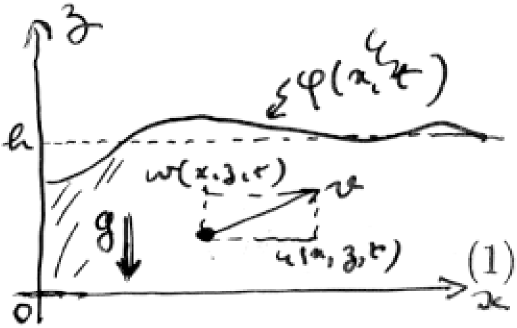
physics/0303085

Euler equations:



$$u_t + uu_x + wu_z = -p_x \tag{1}$$

$$w_t + ww_x + ww_z = -p_z - g \tag{2}$$

where $p(x, z, t)$ is the pressure field and g the acceleration of gravity.

The surface is decribed by the function $\varphi(x, t)$ which obeys the standard conti-uity equation

$$\varphi_t + u\varphi_x - w = 0 \quad \text{at} \quad z = \varphi(x, t) \tag{3}$$

The other boundary condition at the surface is

$$p(x, \varphi, t) = p_0 - \frac{T\phi_{xx}}{(1 + \phi_x^2)^{3/2}} \tag{4}$$

We take the bottom to be flat and fixed at $z = 0$ which gives the boundary ondition

$$w(x, 0, t) = 0 \tag{5}$$

Linearisation: dispersion relation

$$\Omega^2 = (gk + Tk^3)\tanh kh \tag{6}$$

Green-Naghdi approximation:

$$u = u(x, t) \tag{7}$$

$$w = -zu_x(x, t) \tag{8}$$

$$\Omega^2 = \frac{ghk^2 + Thk^4}{1 + \frac{h^2k^2}{3}} \tag{9}$$

Multiscale analysis:

- **long waves** (k small)

$$\Omega = \sqrt{gh}\left[k - \frac{1}{6}k^3h^2 + O(k^5)\right]$$

$$v = \frac{\Omega}{k} = \sqrt{gh} \xrightarrow{x \to x - vt,\ t} \Omega = -\frac{1}{6}k^3h^2\sqrt{gh}$$

$$k = O(\varepsilon) \simeq \frac{\partial}{\partial x} \qquad \Omega = O(\varepsilon^3) \simeq \frac{\partial}{\partial t}$$

rescale x and t to arrive at

$$\varepsilon^3 \frac{\partial u}{\partial t} = \frac{1}{6}h^2\sqrt{gh}\frac{\partial^3 u}{\partial x^3}\varepsilon^3$$

then rescale u so that first non linear term is $\varepsilon^3 u u_x \rightarrow$ KdV (or higher KdV)

- **short waves** (k large) (M. MANNA)

$$\Omega = \sqrt{\frac{3T}{h}}\,k + \frac{A}{k} + O\left(\frac{1}{k^3}\right)$$

$$k = O\left(\frac{1}{\varepsilon}\right) \simeq \frac{\partial}{\partial x} \qquad \Omega \simeq O(\varepsilon) \simeq \frac{\partial}{\partial t}$$

$$v = \sqrt{\frac{3T}{h}} \xrightarrow{(x,t) \to (x - vt,\ t)} \Omega = \frac{A}{k}$$

$$\frac{\partial^2 u}{\partial x \partial t} = -Au$$

then rescalings to obtain non linear terms of the same order.

Short-wave expansion:

Asymptotic variables

$$\zeta = (1/\epsilon)(x - v_p t) \tag{10}$$

$$\tau = \epsilon t \tag{11}$$

$$u = \epsilon^2 u_0 \tag{12}$$

$$u_{xt} = u - u u_{xx} - \frac{1}{2} u_x^2 + \frac{\lambda}{2} u_{xx} u_x^2 \tag{13}$$

Lagrangian:

$$\mathcal{L} = \frac{1}{2} u_x u_t + \frac{1}{2} u^2 + \frac{1}{2} u u_x^2 - \frac{\lambda}{24} u_x^4 \tag{14}$$

