A construction of local operators in the lattice sine-Gordon model

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1. Introduction

XXZ spin chain

Inverse mapping (Maillet et al., . . .)

$$\sigma_n^a \leftarrow T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

$\Rightarrow$ Form factors, Correlation functions, . . .

Other models?

Sine-Gordon model

$$H = \int dx \left(\frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{m^2}{\beta^2}(1 - \cos\beta u) \right).$$

UV regularization $\Rightarrow$ Lattice sine-Gordon (LSG) model

"Inverse mapping" for some local operators?

Plan of Talk

1. Introduction
2. The LSG model
3. A construction of local operators
4. Form Factors
5. Local operators for the LSG model
6. Discussion

2. The LSG model

Various lattice regularizations

- A. Luther (1972)
- G. I. Japaridze, A. A. Nersesian and P. B. Wiegmann (1983),
- E. K. Sklyanin, L. A. Takhtadzhyan and L. D. Faddeev (1979), A. G. Izergin and V. E. Korepin (1982)
- A. Yu. Volkov and L. D. Faddeev (1992)
- $\cdots$

$$L_n(\lambda) = \begin{pmatrix} \pi_n^{-1/2}\varphi(u_n)\pi_n^{-1/2} & -i(m\Delta/2)\sin((\beta/2)u_n + i\lambda) \\ -i(m\Delta/2)\sin((\beta/2)u_n - i\lambda) & \pi_n^{1/2}\varphi(u_n)\pi_n^{1/2} \end{pmatrix},$$

$$\pi_n = \exp\left(\frac{i}{4}\beta p_n\right), \qquad \varphi(u_n) = (1 + 2r\cos\beta u_n)^{1/2}, \qquad r = \left(\frac{m\Delta}{4}\right)^2.$$

$$[u_n, p_m] = i\delta_{nm}.$$

$$u_n \to u(x), \qquad p_n \to \pi(x)\Delta, \qquad (\Delta \to 0).$$

$$R(\lambda,\mu)(L_n(\lambda)\otimes L_n(\mu)) = (L_n(\mu)\otimes L_n(\lambda))R(\lambda,\mu).$$

$$R(\lambda,\mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda,\mu) & c(\lambda,\mu) & 0 \\ 0 & c(\lambda,\mu) & b(\lambda,\mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad R(\lambda,\lambda) = 1.$$

$$b(\lambda,\mu) = \frac{i\sin\gamma}{\sinh(\lambda-\mu+i\gamma)}, \qquad c(\lambda,\mu) = \frac{\sinh(\lambda-\mu)}{\sinh(\lambda-\mu+i\gamma)}, \qquad \gamma = \beta^2/8.$$

The monodromy matrix

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = L_N(\lambda)L_{N-1}(\lambda)\ldots L_1(\lambda).$$

3. A construction of local operators

• Simple idea for construction:

Quantum determinant

$$\det_q(L_n(\lambda)) = (L_n(\lambda))_{11}(L_n(\lambda - i\gamma))_{22} - (L_n(\lambda))_{12}(L_n(\lambda - i\gamma))_{21}.$$

For a spectral parameter at which $\det_q(L_n(\lambda))\Big|_{\lambda=\nu} = 0$,

$$(L_n(\lambda))_{ij}\Big|_{\lambda=\nu} = P_i(n)Q_j(n), \qquad i, j = 1, 2.$$

Then

$$A(\nu) = P_1(N)WQ_1(1), \qquad B(\nu) = P_1(N)WQ_2(1),$$

$$C(\nu) = P_2(N)WQ_1(1), \qquad D(\nu) = P_2(N)WQ_2(1).$$

Here

$$W = w(N|N-1)w(N-1|N-2)\dots w(2|1), \qquad w(n+1|n) = \sum_{i=1}^{2} Q_i(n+1)P_i(n).$$

