AdS/CFT anomalies from the functional Schroedinger equation.

The Maldacena conjecture is tested at sub-leading order by computing the Weyl anomaly for the boundary CFT at one-loop in the bulk Supergravity theory using the functional Schroedinger equation.

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$$\int \mathcal{D}A\, e^{i\{S_{YM}+\int d^4x\, j\Omega(A)\}} = \int \mathcal{D}\Phi\, e^{iS_{IIB}}\Big|_{\Phi(y=\epsilon)=\hat{\Phi}}$$

Boundary sources perturb the AdS metric.

e.g. the metric on the boundary of AdS is

$$G_{ij}|_{y=\epsilon} = \frac{\eta_{ij}}{\epsilon^2}$$

Taking $\Omega =$ Yang-Mills Stress-Tensor gives j the interpretation of being a perturbation of boundary metric.

The Yang-Mills theory sees a curved metric, g_{ij}.

g_{ij} is not dynamical.

In the bulk theory we have to solve Einstein's equation with new boundary condition,

$$G_{ij}|_{y=\epsilon} = g_{ij}$$

The S^5 part of the compactification goes through as before.

Maldacena's conjecture $\Rightarrow$

$$e^{iF_{YM}[g_{ij}]} = \int \mathcal{D}A..e^{iS_{YM}[g_{ij}]} = \int \mathcal{D}\Phi..e^{iS_{IIB\text{String}}}$$

for large N small g_s this can be calculated in SUGRA perturbation theory as a saddle-point approx:

$$\sim \int \mathcal{D}\phi..e^{iS_{IIB\text{SUGRA}}} \sim e^{iS_{EH}} \prod \text{Det..}$$

$$\equiv \text{Tree level} \times 1\,\text{loop} + ..\text{in bulk}$$

to leading order

$$F_{YM}[g_{ij}] \sim S_{EH}[G_{\mu\nu}]\,, \qquad G_{\mu\nu}\,\text{on shell}$$

$$g_{ij} = G_{ij}|_{y=\epsilon}$$

Use $F_{YM}[g_{ij}] \sim S_{EH}[G_{\mu\nu}]$ to **test the Maldacena conjecture.**

- **Weyl Anomaly**: for scaling $\delta g_{ij} = 2\delta\sigma g_{ij}$ the change in $F_{YM}[g]$ is known exactly

$$\delta F_{YM} = \frac{N^2-1}{16\pi^2}\int d^4x\,\sqrt{g}\,(6s+2f+v_g)\delta\sigma$$

s, f, v_g scalar, fermion , gauge vector heat kernel coefficients.

$$\delta F_{YM} = \int d^4x\,\sqrt{g}\,(a\,E + c\,I)\,\delta\sigma$$

$$E = (\mathrm{Riem}^2 - 4\,\mathrm{Ric}^2 + R^2)/64$$

$$I = (-\mathrm{Riem}^2 + 2\,\mathrm{Ric}^2 - R^2/3)/64$$

$$a = c = -(N^2-1)/\pi^2, \quad \text{1 loop exact}$$

- For the gravity calculation Henningson and Skenderis solve boundary value problem in perturbation theory to obtain

$$\delta S_{EH}[G] = \int d^4x\,\sqrt{g}\,(a\,E + c\,I)\,\delta\sigma$$

$$a = c = -N^2/\pi^2 \quad \text{Agreement to leading order!}$$

Rather than perturbation theory use exact solution to the bulk Einstein equation $[\eta_{ij} \to e^{\rho}\widehat{g}_{ij}]$

$$ds^2 = G_{\mu\nu}dX^{\mu}dX^{\nu} = \frac{l^2 dy^2}{y^2} + \frac{e^{\rho}\widehat{g}_{ij}dx^i dx^j}{y^2}$$

$$e^{\rho} = (1 - Cy^2)^2, \quad C = l^2\widehat{R}/48$$

where $\widehat{g}$ is a 4-dimensional Einstein-metric ($\widehat{R}_{ij} = \widehat{R}\widehat{g}_{ij}/4$ with constant $\widehat{R}$). The Einstein-Hilbert action evaluated in this metric is

