Universal Solution to Reflection Equations

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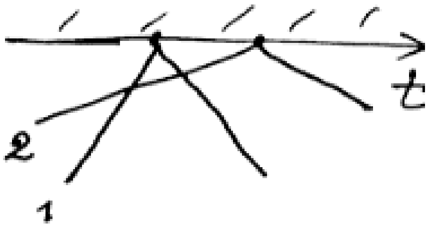
- Introduction YBE & RE

$$R(1,2)\,K(1)\,R(2,1)\,K(2) = K(2)\,R(1,2)\,K(1)\,R(\ldots)$$



- Quasitriangular Hopf algebras, QG [Drinfeld, Jimbo]

  $\mathcal{H}(m, \Delta, \ldots; \mathcal{R})$ ← univ. R-matrix

- Twisting of Hopf algebras by $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$:

$$\Delta(x) \rightsquigarrow \tilde{\Delta}(x) = \mathcal{F}^{-1} \Delta(x)\, \mathcal{F}$$

- Twisted tensor product of Hopf algebras:

$$\mathcal{H}' \equiv \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \xrightarrow{\mathcal{F}} \tilde{\mathcal{H}}', \quad \mathcal{F} \in \mathcal{H}^{(2)} \otimes \mathcal{H}^{(1)} \ldots$$

- Twisted $\otimes$ of H-module algebras

  $\mathcal{A} \sim \mathcal{H} \xrightarrow{\mathcal{F}} \tilde{\mathcal{H}} \sim \tilde{\mathcal{A}}$ with NEW multiplication $\tilde{m}$

- Universal K-matrix $\mathcal{K}$ as canonical element of $\mathcal{H} \otimes \tilde{\mathcal{H}}^*$ twisted $\mathcal{H}$-bimodule

- Characteristic eq. for $\mathcal{K}$, fusion and Lattice current algebra (by twisting)

Quasitriangular Hopf algebras

$\mathcal{H}\,(m, \Delta, \varepsilon, \gamma; \mathcal{R})$ ← antipode : $\mathcal{H} \to \mathcal{H}$

multiplication (associative)

coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$

(alg. [illegible] .. not [illegible]!)

$\varepsilon : \mathcal{H} \to k$ ($\mathbb{C}$) 1-dim. rep.

$\Delta(x) = \sum_i x_i \otimes y_i := x_{(1)} \otimes x_{(2)}$ $\to \Delta^{op}$

Ex: $\mathcal{U}(\mathfrak{g})$; $\mathrm{Fun}(G) \ni f(g)$ $(H, H_{op}, H^{op}, H^{op}_{op})$

UEA

$H^* : \{\alpha, \langle \alpha, x \rangle \in k\}$

universal R-matrix $\mathcal{R} \in \mathcal{H} \otimes \mathcal{H}$, $\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2$

$$\mathcal{R}\,\Delta(x) = \Delta^{op}(x)\,\mathcal{R}, \quad (\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} =$$

$$= \underbrace{\mathcal{R}_1 \otimes \mathcal{R}_{1'}}_{} \otimes \underbrace{\mathcal{R}_2 \mathcal{R}_{2'}}_{} \quad \Sigma \quad \Sigma$$

$$(id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} = \ldots$$

YBE $\quad \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$

$(\rho \otimes \rho)\mathcal{R} = R$

matrix solution

Canonical element T-matrix

of $\mathcal{H} \otimes \mathcal{H}^* \ni \sum e_i \otimes e^i = \mathcal{T}$ Bicharacter, e.g.

$\langle e_i, e^j \rangle = \delta_i^{\,j}$ $\qquad (\Delta \otimes id)\mathcal{T} = \mathcal{T}_1\mathcal{T}_2 = \sum e_i \otimes e_j \otimes e^i e^j$

$\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}^*$

$$\boxed{R_{12}\,T_1\,T_2 = T_2\,T_1\,R_{12}} \quad \text{RTT-rel.}$$

$$\boxed{R_{21}\,K_1\,R_{12}\,K_2 = K_2\,R_{21}\,K_1\,R_{12}} \quad \text{reflection equation (RE)}$$

Twisting of Hopf algebras [Dr]

coassociative

coproduct $\Delta^1 := id$, $\Delta^2 := \Delta$, $\Delta^3 := (\Delta \otimes id)\Delta$, ...

