Correlation functions of the XXZ spin-1/2 chain: recent progress

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Quantum integrable models

Interests

- Exacts results not accessible by usual techniques
- Direct applications : condensed matter, solid state physics,...
- Mathematics : quantum groups, knot theory,...

What can we compute?

Hamiltonian spectrum, scattering matrix , partition function, critical exponents,....
(Bethe, Onsager, Yang, Baxter, McCoy, Faddeev, Zamolodchikov,...)

Correlation functions?

$$\langle \ \mathcal{O} \ \rangle = \frac{\mathrm{tr}_{\mathcal{H}} \left(\mathcal{O} \, e^{-\mathrm{H}/kT} \right)}{\mathrm{tr}_{\mathcal{H}} \left(e^{-\mathrm{H}/kT} \right)}$$

Correlation functions

Why is it so difficult? (Bethe ansatz already 70 years old...!)

For example at T=0 only the ground state $|\psi_g\rangle$ of H contributes :

$$\langle\psi_g|\, \mathcal{O}\, |\psi_g\rangle$$

Three main problems to be solved :

- Compute exact eigenstates and energy levels of the Hamiltonian (Bethe ansatz)
- Obtain the action of local operators on the eigenstates : main problem since eigenstates are highly non-local!
- Compute the resulting scalar products with the eigenstates

Beyond Ising type models (there, use the free fermion algebra)...?
(Lieb, Shultz, Mattis, Wu, McCoy,...)

A very brief history...

Three main lines of approach have been developped :

- Bootstrap methods for the form factors + sums over all eigenstates (Karowski, Weisz, Smirnov,....)
- Infinite volume + non-abelian quantum symmetries (Jimbo, Miwa,....)
- Finite volume + algebraic Bethe ansatz (Izergin, Korepin,...)

$\longrightarrow$ More recently a new approach :
Algebraic Bethe ansatz in finite volume + solution of the quantum inverse scattering problem (Kitanine, Maillet, Terras)

$\longrightarrow$ At zero temperature, and in the thermodynamic limit, it leads to multiple integral representations of the correlation functions for the XXZ spin-1/2 chain

The spin-1/2 XXZ Heisenberg chain

$$H = \sum_{m=1}^{M} \left(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta \left(\sigma_m^z \sigma_{m+1}^z - 1 \right) \right) - \frac{h}{2} \sum_{m=1}^{M} \sigma_m^z$$

- Hamiltonian eigenstates

  Algebraic Bethe ansatz : $\sigma_m^\alpha \longrightarrow T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$

  $$T(\lambda) \equiv T_{a,1\ldots N}(\lambda) = L_{aN}(\lambda - \xi_N) \ldots L_{a1}(\lambda - \xi_1)$$

  $L_{an}(\lambda)$ being 2×2 matrices with entries function of $\sigma_n^{x,y,z}$ operators in site n.

  Yang-Baxter algebra : $R_{12}(\lambda_1, \lambda_2) T_1(\lambda_1) T_2(\lambda_2) = T_2(\lambda_2) T_1(\lambda_1) R_{12}(\lambda_1, \lambda_2)$

  Commuting conserved charges : $t(\lambda) = A(\lambda) + D(\lambda), \qquad [t(\lambda), t(\mu)] = 0$

Hamiltonian : $H = 2\sinh\eta \, \frac{\partial}{\partial\lambda} \log t(\lambda)\big|_{\lambda=\frac{\eta}{2}} + c$ for all $\xi_j = 0$.

Eigenstates of $t(\mu)$: $|\psi\rangle = \prod_k B(\lambda_k)|0\rangle$ with $\{\lambda_k\}$ solution of the Bethe equations.