Lax pair:

$$L = \frac{\partial}{\partial x} + i\sqrt{E} F \sigma_3 + \frac{1}{2} \frac{u_{xxx}\sqrt{1-\lambda}}{F^2} \sigma_1 \tag{15}$$

$$M = -\frac{1}{2} \left(u - \frac{1}{2} \lambda u_x^2 \right) \frac{u_{xxx}\sqrt{1-\lambda}}{F^2} \sigma_1 \tag{16}$$

$$-i\sqrt{E} \left(u - \frac{1}{2} \lambda u_x^2 \right) F \sigma_3 - \frac{i}{4\sqrt{E}} \frac{1 - u_{xx}}{F} \sigma_3 + \frac{1}{4\sqrt{E}} \frac{u_{xx}\sqrt{1-\lambda}}{F} \sigma_2 \tag{17}$$

where σ are the usual Pauli matrices, E the "eigenvalue" and

$$F^2 = 1 - 2u_{xx} + \lambda u_{xx}^2 \tag{18}$$

Non-trivial conserved quantities:

$$\partial_t F = \partial_x \left[\left(u - \frac{\lambda}{2} u_x^2 \right) F \right] \tag{19}$$

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Change of function: from $u(x,t)$ to

$$g(y,t) = \frac{1}{\sqrt{1-\lambda}}\text{Argtanh}\frac{u_{xx}\sqrt{1-\lambda}}{1-u_{xx}} \tag{20}$$

with

$$y = \int^x F dx \tag{21}$$

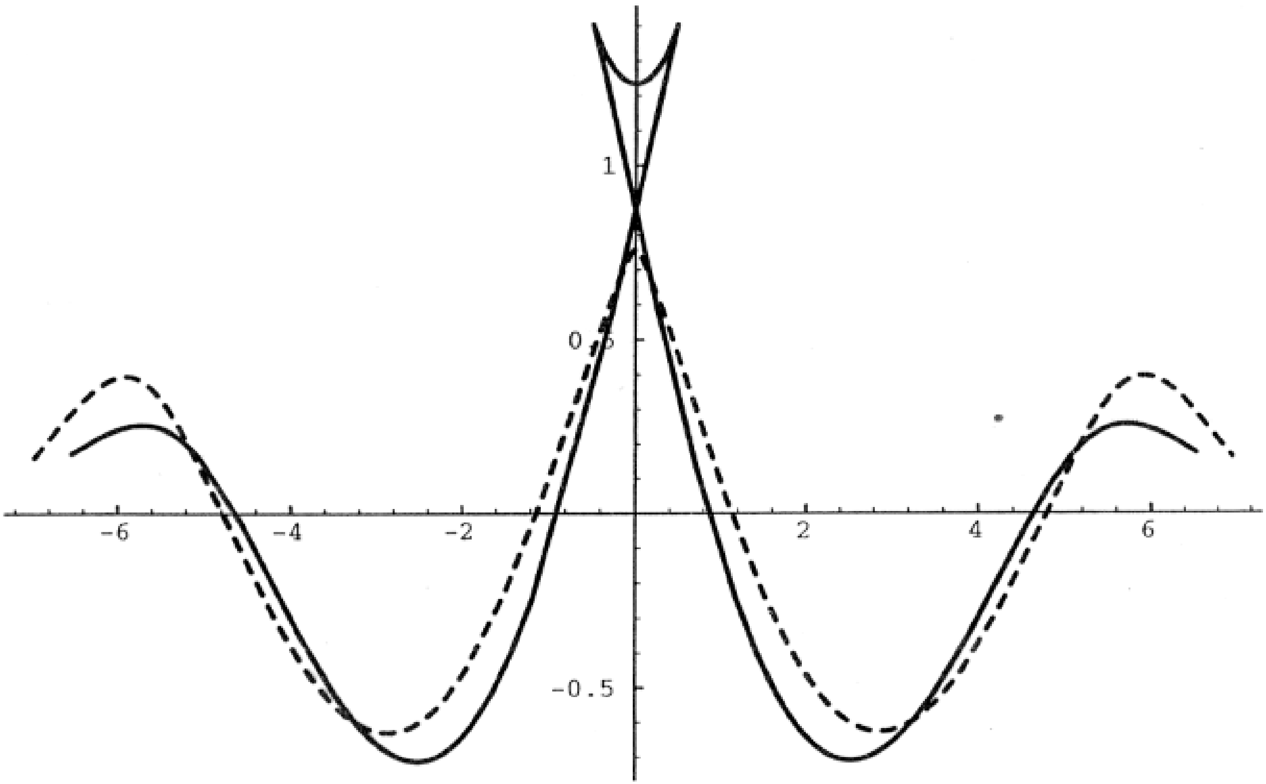
ne finds that g satisfies the sinh-Gordon equation

$$\frac{\partial^2 g}{\partial y \partial t} = \frac{1}{\sqrt{1-\lambda}}\sinh\sqrt{1-\lambda}g \tag{22}$$

Lorentz invariance:

$$x \to ax, t \to t/a, u \to a^2 u \tag{23}$$

for arbitrary real a


1
0.5
-0.5
-6
-4
-2
2
4
6