$\mathcal{H} \to \mathcal{H}$, $\mathcal{H} \to \mathcal{H}^{\otimes 2}$, $\mathcal{H} \to \mathcal{H}^{\otimes 3}$, ...

increase # of deg. freed.

$$(id \otimes \Delta^N) R = R_{0N} R_{0N-1} \dots R_{02} R_{01}$$

auxiliary factor/space

QISM & L-operator

Change of Δ by a similarity transformation (TWIST)

$$\Delta \rightsquigarrow \tilde{\Delta} = \mathcal{F}^{-1} \Delta \mathcal{F}, \quad \mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$$

$$\boxed{(\Delta \otimes id)\mathcal{F} \cdot \mathcal{F}_{12} = (id \otimes \Delta)\mathcal{F} \cdot \mathcal{F}_{23}} \in \mathcal{H}^{\otimes 3} \quad \text{(TE)}$$

twist eq.

~ cocycle

$$(\varepsilon \otimes id)\mathcal{F} = (id \otimes \varepsilon)\mathcal{F} = 1 \otimes 1$$

$\gamma \to \tilde{\gamma} = u^{-1} \gamma u$, ...

$$\mathcal{H}(m, \Delta, \varepsilon, \gamma; R) \xrightarrow{\mathcal{F}} \tilde{\mathcal{H}}(m, \tilde{\Delta}, \varepsilon, \tilde{\gamma}; \tilde{R})$$

$$\tilde{R} = \mathcal{F}_{21}^{-1} R \mathcal{F}$$

$\mathcal{H}$ & $\tilde{\mathcal{H}}$ are isomorphic as an alg.

Twist establishes an equivalence rel. among H-algs.

$\mathcal{A}$ is a left $\mathcal{H}$-mod. alg.

$x \triangleright \alpha\beta = (x_{(1)} \triangleright \alpha)(x_{(2)} \triangleright \beta)$, $x \in \mathcal{H}$; $\mathcal{A} \xrightarrow{\mathcal{F}} \tilde{\mathcal{A}}$ ← new multiplication, "$\star$-product"

$$\mathcal{F} = \sum f_1 \otimes f_2 \in \mathcal{H}^{\otimes 2} \qquad \boxed{\alpha \circ \beta = (f_1 \triangleright \alpha)(f_2 \triangleright \beta)}$$

$\tilde{\mathcal{H}}$-mod. alg.

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$\mathcal{H} \to \mathcal{H}$, $\mathcal{H} \to \mathcal{H}^{\otimes 2}$, $\mathcal{H} \to \mathcal{H}^{\otimes 3}$, ...

increase # of deg. freed

$$(id \otimes \Delta^N) R = \mathcal{R}_{0N} \mathcal{R}_{0N-1} \cdots \mathcal{R}_{02} \mathcal{R}_{01}$$

auxiliary factor/space

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~ cocycle

$$(\varepsilon \otimes id)\mathcal{F} = (id \otimes \varepsilon)\mathcal{F} = 1 \otimes 1$$

$\gamma \to \tilde{\gamma} = u^{-1} \gamma u$, ...

$$\mathcal{H}(m, \Delta, \varepsilon, \gamma; \mathcal{R}) \xrightarrow{\mathcal{F}} \tilde{\mathcal{H}}(m, \tilde{\Delta}, \varepsilon, \tilde{\gamma}; \tilde{\mathcal{R}})$$

$$\tilde{\mathcal{R}} = \mathcal{F}_{21}^{-1} \mathcal{R} \mathcal{F}$$

$\mathcal{H}$ & $\tilde{\mathcal{H}}$ are isomorphic as an. alg.

Twist establishes an equivalence rel. among H-algs.