- Action of local operators on eigenstates

Resolution of the quantum inverse scattering problem : $\sigma_m^\alpha \longleftarrow T(\lambda)$

$$\sigma_j^- = \prod_{k=1}^{j-1} t(\xi_k) \cdot B(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k),$$
$$\sigma_j^+ = \prod_{k=1}^{j-1} t(\xi_k) \cdot C(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k),$$
$$\sigma_j^z = \prod_{k=1}^{j-1} t(\xi_k) \cdot (A - D)(\xi_j) \cdot \prod_{k=1}^{j} t^{-1}(\xi_k),$$

+ Yang-Baxter algebra for A, B, C, D to get the action on arbitrary states, for example

$$\langle 0| \prod_{k=1}^{N} C(\lambda_k)\, A(\lambda_{N+1}) = \sum_{a'=1}^{N+1} \Lambda_{a'}\, \langle 0| \prod_{\substack{k=1 \\ k\neq a'}}^{N+1} C(\lambda_k)$$

- Scalar products :

$$\langle 0|\ \prod_{j=1}^{N} C(\mu_j)\ \prod_{k=1}^{N} B(\lambda_k)\ |0\rangle = \frac{\det U(\{\mu_j\},\{\lambda_k\})}{\det V(\{\mu_j\},\{\lambda_k\})}$$

for $\{\lambda_k\}$ a solution of Bethe equations and $\{\mu_j\}$ an arbitrary set of parameters, :

$$U_{ab} = \partial_{\lambda_a}\tau(\mu_b,\{\lambda_k\}), \qquad V_{ab} = \frac{1}{\sinh(\mu_b - \lambda_a)}, \qquad 1 \leqslant a, b \leqslant N,$$

where $\tau(\mu_b,\{\lambda_k\})$ is the eigenvalue of the transfer matrix $t(\mu_b)$

Matrix elements of local operators

For example :

$$\langle 0| \prod_{j=1}^{N} C(\mu_j)\ \sigma_n^z \prod_{k=1}^{N} B(\lambda_k)\ |0\rangle =$$

$$= \langle 0| \prod_{j=1}^{N} C(\mu_j) \prod_{k=1}^{n-1} t(\xi_k) \cdot \big(A - D\big)(\xi_j) \cdot \prod_{k=1}^{n} t^{-1}(\xi_k) \prod_{k=1}^{N} B(\lambda_k)\ |0\rangle$$

Here the sets $\{\lambda_k\}$ and $\{\mu_j\}$ are both solutions of Bethe equations $\longrightarrow$

$$\langle 0| \prod_{j=1}^{N} C(\mu_j)\ \sigma_n^z \prod_{k=1}^{N} B(\lambda_k)\ |0\rangle = \Phi_n\ \langle 0| \prod_{j=1}^{N} C(\mu_j)\ \big(A - D\big)(\xi_j) \prod_{k=1}^{N} B(\lambda_k)\ |0\rangle$$

Hence it leads to determinant representations of these matrix elements (using the scalar product formula)

Correlation functions : elementary blocks

$$F_m(\{\epsilon_j, \epsilon'_j\}) = \frac{\langle \psi_g | \prod_{j=1}^{m} E_j^{\epsilon'_j, \epsilon_j} | \psi_g \rangle}{\langle \psi_g | \psi_g \rangle} \qquad E_{lk}^{\epsilon', \epsilon} = \delta_{l,\epsilon'} \delta_{k,\epsilon}$$

Solution of the quantum inverse scattering problem + Yang-Baxter algebra of operators $T(\lambda)$ $\longrightarrow$ Multiple integral formula for the correlation functions

$$F_m(\{\epsilon_j, \epsilon'_j\}) = (\prod_{k=1}^{m} \int_{C_k^h} d\lambda_k)\ \ \Omega_m(\{\lambda_k\},\ \{\epsilon_j, \epsilon'_j\})\ S_h(\{\lambda_k\})$$

where $\Omega_m(\{\lambda_k\},\ \{\epsilon_j, \epsilon'_j\})$ is purely algebraic and $S_h(\{\lambda_k\})$, C_k^h are depending on the regime and the magnetic field h.

