# The emergence of extra quantum numbers in the Coulomb gas.

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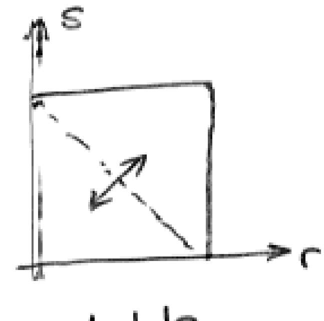
- Concentrate on Virasoro $c_{p,q}$ models.
- Extra structure within degenerate operators
- Special infinite subset of locally chiral operators with extra internal $SU(2)$ quantum numbers.
  - Rational correlation f.$^{ns}$
  - Free field rep.$^{n}$
- One-one correspondence with special rep.$^{ns}$ in quantum group. Extra quantum numbers encode transf.$^{n}$ under center.

# Question: Can we "make sense" of the extended Virasoro minimal models?

2D conformally invt. models. Algebra: $$T(z)T(w) \sim \frac{c}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

Central charge $$c_{p,q} = 1 - 6\frac{(p-q)^2}{pq} \qquad p,q \in \mathbb{N}$$




Kac-table

Primary operators $$\phi(w) = \phi(z)\left(\frac{dw}{dz}\right)^{-h}$$

Degenerate states $$h_{r,s} = \frac{(pr-qs)^2-(p-q)^2}{4pq} \qquad r,s \in \mathbb{N}$$

Null Vectors
Orthog. to all other states.

Minimal models: (BPZ)

$$1 \leqslant r \leqslant q-1 \qquad h_{r,s} = h_{q-r,p-s}$$
$$1 \leqslant s \leqslant p-1$$

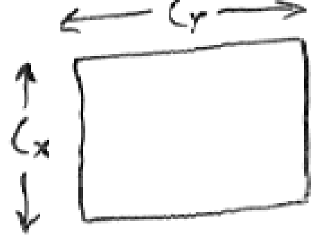
Rational CFT $$\phi_i \otimes \phi_j = \sum_{\text{finite}} C_{ij}^{k}\,[\phi_k]$$

"Extended"

e.g $c_{2,3} = 0$ Minimal set $h_{1,1} = 0$

BUT $h_{2,1} = 0$ Non-trivial

Important for percolation. (Cardy)




At criticality $P_{cross}\left(\frac{L_y}{L_x}\right)$

Problem : Fusion $\longrightarrow$ Infinite set of Vir. primaries
NOT RCFT.

(Solution:) Extended chiral algebra - regroup fields
(RCFT?)

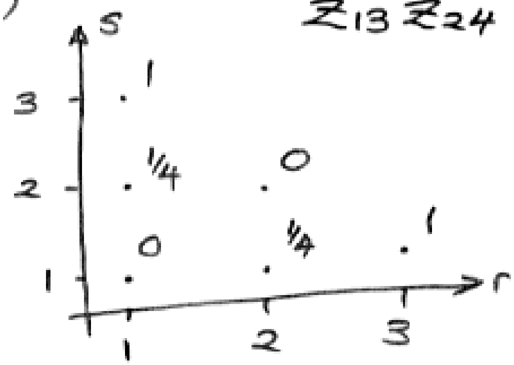
Generated by some degenerate field

$$\langle \phi(0)\phi(z)\phi(1)\phi(\infty)\rangle = \sum_{i=1}^{2h} \frac{a_n}{z^n} + \frac{b_n}{(z-1)^n} + c \quad \text{Rational solution}$$

($h_{\text{fields}} \geq 0$) $\qquad z = \frac{z_{12} z_{34}}{z_{13} z_{24}}$

$c_{1,1} = 1 \quad h_{3,1} = 1$

$3^{rd}$ order ODE




$\longrightarrow$ 3 solutions :-

$$F^{(1)} = \frac{z^2}{(1-z)^2} \qquad F^{(2)} = \frac{(1-z+z^2)^2}{z^2(1-z)^2} \qquad F^{(3)} = \frac{(1-z)^2}{z^2}$$

$$F^{(1)}(1-z) = F^{(3)} \qquad z^{-2h}F^{(1)}\left(\tfrac{1}{z}\right) = -F^{(1)} + 2F^{(2)} - F^{(3)}$$

$$F^{(2)}(1-z) = F^{(2)} \qquad z^{-2h}F^{(2)}\left(\tfrac{1}{z}\right) = F^{(2)}$$

$$F^{(3)}(1-z) = F^{(1)} \qquad z^{-2h}F^{(3)}\left(\tfrac{1}{z}\right) = -F^{(1)} + F^{(3)}$$

Single chiral field $\Rightarrow$ 1 rat! sol$^{n}$.

- Claim: Can have 3 chiral fields with extra internal SU(2) quantum numbers $(\omega^+, \omega^-, \omega^3)$.