$\mathcal{A}$ is a left $\mathcal{H}$-mod. alg.

$$x \triangleright \alpha\beta = (x_{(1)} \triangleright \alpha)(x_{(2)} \triangleright \beta), \quad x \in \mathcal{H}; \quad \mathcal{A} \xrightarrow{\mathcal{F}} \tilde{\mathcal{A}}$$

← new multiplication, "$\ast$-product", $\tilde{\mathcal{H}}$-mod. alg.

$$\mathcal{F} = \sum f_1 \otimes f_2 \in \mathcal{H}^{\otimes 2} \quad \boxed{\alpha \circ \beta = (f_1 \triangleright \alpha)(f_2 \triangleright \beta)}$$

Ex. jordanian twist of Borel subalg. $b_2 \subset sl(2)$

$[h, z] = z, \quad \exp(h \otimes \sigma) \in U(b_2) \otimes U(b_2)$

$$\sigma = \ln(1+z)$$

TE $\quad (\Delta \otimes id)\mathcal{F} \cdot \mathcal{F}_{12} = (id \otimes \Delta)\mathcal{F}\,\mathcal{F}_{23}$ diff.

$$\exp\{(h\otimes 1 + 1 \otimes h)\otimes\sigma\}\, e^{h\otimes\sigma\otimes 1} = (id\otimes\Delta)\mathcal{F}\,\mathcal{F}_{23}$$

Factorizable twists

$$e^{h\otimes 1\otimes\sigma}\, e^{h\otimes\sigma\otimes 1} \overset{?}{=} \mathcal{F}_{23}^{-1}(id\otimes\Delta)\mathcal{F}\,\mathcal{F}_{23} :=$$

$$= (id\otimes\tilde{\Delta})\mathcal{F} = \exp(h\otimes(\sigma\otimes 1 + 1\otimes\sigma))!$$

O'k

Twisted tensor product of Hopf algebras

$\mathcal{H}^{(i)}, i = 1, 2; \quad \mathcal{H}' = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$

$\mathcal{F} \in \mathcal{H}^{(2)} \otimes \mathcal{H}^{(1)} \subset (1\otimes\mathcal{H}^{(2)})\otimes(\mathcal{H}^{(1)}\otimes 1) \subset \mathcal{H}'\otimes\mathcal{H}'$

If $\mathcal{F}$ is factorizable (à la bicharacter):

$$(\Delta^{(2)}\otimes id)\mathcal{F} = \mathcal{F}_{13}\,\mathcal{F}_{23} \in \mathcal{H}^{(2)}\otimes\mathcal{H}^{(2)}\otimes\mathcal{H}^{(1)}$$

$$(id\otimes\Delta^{(1)})\mathcal{F} = \mathcal{F}_{13}\,\mathcal{F}_{12} \in \mathcal{H}^{(2)}\otimes\mathcal{H}^{(1)}\otimes\mathcal{H}^{(1)}$$

$\mathcal{F}$ satisfies TE! $\rightarrow$

twisted $\quad \tilde{\mathcal{H}}' \equiv \mathcal{H}^{(1)} \overset{\mathcal{F}}{\otimes} \mathcal{H}^{(2)}$

Twisted tensor square of quasitr. H-alg. [R-S-TS]

$\mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$, $\mathcal{R}\in(1\otimes\underset{2}{\mathcal{H}})\otimes(\underset{3}{\mathcal{H}}\otimes 1)\subset(\mathcal{H}\otimes\mathcal{H})^{\otimes 2}$

univ. R-matrix of $\mathcal{H}^{\otimes 2}$ is $\mathbb{R} = \mathcal{R}_{31}^{-1}\mathcal{R}_{24}\in\mathcal{H}^{\otimes 4}$

— " — " — " of twisted $\otimes$ square   factor indices

$$\tilde{\mathbb{R}} = \tilde{\mathcal{F}}_{21}^{-1}\,\mathbb{R}\,\tilde{\mathcal{F}} = \mathcal{R}_{41}^{-1}\mathcal{R}_{31}^{-1}\mathcal{R}_{24}\mathcal{R}_{23}$$