$\longrightarrow$ Proof of the results and conjectures of Jimbo, Miwa et al. and extension to the non zero magnetic field h (a case where the quantum affine symmetry is broken)

Physical correlation functions

Example :

$$\langle \psi_g | \sigma_1^+ \, \sigma_{m+1}^- | \psi_g \rangle \equiv \langle \psi_g | E_1^{12} \prod_{j=2}^{m} (E_j^{11} + E_j^{22}) \, E_{m+1}^{21} | \psi_g \rangle$$

$\longrightarrow$ sum of 2^{m-1} elementary blocks: $\langle \psi_g | C(\frac{\eta}{2}) \, (A+D)^{m-1}(\frac{\eta}{2}) \, B(\frac{\eta}{2}) | \psi_g \rangle$

Compact and symmetrized formula for the multiple action of $A + D$ operators

$\longrightarrow$ Two point correlation functions :

$$\langle \sigma_1^\alpha \sigma_{m+1}^\beta \rangle = \sum_{n=0}^{m-1} \oint_{C_z} d^{n+1} z \int_{C_\lambda} d^n \lambda \int_{C_\mu} d^2 \mu \; [f(\{\lambda, z\})]^m \; \Gamma_n^{\alpha\beta}(\{\lambda, \mu, z\}) \; S_h(\{\lambda, z\})$$

$\longrightarrow$ Gives directly known free fermion results setting $\Delta = 0$

$\longrightarrow$ Asymptotics of two point functions for general Δ?

Emptiness formation probability

$$\tau(m) = \langle \psi_g | \prod_{k=1}^{m} \frac{1-\sigma_k^z}{2} | \psi_g \rangle$$

for $\Delta = \cos\zeta,\ 0 < \zeta < \pi$:

$$\tau(m) = \lim_{\xi_1,\ldots\xi_m \to -\frac{i\zeta}{2}} \frac{1}{m!} \int_{-\infty}^{\infty} \frac{Z_m(\{\lambda\},\{\xi\})}{\prod_{a<b}^{m} \sinh(\xi_a - \xi_b)} \det_m \left(\frac{i}{2\zeta \sinh \frac{\pi}{\zeta}(\lambda_j - \xi_k)} \right) d^m\lambda$$

$$Z_m(\{\lambda\},\{\xi\}) = \prod_{a=1}^{m} \prod_{b=1}^{m} \frac{\sinh(\lambda_a - \xi_b)\sinh(\lambda_a - \xi_b - i\zeta)}{\sinh(\lambda_a - \lambda_b - i\zeta)} \cdot \frac{\det_m \left(\frac{-i\sin\zeta}{\sinh(\lambda_j - \xi_k)\sinh(\lambda_j - \xi_k - i\zeta)} \right)}{\prod_{a>b}^{m} \sinh(\xi_a - \xi_b)}$$

For $\zeta = \pi/3$: $\tau(m) = \left(\frac{1}{2}\right)^{m^2} \prod_{k=0}^{m-1} \frac{(3k+1)!}{(m+k)!} \to c \left(\frac{\sqrt{3}}{2}\right)^{3m^2} m^{-\frac{5}{36}}, \quad m \to \infty$

Idea of the proof : Observe first that for $\zeta = \pi/3$,

$$Z_m(\{\lambda\},\{\xi\}) = \frac{(-1)^{\frac{m^2-m}{2}}}{2^{m^2+m}} \prod_{a>b}^{m} \frac{\sinh 3(\xi_b - \xi_a)}{\sinh(\xi_b - \xi_a)\sinh(\xi_a - \xi_b)}$$

$$\times \det_m\left(\frac{1}{\sinh(\lambda_j - \xi_k)\sinh(\lambda_j - \xi_k - i\zeta)}\right) \frac{\det_m\left(\frac{1}{\sinh(\lambda_j - \xi_k + \frac{i\pi}{3})}\right)}{\det_m\left(\frac{1}{\sinh 3(\lambda_j - \xi_k)}\right)}.$$

Using $\sinh(3x) = 4\sinh(x)\sinh(x + i\pi/3)\sinh(x - i\pi/3)$.