Exactly fields $J^a(z)$ of $SU(2)_1$ - RCFT.

$$J^a(z)J^b(\omega) \sim \frac{\delta^{ab}}{(z-\omega)^2} + f^{ab}{}_c \frac{J^c(\omega)}{z-\omega}.$$

$$\langle J^+(0)\,J^+(z)\,J^-(1)\,J^-(\infty)\rangle = F^{(1)}$$
$$\langle J^3 \quad J^3 \quad J^3 \quad J^3 \quad \rangle = F^{(2)}$$
$$\langle J^+ \quad J^- \quad J^- \quad J^+ \quad \rangle = F^{(3)}.$$

$h_{2,1} = \frac{1}{4}$ SU(2) doublet : Parafermionic.

$C_{2,1} = -2$ $\qquad T = \xi \partial \eta \qquad \xi(z)\eta(\omega) \sim \frac{1}{z-\omega}.$

- $h_{2,1} = 1 = h_{1,5}$ 2 rat! sol$^{N}$s

$$F^{(1)} = 1 - \frac{1}{z^2} \qquad F^{(2)} = 1 - \frac{1}{(1-z)^2}$$

Transf.$^N$ properties $\Rightarrow \Psi^{\pm}(z)$ fermionic

- $h_{3,1} = 3 = h_{1,7}$ 3 rat! sol$^{N}$s Triplet $W^a(z)$ bosonic

$$h_{3,1} \otimes h_{3,1} = [h_{1,1}] + [h_{3,1}] + [h_{5,1}]$$
$$h_{1,7} \otimes h_{1,7} = [h_{1,1}] + [h_{1,3}] + \cdots + [h_{1,13}]$$

$$\Rightarrow 3 \otimes 3 = [0] + [3] + \cancel{[10]}$$

$$T(z)W^a(w) \sim \frac{3W^a(w)}{(z-w)^2} + \frac{\partial W^a(w)}{z-w}$$

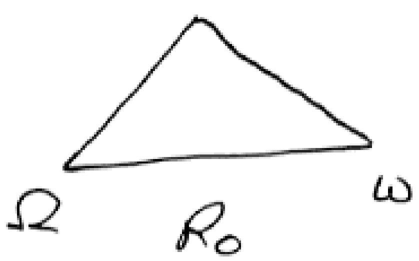
$$W^a(z)W^b(w) \sim \delta^{ab}\left[\frac{1}{(z-w)^6} - \frac{3T}{(z-w)^4} - \frac{3}{2}\frac{\partial T}{(z-w)^3} + \frac{3}{2}\frac{\partial^2 T}{(z-w)^2} - \frac{4T^2}{(z-w)^2} + \frac{1}{6}\frac{\partial^3 T}{z-w} - 4\frac{\partial(T^2)}{z-w}\right]$$

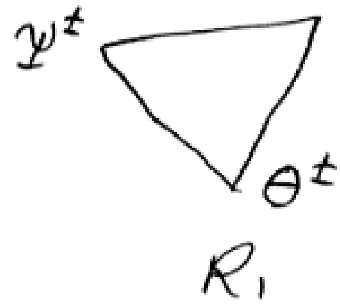
$$-5f^{ab}_c\left[\frac{W^c}{(z-w)^3} + \frac{1}{2}\frac{\partial W^c}{(z-w)^2} + \frac{1}{25}\frac{\partial^2 W^c + TW^c}{z-w}\right]$$

Associativity $\longrightarrow$ Null Vectors $\longrightarrow$ Rep$^{n}$s and fusion rules.

(Gaberdiel + Kausch)

$\underbrace{\Omega, \mu, \nu^\pm, \Psi^\pm}_{\text{irreducible}}, \underbrace{R_0, R_1}_{\text{indecomposable}}$ Rational




Extra indices permeate theory.

- Locality? (Gaberdiel + Kausch)
- Modular invariance? (Flohr)

$c_{2,3} = 0$

| | | | | |
|---|---|---|---|---|
| $h_{1,3} = 2 = h_{5,1}$ | 1 rat! sol$^N$ | $T(z)$ primary! | bosonic | |
| $h_{1,5} = 7 = h_{8,1}$ | 2 rat! sol$^N$s | $\psi^{\pm}(z)$ | fermionic | (related to twisted $\mathcal{N}=2$ SUSY.) |
| $h_{1,7} = 15 = h_{11,1}$ | 3 rat! sol$^N$s | $W^a(z)$ | bosonic | |

$$W^a \otimes W^b = \delta^{ab}[T] + f^{ab}_{\ \ c}[W^c]$$

What is the structure of this theory??