Let introduce notation: $\mathcal{H}^{\tilde{\otimes}2} := \mathcal{H}\overset{\mathcal{R}}{\otimes}\mathcal{H}$

Prop. $\mathcal{H}^{\tilde{\otimes}2}$ is twist equivalent to $\mathcal{H}^{op}\otimes\mathcal{H}$

via cocycle $\mathcal{F} = \mathcal{R}_{13}\mathcal{R}_{23}$

$$\begin{array}{c} \downarrow \mathcal{R}_{13} \\ \mathcal{H}\otimes\mathcal{H} \\ \downarrow \mathcal{R}_{23} \\ \mathcal{H}^{\tilde{\otimes}2} \end{array}$$

Twisted $\otimes$ of H-module algebras

$\mathcal{H}^{(i)}$ $i=1,2$, $\mathcal{A}^{(i)}$ are corresp. mod. algs.

$\mathcal{H}' := \mathcal{H}^{(1)}\overset{\mathcal{F}}{\otimes}\mathcal{H}^{(2)}$ by a bicharacter $\mathcal{F}\in\mathcal{H}^{(2)}\otimes\mathcal{H}^{(1)}$

$\mathcal{H}'$ - module algebra $\mathcal{A}^{(1)}\underset{\mathcal{F}}{\otimes}\mathcal{A}^{(2)} \sim \mathcal{A}^{(1)}\otimes\mathcal{A}^{(2)}$ as module

$$(a_1^{(1)}\otimes a_1^{(2)})\,\tilde{\cdot}\,(a_2^{(1)}\otimes a_2^{(2)}) = (a_1^{(1)} f_2\triangleright a_2^{(1)})\otimes(f_1\triangleright a_1^{(2)})\,a_2^{(2)}$$

$$\mathcal{F} = (1\otimes f_1)\otimes(f_2\otimes 1)\in(1\otimes\mathcal{H}^{(2)})\otimes(\mathcal{H}^{(1)}\otimes 1)\subset(\mathcal{H}^{(1)}\otimes\mathcal{H}^{(2)})^{\otimes 2}$$

Reflection Equation dual algebra $\mathcal{H}^*$

$\mathcal{H}'$ is quasitriag. H-alg.; $\mathcal{R}'$, $\mathcal{A}$ is l-$\mathcal{H}'$-m.alg

Def. $\mathcal{A}$ is quasi-commutative if

$$(\mathcal{R}'_2 \blacktriangleright b)(\mathcal{R}'_1 \blacktriangleright a) = \underbrace{ab}_{\text{multiplic. in } \mathcal{A}}$$

Prop. If $\mathcal{H}' \xrightarrow{\mathcal{F}} \tilde{\mathcal{H}}'$ and $\mathcal{A}$ is quasi-commut. $\mathcal{H}'$-mod. alg., then $\tilde{\mathcal{A}}_{\mathcal{F}}$ — " — " — " also.

This is valid for twisted $\otimes$ of $\mathcal{H}^{(i)}$: $\mathcal{H}^{(1)} \overset{\mathcal{F}}{\otimes} \mathcal{H}^{(2)}$.

Dual to (quasitriag.) H-algebra $\mathcal{H} \to \mathcal{H}^*$ is $\mathcal{H}$-bimodule w.r.t. coregular actions (lin. functionals)

$$x \triangleright \alpha = \alpha_{(1)} \langle x, \alpha_{(2)} \rangle, \quad \alpha \triangleleft y = \langle y, \alpha_{(1)} \rangle \alpha_{(2)}.$$

$\mathcal{H}^*$ is left $\underbrace{\mathcal{H}^{op} \otimes \mathcal{H}}$-module algebra

$\leftarrow$ univ. R-matrix $\mathcal{R}^{-1}_{13} \mathcal{R}_{24}$

Action of $\mathcal{H}^{op} \otimes \mathcal{H}$!