$$\tau(m,\{\xi_j\}) = \left(\frac{3i}{4\pi}\right)^m \frac{(-1)^{\frac{m^2-m}{2}}}{2^{m^2} m!} \prod_{a>b}^{m} \frac{\sinh 3(\xi_b - \xi_a)}{\sinh(\xi_b - \xi_a)} \prod_{\substack{a,b=1\\a\neq b}}^{m} \sinh^{-1}(\xi_a - \xi_b)$$

$$\times \int_{-\infty}^{\infty} d^m\lambda \quad \det_m\left(\frac{1}{\sinh(\lambda_j - \xi_k + \frac{i\pi}{3})}\right) \det_m\left(\frac{1}{\sinh(\lambda_j - \xi_k)\sinh(\lambda_j - \xi_k - \frac{i\pi}{3})}\right)$$

$$\tau(m,\{\varepsilon_j\}) = (-1)^{\frac{m^2-m}{2}} 3^m 2^{-m^2} \prod_{a>b}^{m} \frac{\sinh 3(\varepsilon_b-\varepsilon_a)}{\sinh(\varepsilon_b-\varepsilon_a)} \prod_{\substack{a,b=1\\a\neq b}}^{m} \frac{1}{\sinh(\varepsilon_a-\varepsilon_b)}$$

$$\times \quad \det_m \left(\int_{-\infty}^{\infty} \frac{d\lambda}{4\pi\cosh(\lambda-\varepsilon_j)\sinh(\lambda-\varepsilon_k-\frac{i\pi}{6})\sinh(\lambda-\varepsilon_k+\frac{i\pi}{6})} \right)$$

$$\tau(m,\{\varepsilon_j\}) = \frac{(-1)^{\frac{m^2-m}{2}}}{2^{m^2}} \prod_{a>b}^{m} \frac{\sinh 3(\varepsilon_b-\varepsilon_a)}{\sinh(\varepsilon_b-\varepsilon_a)} \prod_{\substack{a,b=1\\a\neq b}}^{m} \frac{1}{\sinh(\varepsilon_a-\varepsilon_b)} \cdot \det_m \left(\frac{3\sinh\frac{\varepsilon_j-\varepsilon_k}{2}}{\sinh\frac{3(\varepsilon_j-\varepsilon_k)}{2}} \right)$$

Take the homogeneous limit $\varepsilon_j \longrightarrow 0$ ($\xi_k = \varepsilon_k - i\pi/6$) :

$$\tau(m) = (-1)^{\frac{m^2-m}{2}} 3^{\frac{m^2+m}{2}} 2^{-m^2} \prod_{n=0}^{m-1} (n!)^{-2} \det_m \left[\frac{\partial^{j+k-2}}{\partial x^{j+k-2}} \frac{\sinh\frac{x}{2}}{\sinh\frac{3x}{2}} \right]_{x=0}$$

Can be computed using Kuperberg determinant identity

For arbitrary ζ :

$$\lim_{m\to\infty}\frac{\log\tau(m)}{m^2}=\log\frac{\pi}{\zeta}+\frac{1}{2}\int_{\mathbb{R}-i0}\frac{d\omega}{\omega}\frac{\sinh\frac{\omega}{2}(\pi-\zeta)\cosh^2\frac{\omega\zeta}{2}}{\sinh\frac{\pi\omega}{2}\sinh\frac{\omega\zeta}{2}\cosh\omega\zeta},$$

where $\cosh\eta=\cos\zeta=\Delta$, $0<\zeta<\pi$.

$$\lim_{m\to\infty}\frac{\log\tau(m)}{m^2}=-\frac{1}{2}\log 2,\qquad \Delta=0,$$
$$\lim_{m\to\infty}\frac{\log\tau(m)}{m^2}=\frac{3}{2}\log 3-3\log 2,\qquad \Delta=\frac{1}{2},$$

For the XXX chain ($\Delta=1$, $\zeta=0$):

$$\lim_{m\to\infty}\frac{\log\tau(m)}{m^2}=\log\left(\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})}\right)\approx\log(0.5991),$$

