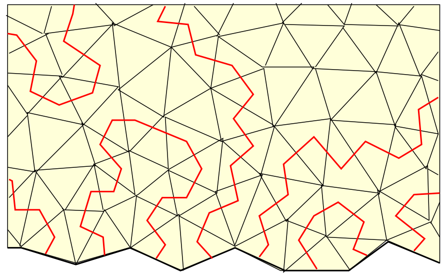
$b \leftrightarrow 1/b$, electric $\leftrightarrow$ magnetic duality.

- Only the dilute phase matches with the functional integral

- Boundary conditions for the height variable χ:

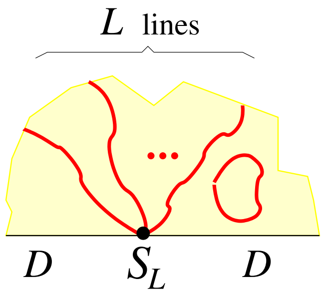


Dirichlet (fixed) b.c.

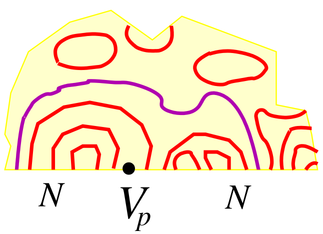


Neumann (free) b.c.

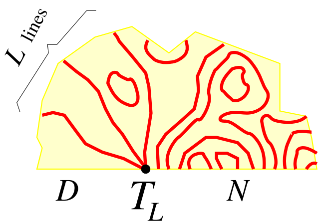
- Boundary operators in the loop gas representation:



magnetic



electric



twist

$$\Delta_{\rm flat} = \frac{[(L+1)/b-b]^2}{4} - \frac{e_0^2}{4} \qquad \frac{P^2}{4} - \frac{e_0^2}{4} \qquad \frac{[(L+1/2)^2/b]}{4} - \frac{e_0^2}{4}$$

$$(r,s) \;=\; (L+1,1) \qquad \text{diagonal} \qquad (L+\tfrac{1}{2},0)$$

3. Boundary correlation functions in the discrete approach

- Consider *complex* boundary cosmological constants

$$\boxed{\mu_B \longrightarrow z \in \mathbf{C}}$$

- The boundary correlators are *meromorphic* functions in z with a cut $-\infty < z < -M$.

- Integral equations $\Rightarrow$ boundary conditions on the cut $\Rightarrow$ monodromy relations in the variable z.

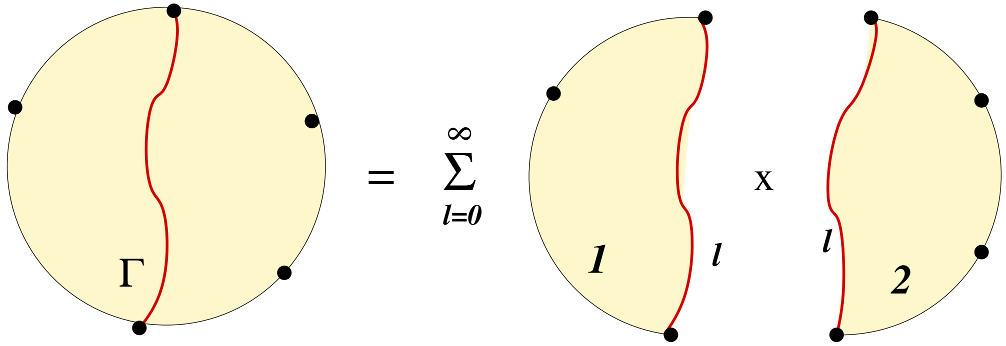
- The cut is unfolded by introducing the uniformization variable τ:

$$\boxed{z = M \cosh \tau}.$$

monodromy relations $\Rightarrow$ *finite-difference functional equations.*

4. Factorization formula [Migdal]^{1988}

- The integral equations follow from the following *factorization property* of the integration measure over surfaces



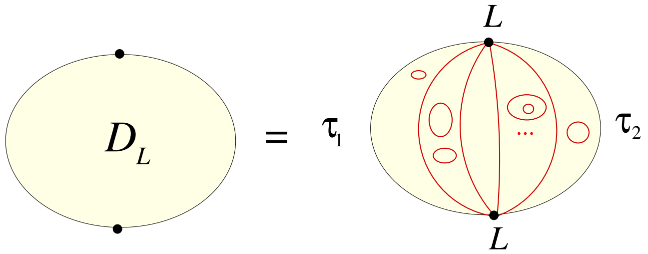
$$\int \mathcal{D}g_{ab}(\mathcal{M}) \int_{\mathcal{M}} \mathcal{D}\Gamma = \int_0^\infty d\ell \int \mathcal{D}g_{ab}(\mathcal{M}_1) \int \mathcal{D}g_{ab}(\mathcal{M}_2)$$