$$(x \otimes y) \blacktriangleright \alpha = y \triangleright \alpha \triangleleft \gamma(x), \quad \alpha \in \mathcal{H}^*$$

quasi-commutative: $\underline{(\mathcal{R}_2 \triangleright \beta)(\mathcal{R}_1 \triangleright \alpha) = (\alpha \triangleleft \mathcal{R}_1)(\beta \triangleleft \mathcal{R}_2)}$ (*)

$\mathcal{T} = \sum e_i \otimes e^i \in \mathcal{H} \otimes \mathcal{H}^*$, $\sum_j e_j \otimes e^j \triangleleft x = (x \otimes 1) \sum_j e_j \otimes e^j$ $\mathcal{T}$!

$$(\mathcal{R}_1 \otimes \mathcal{R}_2)\, \mathcal{T}_1 \mathcal{T}_2 = \mathcal{T}_2 \mathcal{T}_1 (\mathcal{R}_1 \otimes \mathcal{R}_2)$$

Universal K-matrix $\mathcal{K}$ and fusion

$\mathcal{H}^*$ is $\mathcal{H}^{op} \otimes \mathcal{H}$ - module algebra

Twist of $\mathcal{H}^{op} \otimes \mathcal{H}$ by $\mathcal{R}_{13}\mathcal{R}_{23} \rightsquigarrow \mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H} \equiv \mathcal{H}^{\tilde{\otimes} 2}$

corresp. twisting of module algebra $\tilde{\mathcal{H}}^*$

$\tilde{\mathcal{H}}^*$ is quasi-commutative $\mathcal{H} \overset{\mathcal{R}}{\otimes} \mathcal{H}$-mod. algebra

$$(\mathcal{R}_{1'} \triangleright \alpha \triangleleft \mathcal{R}_2)(\beta \triangleleft \mathcal{R}_1 \triangleleft \mathcal{R}_{2'}) = (\mathcal{R}_{1'} \triangleright \mathcal{R}_2 \triangleright \beta)(\mathcal{R}_1 \triangleright \alpha \triangleleft \mathcal{R}_{2'})$$

(dual basis of $\mathcal{H}^*$ e^i e^j $\sum e_i \otimes e_j \otimes \tilde{\mathcal{H}}^*$) $\mathcal{H} \otimes \mathcal{H} \otimes \tilde{\mathcal{H}}^*$

$$\boxed{\mathcal{R}_{21}\mathcal{K}_1\mathcal{R}_{12}\mathcal{K}_2 = \mathcal{K}_2\mathcal{R}_{21}\mathcal{K}\mathcal{R}_{12}} \quad \text{URE}$$

$$\boxed{\mathcal{K} = \sum e_i \otimes e^i} \in \mathcal{H} \otimes \tilde{\mathcal{H}}^* \quad \text{universal K-matrix}$$

Fusion

$$(\Delta \otimes id)\mathcal{K} = \mathcal{R}^{-1}\mathcal{K}_1\mathcal{R}\mathcal{K}_2 \xrightarrow{\Delta^{op}} \text{URE}$$

Irrep of $\mathcal{H} \xrightarrow{\rho} \operatorname{End} V$, $\rho^{\otimes 2}(\Delta \otimes id)\mathcal{K} = R_{12}^{-1} K_1 R_{12} K_2$,

K is $n \times n$ matrix with entries $\in$ REA $\tilde{\mathcal{H}}^*$

Projection: $P_+ (\check{R}_{12})^{-1} K_2 \check{R}_{12} K_2 \sim P_+ \underbrace{K_2 \check{R}_{12} K_2}$

$V \otimes V = V_+ \oplus \dots$

Br-group rep.