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N}\int d^5X\sqrt{G}\left(R^{(5)} - \frac{12}{l^2}\right)$$

$$= -\frac{1}{2\pi G_N l}\int_{y>\epsilon}\frac{dy\, d^4x}{y^5}\sqrt{\widehat{g}}\,(1 - Cy^2)^4$$

Varying the cut-off, ϵ changes the boundary metric $g_{ij} \approx \widehat{g}_{ij}/\epsilon^2$ by $\delta g_{ij} = -2\,(\delta\epsilon/\epsilon)\,g_{ij} = 2\,\delta\sigma\,g_{ij}$

$$\delta S_{EH} = -\delta\sigma\,\epsilon\frac{\partial}{\partial\epsilon}S_{\mathrm{EH}} = -\frac{\delta\sigma}{2\pi G_N l}\int\frac{d^4x}{\epsilon^4}\sqrt{\widehat{g}}\,(1 - C\epsilon^2)^4$$

$$= -\frac{\delta\sigma}{2\pi G_N l}\int d^4x\sqrt{\widehat{g}}\left(.. + 3C^2 + ..\right)$$

$$= .. - \frac{\delta\sigma N^2}{\pi^2}\int d^4x\,\sqrt{g}\,(E + I) + ..$$

Using $G_N = \pi l^3/(2N^2)$, and $C = l^2\widehat{R}/48$.

$$\delta S_{EH} = -\frac{N^2}{\pi^2} \int d^4x \sqrt{g}\,(E + I)\,\delta\sigma\,,$$

$$\delta F_{YM} = -\frac{N^2-1}{\pi^2} \int d^4x \sqrt{g}\,(\,E +\ I)\,\delta\sigma$$

Agreement to order $1/N^2$ between 1-loop SYM and a calculation in classical gravity!

But this does not test the full content of IIB Superstrings.

For a more stringent test check the sub-leading contribution to Weyl anomaly, i.e. $-\mathbf{1}$.

Approximate Superstrings by IIB Supergravity

From 1-loop SUGRA in bulk

$$e^{iF_{YM}} \sim e^{iS_{EH}} \prod \text{Det}$$

One loop contributions come from quadratic fluctuations $\sim$ free fields, all we need to know is their masses

Mass spectrum. The supermultiplets (irreps of U(2,2/4)) are labelled by the integer p. The (a,b,c) representation of $SU(4)$ has dimension $r = (a+1)(b+1)(c+1)(a+b+2)(b+c+2)(a+b+c+3)/12$, and a subscript c indicates that the representation is complex.

	Field	$SU(4)$ rep$^{\mathrm{n}}$	Mass on S^5
$p \geq 2$	$\phi^{(1)}$	$(0, p, 0)$	$m^2 = p(p-4)$
	$\psi^{(1)}$	$(0, p-1, 1)_c$	$m = p - 3/2$
	$A^{(1)}_{\mu\nu}$	$(0, p-1, 0)_c$	$m^2 = (p-1)^2$
	$\phi^{(2)}$	$(0, p-2, 2)_c$	$m^2 = (p+1)(p-3)$
	$\phi^{(3)}$	$(0, p-2, 0)_c$	$m^2 = (p+2)(p-2)$
	$\psi^{(2)}$	$(0, p-2, 1)_c$	$m = p - 1/2$
	$A^{(1)}_{\mu}$	$(1, p-2, 1)$	$m^2 = p(p-2)$
	$\psi^{(1)}_{\mu}$	$(1, p-2, 0)_c$	$m = p - 1/2$
	$h_{\mu\nu}$	$(0, p-2, 0)$	$m^2 = (p+2)(p-2)$
$p \geq 3$	$\psi^{(3)}$	$(2, p-3, 1)_c$	$m = p - 1/2$
	$\psi^{(4)}$	$(0, p-3, 1)_c$	$m = p + 1/2$
	$A^{(2)}_{\mu}$	$(1, p-3, 1)_c$	$m^2 = (p+1)(p-1)$
	$A^{(2)}_{\mu\nu}$	$(2, p-3, 0)_c$	$m^2 = p^2$
	$A^{(3)}_{\mu\nu}$	$(0, p-3, 0)_c$	$m^2 = (p+1)^2$
	$\psi^{(2)}_{\mu}$	$(1, p-3, 0)_c$	$m = p + 1/2$
$p \geq 4$	$\phi^{(4)}$	$(2, p-4, 2)$	$m^2 = (p+2)(p-2)$
	$\phi^{(5)}$	$(0, p-4, 2)_c$	$m^2 = (p+3)(p-1)$
	$\phi^{(6)}$	$(0, p-4, 0)$	$m^2 = p(p+4)$
	$\psi^{(5)}$	$(2, p-4, 1)_c$	$m = p + 1/2$
	$\psi^{(6)}$	$(0, p-4, 1)_c$	$m = p + 3/2$
	$A^{(3)}_{\mu}$	$(1, p-4, 1)$	$m^2 = p(p+2)$