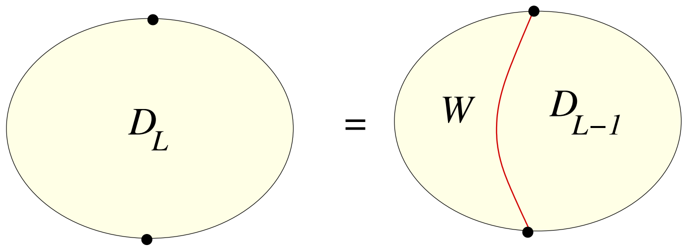
the red line Γ represents a self-avoiding line on the random graph

ℓ – the length of the line

Boundary 2p function of magnetic operators [I.K.^{02}]

$D_L(z_1, z_2) = \langle\, [\mathbf{S}_L]^{z_1 z_2} [\mathbf{S}_L]^{z_2 z_1} \,\rangle_{\rm disc}$





$\hat{D}_L(\ell_1, \ell_2) = \int_0^\infty d\ell\ \hat{W}(\ell_1 + \ell)\hat{D}_{L-1}(\ell, \ell_2)$

$$\Rightarrow D_L(z_1, z_2) = \oint \frac{dz}{2\pi i}\ \frac{W(z_1) - W(z)}{z_1 - z}\ D_{L-1}(-z, z_2)$$

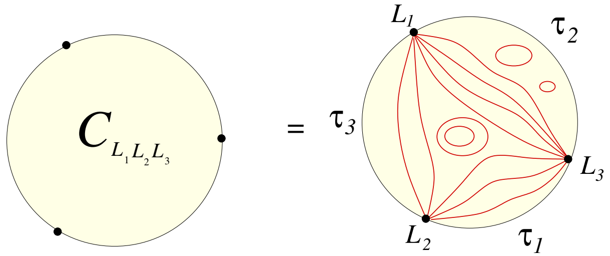
$\Rightarrow$ Equation for the discontinuity along the cut. Going to the τ-variable $(z = M\cosh\tau)$ we get the finite difference equation

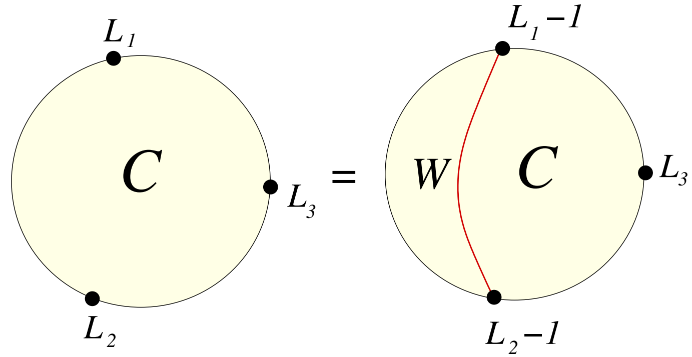
$$D_L(\tau_1 + i\pi, \tau_2) - D_L(\tau_1 - i\pi, \tau_2) = i\sinh g\tau_1\ D_{L-1}(\tau_1, \tau_2)$$

initial condition: $D_0(\tau_1, \tau_2) = \frac{W(z_1) - W(z_2)}{z_1 - z_2}$ $\Rightarrow$ explicit solution

Boundary 3p function of magnetic operators [KPS.02]

$$C_{L_1,L_2,L_3}(\tau_1,\tau_2,\tau_3) = \langle\, [\mathbf{S}_{L_1}]^{\tau_2\tau_3} [\mathbf{S}_{L_2}]^{\tau_3\tau_1} [\mathbf{S}_{L_3}]^{\tau_1\tau_2}\, \rangle_{\text{disc}}$$





⇒ Difference equation:

$$C_{L_1,L_2,L_3}(\tau_3+i\pi,\tau_1,\tau_2) - C_{L_1,L_2,L_3}(\tau_3-i\pi,\tau_1,\tau_2)$$
$$= (W(\tau_3+i\pi) - W(\tau_3-i\pi))\; C_{L_1-1,L_2-1,L_3}(\tau_3,\tau_1,\tau_2).$$