$[\check{R}, \Delta] = 0$

kenmosh receptie

$x\_ \otimes n \dots$

~90 [MN, KSkl]

Characteristic eq. of Universal K-matrix

$$\mathcal{K} = \sum_i e_i \otimes e^i \in \mathcal{H} \otimes \tilde{\mathcal{H}}^*$$ „Bicharacter"

$$(\Delta \otimes id)\mathcal{K} = \sum_i \Delta(e_i) \otimes e^i = \mathcal{R}^{-1}\mathcal{K}_1 \mathcal{R} \mathcal{K}_2 \in \mathcal{H} \otimes \mathcal{H} \otimes \tilde{\mathcal{H}}^*$$

$$\mathcal{R}\,\Delta = \Delta^{op}\mathcal{R} \qquad \Leftarrow \quad \mathcal{R}_{12} \times$$

$$(\Delta^{op} \otimes id)\mathcal{K} = \mathcal{R}_{21}^{-1}\mathcal{K}_2 \mathcal{R}_{21} \mathcal{K}_1 \qquad \times \mathcal{R}_{12}$$

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$$

$$\underbrace{(\Delta \otimes id \otimes id)}_{(\Delta \otimes id)}\mathcal{K} = (\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23})^{-1}\mathcal{K}_1 \mathcal{R}_{12}\mathcal{K}_2 \times$$

$$\times\, \mathcal{R}_{13}\mathcal{R}_{23}\mathcal{K}_3 \,;$$

$$(\Delta^{(N)} \otimes id)\mathcal{K} = \left(\prod \mathcal{R}_{ij}\right)^{-1}\mathcal{K}_1 \prod \mathcal{R}\,\mathcal{K}_2 \cdots \prod \mathcal{R}\,\mathcal{K}_N$$

structure of N-fold map: $\mathcal{H} \otimes \tilde{\mathcal{H}}^* \to \mathcal{H}^{\otimes N} \otimes \tilde{\mathcal{H}}^*$

Lattice current algebras

Twist of $\mathcal{H}^{op}\otimes\mathcal{H} \xrightarrow{R_{13}} \mathcal{H}\otimes\mathcal{H}$

Corresponding twist of $\mathcal{H}^{op}\otimes\mathcal{H}$-mod. algebra $\mathcal{H}^* \to \hat{\mathcal{H}}^*$

$a, b \in \mathcal{H}^*$ $\quad m(a\otimes b) = \hat{m}\,(a \triangleleft \gamma(\mathcal{R}_1)) \otimes (b \triangleleft \mathcal{R}_2)$

$\mathcal{L} := \sum_i e_i \otimes e^i \in \mathcal{H}\otimes\hat{\mathcal{H}}^*$, $\quad \boxed{\Delta\mathcal{L} = \mathcal{R}_{12}^{-1}\mathcal{L}_1\mathcal{L}_2}$

$\mathcal{R}\,\Delta(\cdot) = \Delta^{op}(\cdot)\,\mathcal{R} \quad \rightarrow \quad \boxed{\mathcal{R}_{21}\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2\mathcal{L}_1\mathcal{R}_{12}}$

Twisted $\otimes$ of $\mathcal{H}^{\otimes 2}$-module algebras: $(\hat{\mathcal{H}}^*)^{\hat{\otimes} n}$

$\mathcal{L}^{(i)}$ the embedding of $\mathcal{L}$ into $\mathcal{H}\otimes(\hat{\mathcal{H}}^*)^{\hat{\otimes} n}$ i-th factor

" $\sum e_k \otimes \underset{\in}{e^k}$ " ← i-th factor

<u>Prop.</u> The elements $\mathcal{L}^{(i)}$ satisfy the lattice current alg.

$$\mathcal{R}_{21}\mathcal{L}_1^{(i)}\mathcal{L}_2^{(i)} = \mathcal{L}_2^{(i)}\mathcal{L}_1^{(i)}\mathcal{R},$$

$$\mathcal{L}_2^{(i)}\mathcal{R}^{-1}\mathcal{L}_1^{(i+1)} = \mathcal{L}_1^{(i+1)}\mathcal{L}_2^{(i)},$$

$$\mathcal{L}_2^{(i)}\mathcal{L}_1^{(j)} = \mathcal{L}_1^{(j)}\mathcal{L}_2^{(i)}, \quad |i-j|\geq 2.$$

[AFS-TSK]
~1990
CMP