- Example: Free scalar field with action

$$S = \frac{1}{2} \int d^4x \, dy \, \sqrt{G} \left(G^{\mu\nu} \, \partial_\mu \phi \, \partial_\nu \phi + m^2 \phi^2 \right)$$

- Wick rotate and compute

$$\Psi = \int \mathcal{D}\phi \, e^{-S} \qquad \text{with b.c. } \phi|_{y=\epsilon} = \varphi$$

$$= e^{-F[g] + \frac{1}{2} \int d^4x \, \varphi \Gamma \varphi} = 1 \text{ loop Det} \times \text{tree level}$$

- If $\widehat{g}_{ij} = \eta_{ij}$:

$$\Gamma \sim |x - y|^{-2\Delta}, \qquad \Delta = 2 + \sqrt{l^2 m^2 + 4}$$

- To compute the 1-loop functional determinant we need the heat-kernel for the scalar Laplacian on Anti-de Sitter space, a five dimensional manifold with boundary.

- Avoid 5-diml problems using a Hamiltonian formulation.

- Interpret

$$\Psi = \int \mathcal{D}\phi \, e^{-S} \quad \text{b.c.s } \phi|_{y=\epsilon} = \varphi$$

as the Schrödinger functional

$$T\langle \, N \, | \, e^{-\int_\epsilon^\infty dy \, H} | \, \varphi \, \rangle$$

where y is 'Euclidean time'

- Rewrite 5 diml theory in terms of 4 dimensional quantities with known properties.

- Ψ satisfies the Schrödinger equation. **Schematically:**

$$\epsilon \frac{\partial}{\partial \epsilon} \Psi = \frac{l}{2} \int d^4x \Big\{ - \frac{\delta^2}{\delta \varphi^2} + \varphi \, \Omega \, \varphi + \tilde{m}^2 \varphi^2 \Big\} \, \Psi \, ,$$

where $\Omega \propto \Box + \hat{R}/6$ is an operator in 4 dimensions and $l\tilde{m} = \Delta - 2$

- Substitute $\Psi = \exp(-F + \frac{1}{2}\int d^4x\, \varphi\, \Gamma\, \varphi)$ into the Schrödinger equation

$$\epsilon\frac{\partial}{\partial\epsilon}\Psi = \frac{l}{2}\int d^4x \Big\{ -\frac{\delta^2}{\delta\varphi^2} + \varphi\,\Omega\,\varphi + \tilde{m}^2\varphi^2 \Big\}\Psi\,,$$

- Fixes Γ: $\epsilon\frac{\partial}{\partial\epsilon}\Gamma = -\Gamma^2 + \Omega + \tilde{m}^2 \quad \Rightarrow$

$$\Gamma = -\tilde{m} + b_1(\epsilon)\,\Omega + b_2(\epsilon)\,\Omega^2..$$

- Fixes F: $\epsilon\frac{\partial}{\partial\epsilon}F = \frac{l}{2}\mathrm{Tr}\,\Gamma$