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Numerical work:

CONJECTURE: In all $c_{p,q}$ models $[\gcd(p,q)=1]$

for $2j \in \mathbb{N}$ find:

$$h_{1,(2j+2)p-1} = [(j+1)p-1][(j+1)q-1] = h_{(2j+2)q-1,1}$$

has $2j+1$ rat! sol$^N$s $\longrightarrow$ Chiral field $\Phi^m_{(j)}(z)$

$m=-j,\dots,j$ in $2j+1$ dim! rep$^N$ of $SU(2)$.

$j=0$ $h=(p-1)(q-1)$ is Virasoro vacuum null vector

$$\mathcal{L}_{-(p-1)(q-1)}|0\rangle \equiv 0 \longrightarrow \text{minimal models.}$$

CAN WE EXPLAIN THIS PATTERN?

Coulomb gas: $\phi(z)\phi(\omega) \sim -\ln(z-\omega)$
(Dotsenko + Fateev)

$$T(z) = -\frac{1}{2}\partial\phi\partial\phi + i\sqrt{2}\,\alpha_0\,\partial^2\phi \qquad \alpha_0 = \frac{p-q}{2\sqrt{pq}} \quad c = c_{p,q}$$

Vertex operators: $e^{i\sqrt{2}\,\alpha\phi}$ $\qquad h = \alpha^2 - 2\alpha\alpha_0$

Charge neutrality: $\sum_k \alpha_k = 2\alpha_0$ Non-vanishing $\langle \phi\phi\ldots\rangle$

Screening charges: $Q_\pm = \oint dz\, e^{i\sqrt{2}\,\alpha_\pm\phi}$ $\qquad \alpha_+ = \sqrt{\frac{p}{q}}$

$$[T(z), Q_\pm] = 0. \qquad \alpha_- = -\sqrt{\frac{q}{p}}$$

Degenerate operators: $\alpha_{r,s} = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_- \longrightarrow h_{r,s}$

- How do we get $\Phi^m_{(j)}$ ?

$$h_{r,s} + rs = [(j+1)p-1][(j+1)q-1]$$

Find $2j+1$ sol$^{n}$s for $r,s \in \mathbb{N}$.

$$\begin{aligned} r &= (j+1-m)q-1 \\ s &= (j+1+m)p-1 \end{aligned} \qquad m = -j,\ldots,j \quad h\left(e^{-im\sqrt{2pq}}\right) = h_{r,s}.$$

$$\Phi^m_{(j)} = \mathcal{N}(m)\,\mathcal{L}_{rs}\; e^{-im\sqrt{2pq}\,\phi}$$

OPEs require NO screening!!

- Manifest SU(2) symmetry.

? Degeneracy between winding and momentum

c.f SU(2),.

Rewrite as screened vertex operator:

$$\Phi_{(j)}^{m} = \mathcal{N}(m)\, Q_-^{(j+m+1)\rho-1}\, e^{i\sqrt{2}[-(j+1)\rho+1]\phi}$$

$$\left( \text{OR } \mathcal{N}(m)\, Q_+^{(j-m+1)g-1}\, e^{i\sqrt{2}[-(j+1)g+1]\phi} \right)$$

Fall into rep$^{N}$s of $SU_q(2)$ (Also braiding)

$$q^H S^{\pm} = S^{\pm} q^{H\pm 1} \qquad q = e^{\pi i \frac{g}{\rho}} \quad q^{\rho} = \pm 1$$

$S^-$ is screening chge. $Q_+$.

$$[S^+, S^-] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}} \qquad [x] = \frac{q^x - q^{-x}}{q - q^{-1}}$$

$$|\Phi_{(j)}^{m}\rangle = \frac{(S^-)^{(j+m+1)\rho-1}}{[(j+m+1)\rho-1]!} \Big| \overset{J}{(j+1)\rho-1}\, ;\, \overset{M}{(j+1)\rho-1} \Big\rangle$$

$$[x]! = [x][x-1]! \quad [0]! = 1.$$

$$S^{\pm}|\Phi_{(j)}^{m}\rangle = 0 \qquad \text{'Topological' Rep}^{N}$$

$$\frac{(S^{\pm})^{\rho}}{[\rho]!} |\Phi_{(j)}^{m}\rangle = (j \pm m + 1)\, |\Phi_{(j)}^{m\pm 1}\rangle$$

Transforms in $2j+1$ dim$^{l}$ rep$^{N}$ under

CENTER of $SU_q(2)$!

Further questions :-

- Physical (geometrical ?) implications of extra quantum numbers and chiral fields.

  'Classical' emergent excitations in interacting QFT. – SU(2) NOT $SU_q(2)$.

- Generalisation to all degenerate Coulomb gas models.

- Connections with integrable models?

- Can we use this to build rational extended minimal models (LCFTs) ?

Refs:


A.N hep-th/0205170 JHEP 01 (2003) 022.

A.N. hep-th/0302075 JHEP 08 (2003) 040.

A.N. hep-th/0307050

[Ph.D. thesis hep-th/0210070.]