Initial condition: $C_{000}(z_1,z_2,z_3) = \oint \frac{dz}{2\pi i} \frac{W(z)}{\prod_i (z-z_i)}$ $\Rightarrow$ solution

2. Comparison to the results in boundary Liouville theory

$(V_\alpha(z,\bar{z}) = e^{2\alpha\phi(z,\bar{z})}, \quad B_\beta(x) \equiv e^{\beta\phi(x)})$

1. Bulk one-point function [Fateev, Zamolodchikovs[00]]:

$$\langle V_\alpha(z,\bar{z})\rangle_\tau = \frac{U_\alpha(\tau)}{|z-\bar{z}|^{2\Delta_p}}$$

2. Boundary two-point function [FZZ[00]]:

$$\left\langle [B_{\beta_1}(x)]^{\tau_1\tau_2}[B_{\beta_2}(x')]^{\tau_2\tau_1}\right\rangle = \frac{\delta(\beta_2+\beta_1-Q) + D(\beta_1|\tau_2,\tau_1)\delta(\beta_2-\beta_1)}{|x-x'|^{2\Delta_{\beta_1}}}$$

3. Bulk-boundary two-point function [Hosomichi[01]]:

$$\langle V_\alpha(z,\bar{z})[B_\beta(x)]^{\tau\tau}\rangle = \frac{R_{\alpha,\beta}(\tau)}{|z-\bar{z}|^{2\Delta_\alpha-\Delta_\beta}\,|z-x|^{2\Delta_\beta}}$$

4. Boundary three-point function [Ponsot-Teschner[02]]:

$$\langle [B_{\beta_1}(x_1)]^{\tau_2\tau_3}[B_{\beta_2}(x_2)]^{\tau_3\tau_1}[B_{\beta_3}(x_3)]^{\tau_1\tau_2}\rangle = \frac{C_{\beta_1\beta_2\beta_3}(\tau_1\tau_2\tau_3)}{|x_{21}|^{\Delta_1+\Delta_2-\Delta_3}\,|x_{32}|^{\Delta_2+\Delta_3-\Delta_1}\,|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}.$$

- Boundary 2p function:

$$D(\beta|\tau_1+i\pi,\tau_2)-D(\beta|\tau_1-i\pi,\tau_2) = iM^{1/b^2}\, c(\beta)\sinh\left(\frac{\tau_1}{b^2}\right)\, D(\beta+\frac{1}{2b},\tau_1,\tau_2)$$

with $c(\beta) = \frac{2M^{1/b^2}}{b^2}\frac{\Gamma(1-2\beta/b)}{\Gamma(2+1/b^2-2\beta/b)}$

- Boundary 3p function:

$$C^{(\tau_1,\tau_2,\tau_3+i\pi)}_{\beta_1,\beta_2,\beta_3}-C^{(\tau_1,\tau_2,\tau_3-i\pi)}_{\beta_1,\beta_2,\beta_3} = c(\beta_1,\beta_2,\beta_3)\ \sin\left(\frac{\tau_3}{b^2}\right) C^{(\tau_1,\tau_2,\tau_3)}_{\beta_1+\frac{1}{2b},\beta_2+\frac{1}{2b},\beta_3}$$

with

$$c(\beta_1,\beta_2,\beta_3) = \frac{2}{b^2}M^{1/b^2}\ \frac{\Gamma(1-2\beta_1/b)\Gamma(1-2\beta_2/b)}{\Gamma(1-(\beta_1+\beta_2-\beta_3)/b)\Gamma(2+1/b^2-(\beta_1+\beta_2+\beta_3)/b)}$$

and the corresponding dual equations ($b\to 1/b$)

Conclusions

- The boundary correlation functions for up to 3 operators in 2D QG satisfy *difference equations*, which can be derived by cutting open the the path integral (in the discrete approach) or from the fusion rules with degenerate fields (in the Liouville CFT).

- The difference equations in the discrete approach were derived for a discrete class of operators, but there are indications that they hold for more general cases.

- The normalisation coefficients in quantum gravity and Liouville are different, because of the contribution of the matter $\rightarrow$ compute this contribution.

- The difference equations in quantum gravity hold for arbitrary number of operator insertions (when the factorization of the correlators matter $\times$ Liouville does not hold any more). $\rightarrow$ Find the corresponding operator algebra (Boundary ground ring?)