- Varying the cut-off, ϵ changes the boundary metric $g_{ij} \approx \hat{g}_{ij}/\epsilon^2$ by $\delta g_{ij} = -2(\delta\epsilon/\epsilon)g_{ij} = 2\delta\sigma g_{ij}$ so

$$\delta F = -\delta\sigma\,\epsilon\frac{\partial}{\partial\epsilon}F = -\frac{l\delta\sigma}{2}\mathrm{Tr}\,\Gamma$$

- Regulate using a heat-kernel

$$\mathrm{Tr}\,\Gamma = \mathrm{Tr}\sum b_n\,\Omega^n = \sum b_n\left(-\frac{\partial}{\partial s}\right)^n \mathrm{Tr}\,e^{-s\Omega}$$

- For small s: $\mathrm{Tr}\,e^{-s\Omega} \sim$

$$\int d^4x\,\sqrt{\hat{g}}\frac{1}{16\pi^2 s^2}\left(1 + s\,a_1(x) + s^2\,a_2(x) + ..\right)$$

- This gives a contribution to the Weyl anomaly of

$$\delta F = -\frac{l\delta\sigma}{2}\mathrm{Tr}\Gamma = .. - \int d^4x\,\sqrt{\hat{g}}\,\frac{\delta\sigma\,l\,\tilde{m}\,a_2}{32\pi^2} + ..$$

- Re-write in terms of scaling dimension $l\tilde{m} = \Delta - 2$, so anomaly is

$$-\int d^4x\,\sqrt{\hat{g}}\,\frac{(\Delta-2)a_2}{32\pi^2}$$

- This analysis goes through for all the fields in IIB SUGRA

- Result: 1-loop contribution to the Weyl anomaly from each field in bulk is

$$-\int d^4x \sqrt{\widehat{g}}\, \frac{(\Delta - 2)\, a_2}{32\pi^2}$$

  Δ is the scaling dimension, a_2 is a known heat-kernel coefficient of appropriate 4-dim operator.

- $\Delta \sim m$ for large m so 'heavy' modes cannot be ignored.

- To compute the full 1-loop contribution to the Weyl anomaly from IIB SUGRA we need to sum over all the fields.

$$\text{Anomaly} \propto \sum (\Delta - 2)\, a_2$$

Field	$\Delta - 2$	a_2
ϕ	$\sqrt{m^2+4}$	s
ψ	$\lvert m\rvert$	f
A_μ	$\sqrt{m^2+1}$	v
$A_{\mu\nu}$	m	a
ψ_μ	$\lvert m\rvert$	r
$h_{\mu\nu}$	$\sqrt{m^2+4}$	g

- For generic $SU(4)$ multiplets $p \geq 4$:

$$P(p) \equiv \sum (\Delta - 2)\, a_2 =$$

$$(-4s + f + r + 4a + 2v)p/3$$

$$-(105s + g + 26a + 48v + 72f + 8r)p^3/12$$

$$+(10a + 16v + 4r + g + 25s + 20f)p^5/12$$

- For $p = 3$: $\sum (\Delta - 2)\, a_2 =$

$$244f + 18g + 266s + 218v + 148a + 64r$$

p=2

Original field	Fields	$\Delta-2$	a_2
A_μ	A_i	1	v
(**15** of $SU(4)$)	A_0	2	s
	b_{FP}, c_{FP}	2	-s
ψ_μ	ψ_i^{irr}	3/2	r
	$\gamma^i\psi_i$	5/2	f
(**4** of $SU(4)$)	ψ_0	5/2	f
	λ_{FP}, ρ_{FP}	5/2	-f
	σ_{GF}	5/2	-f
$h_{\mu\nu}$	h_{ij}^{irr}	2	g
($SU(4)$ singlet)	h_{0i}	3	v
	h_{00}, h_μ^μ	cancel	B_0^{FP},C_0^{FP}
	B_0^{FP},C_0^{FP}	cancel	h_{00}, h_μ^μ
	B_i^{FP},C_i^{FP}	3	-v

+42 scalars, 48 spinors, 6 antisymmetric tensors

$$\sum(\Delta-2)\,a_2 =$$

$$12v-6s+6r+6f+2g+12a$$

$\underline{a - c}$

- If we choose the boundary metric g_{ij} to be Ricci-flat $\widehat{R}_{ij} = 0$ and $E = -I$ so

$$\delta F_{YM} = -\frac{N^2 - 1}{\pi^2} \int d^4x \sqrt{g}\, (\, E + \, I)\, \delta\sigma = 0$$

- For the Maldacena conjecture to hold need

$$\sum (\Delta - 2)\, a_2 = 0$$

when the heat-kernel coefficients, a_2, are evaluated in this metric.