Idea of the proof :

$$\tau(m) = \left(\frac{i}{2\zeta\sin\zeta}\right)^m \left(\frac{\pi}{\zeta}\right)^{\frac{m^2-m}{2}} \int_{\mathcal{D}} d^m\lambda \cdot F(\{\lambda\}, m)$$

$$\times \prod_{a>b}^{m} \frac{\sinh\frac{\pi}{\zeta}(\lambda_a - \lambda_b)}{\sinh(\lambda_a - \lambda_b - i\zeta)\sinh(\lambda_a - \lambda_b + i\zeta)} \prod_{a=1}^{m} \left(\frac{\sinh(\lambda_a - \frac{i\zeta}{2})\sinh(\lambda_a + \frac{i\zeta}{2})}{\cosh\frac{\pi}{\zeta}\lambda_a}\right)^m,$$

with

$$F(\{\lambda\}, m) = \lim_{\xi_1,\dots\xi_m \to -\frac{i\zeta}{2}} \frac{1}{\prod\limits_{a>b}^{m}\sinh(\xi_a - \xi_b)} \det_m\left(\frac{-i\sin\zeta}{\sinh(\lambda_j - \xi_k)\sinh(\lambda_j - \xi_k - i\zeta)}\right)$$

Integration domain $\mathcal{D} \equiv -\infty < \lambda_1 < \lambda_2 < \cdots < \lambda_m < \infty$. Then at the saddle point distribution of λ's can be described by a density function $\rho(\lambda')$:

$$\rho(\lambda'_j) = \lim_{m\to\infty} \frac{1}{m(\lambda'_{j+1} - \lambda'_j)}$$

Thus for large m, one can replace sums over the set $\{\lambda'\}$ by integrals. Now estimate the behavior of $F(\{\lambda'\}, m)$ for large m using :

$$\det_m \left(\frac{-i \sin\zeta}{\sinh(\lambda'_j - \xi_k)\sinh(\lambda'_j - \xi_k - i\zeta)} \right)$$

$$\longrightarrow (-2\pi i)^m \det_m \Big(\delta_{jk} - \frac{K(\lambda'_j - \lambda'_k)}{2\pi i m \rho(\lambda'_k)} \Big) \det_m \Big(\frac{i}{2\zeta \sinh \frac{\pi}{\zeta}(\lambda'_j - \xi_k)} \Big),$$

with Lieb kernel

$$K(\lambda) = \frac{-i \sin 2\zeta}{\sinh(\lambda - i\zeta)\sinh(\lambda + i\zeta)}$$

Hence :

$$\tau(m) \longrightarrow \left(\frac{\pi}{\zeta} \right)^{m^2} e^{m^2 S_0 + o(m^2)}, \qquad m \to \infty,$$

with

$$S_0 = \int_{-\infty}^{\infty} d\lambda \rho(\lambda) \log\left(\frac{\sinh(\lambda - i\zeta/2)\sinh(\lambda + i\zeta/2)}{\cosh^2\frac{\pi}{\zeta}\lambda}\right)$$

$$+\frac{1}{2}\int_{-\infty}^{\infty} d\mu d\lambda \rho(\lambda)\rho(\mu) \log\left(\frac{\sinh^2\frac{\pi}{\zeta}(\lambda-\mu)}{\sinh(\lambda-\mu-i\zeta)\sinh(\lambda-\mu+i\zeta)}\right)$$

and the integral equation for the density $\rho(\lambda)$

$$\frac{2\pi}{\zeta}\tanh\frac{\pi\lambda}{\zeta} - \coth(\lambda - i\zeta/2) - \coth(\lambda + i\zeta/2)$$

$$= V.P.\int_{-\infty}^{\infty}\left(\frac{2\pi}{\zeta}\coth\frac{\pi}{\zeta}(\lambda-\mu) - \coth(\lambda-\mu-i\zeta) - \coth(\lambda-\mu+i\zeta)\right)\rho(\mu)\,d\mu$$

Solution :

$$\rho(\lambda) = \frac{\cosh\frac{\pi\lambda}{2\zeta}}{\zeta\sqrt{2}\cosh\frac{\pi\lambda}{\zeta}},$$

Conclusions and Perspectives

New method to obtain correlation functions of quantum integrable models

- Generic tools for a large class of models
- Explicit results for the Heisenberg spin chains

Open new perspectives

- Asymptotic behavior of correlation functions (under study)
- Dynamical correlation functions and depending on temperature (under study)
- Applications to many different models (models with impurities, with boundaries, field theories,...)