- S.Christensen, M.Duff Nucl.Phys B 154(79)301:

$$s = \frac{R_{ijkl}R^{ijkl}}{180}$$

$$f = \frac{7s}{2}\,,\; v = -11s\,,\; a = 33s\,,$$

$$r = -\frac{219s}{2}\,,\; g = 189s$$

- For these values $\sum(\Delta-2)\,a_2=0$ for each multiplet separately, i.e.

- Generic p:

$$(-4s+f+r+4a+2v)p/3$$

$$-(105s+g+26a+48v+72f+8r)p^3/12$$

$$+(10a+16v+4r+g+25s+20f)p^5/12=0$$

- $p=3$:

$$244f+18g+266s+218v+148a+64r=0$$

- $p=2$:

$$12v-6s+6r+6f+2g+12a=0$$

- The Maldacena conjecture survives.

$\underline{-1}$

- Can SUGRA reproduce the -1 in

$$\delta F_{YM} = \frac{N^2-1}{16\pi^2}\int d^4x\,\sqrt{\widehat{g}}\,(6s+2f+v_g)\delta\sigma$$

$$= -\frac{N^2-1}{\pi^2}\int d^4x\sqrt{g}\,(E+I)\,\delta\sigma \quad ?$$

- When the a_2 are evaluated for a general Einstein metric $\widehat{g}_{ij}$ the sum $\sum P(p)$ diverges.

$$P(p) \equiv \sum (\Delta - 2)\, a_2 =$$

$$(-4s+f+r+4a+2v)p/3$$

$$-(105s+g+26a+48v+72f+8r)p^3/12$$

$$+(10a+16v+4r+g+25s+20f)p^5/12$$

- Regulate by weighting the contribution of each multiplet by z^p, and evaluate the sum at $z = 1$ using Cauchy

$$\sum_{p=4}^{\infty} P(p) \to$$

$$\frac{1}{2\pi i} \oint \frac{dz}{z-1} \sum_{p=4}^{\infty} P(p)\, z^p$$

$$= -246f - 20g - 252s - 228v - 160a - 70r$$

- Adding in $p = 2, 3$

$$-246f - 20g - 252s - 228v - 160a - 70r$$

$$+244f + 18g + 266s + 218v + 148a + 64r$$

$$+6f + 2g - 6s + 12v + 12a + 6r$$

$$= 4f + 8s + 2v$$

- This is the same field content as Super-Yang-Mills theory.

- Giving the bulk 1-loop contribution

$$\sum_{SUGRA} \delta F = -\frac{1}{16\pi^2}\int d^4x\sqrt{\widehat{g}}\,(2f+4s+v)\,\delta\sigma$$

- This is to be compared with the -1 part of

$$\delta F_{YM} = \frac{N^2-1}{16\pi^2}\int d^4x\,\sqrt{\widehat{g}}\,(2f+6s+v_g)\,\delta\sigma$$

- Faddeev-Popov ghosts contribute $-2s$.

$$v_g = v - 2s$$

- The Maldacena conjecture survives.

Conclusions

- The Maldacena conjecture is remarkable because it relates very different types of theory. To leading order in large N it states the equivalence of Quantum Super-Yang-Mills and Classical Gravity.

- By reproducing the Super-Yang-Mills Weyl anomaly in SUGRA the conjecture passes a stringent test that is sensitive to the detailed structure of supermultiplets.

- The functional Schrödinger equation is useful in this context because, being Hamiltonian, it formulates a problem in five-dimensional quantum field theory in terms of four-dimensional operators with known properties.