For example, if $D(\nu)$ is invertible then

$$D^{-1}(\nu)C(\nu) = (Q_2(1))^{-1}\, Q_1(1), \qquad B(\nu)D^{-1}(\nu) = P_1(N)\,(P_2(N))^{-1}\,.$$

Under the periodic boundary condition : $L_{n+N}(\lambda) = L_n(\lambda)$,

local operators:

$$Q_n := (Q_2(n))^{-1}Q_1(n) = U_0^{n-1}D^{-1}(\nu)C(\nu)U_0^{-n+1},$$
$$P_n := P_1(n)(P_2(n))^{-1} = U_0^{n}B(\nu)D^{-1}(\nu)U_0^{-n}.$$

U_0: the shift operator. $U_0 u_n U_0^{-1} = u_{n+1}$, $U_0 p_n U_0^{-1} = p_{n+1}$.

4. Form Factors

Off-shell properties of these operators can be extracted from form factors:

$$\langle\Omega| \left(\prod_{k=1}^{M} C(\mu_k) \right) \mathcal{O}_n \left(\prod_{l=1}^{M'} B(\lambda_l) \right) |\Omega\rangle, \qquad \mathcal{O}_n = Q_n \text{ or } P_n.$$

- The action of shift operator on Bethe eigenstates:

$$U_0 \left(\prod_{l=1}^{M} B(\lambda_l) \right) |\Omega\rangle = \left(\prod_{j=1}^{M} r_1(\lambda_j) \right) \left(\prod_{l=1}^{M} B(\lambda_l) \right) |\Omega\rangle,$$

where $r_1(\lambda) = a_1(\lambda)/d_1(\lambda)$,

$$(L_n(\lambda))_{11} |\Omega\rangle = a_1(\lambda)|\Omega\rangle, \qquad (L_n(\lambda))_{22} |\Omega\rangle = d_1(\lambda)|\Omega\rangle.$$

- The action of $D^{-1}(\nu)C(\nu)$ on Bethe states:

If

$$A(\mu)\prod_{l=1}^{M}B(\lambda_l)|\Omega\rangle = a^{(M)}(\mu|\{\lambda_l\})\prod_{l=1}^{M}B(\lambda_l)|\Omega\rangle + \sum_{j=1}^{M}b^{(M)}(\mu|\lambda_j|\{\lambda_l\}_{l\neq j})B(\mu)\prod_{\substack{l=1\\ l\neq j}}^{M}B(\lambda_l)|\Omega\rangle,$$

then

$$D^{-1}(\nu)C(\nu)\prod_{l=1}^{M}B(\lambda_l)|\Omega\rangle = \sum_{j=1}^{M}b^{(M)}(\nu|\lambda_j|\{\lambda_l\}_{l\neq j})\prod_{\substack{l=1\\ l\neq j}}^{M}B(\lambda_l)|\Omega\rangle.$$

A sufficient condition for $\det_q(T(\nu))=0$:

$$A(\nu)-B(\nu)D^{-1}(\nu)C(\nu)=0.$$

- Result:

$$\langle\Omega|\left(\prod_{l=1}^{M}C(\mu_l)\right)D^{-1}(\nu)C(\nu)\left(\prod_{l=1}^{M+1}B(\lambda_l)\right)|\Omega\rangle\Big/\langle\Omega|\Omega\rangle$$

$$=(-1)^M\det{}_M(S_{kl})_{1\le k,l\le M}\prod_{l=1}^{M}d(\mu_l)\prod_{l=1}^{M+1}d(\lambda_l)\prod_{k=1}^{M+1}\prod_{l=1}^{M+1}b^{-1}(\lambda_k,\lambda_l)$$

$$\times\prod_{1\le k<l\le M}g(\mu_l,\mu_k)\prod_{1\le k<l\le M}g(\lambda_k,\lambda_l)\prod_{l=1}^{M}g(\mu_l,\lambda_{M+1})\prod_{j=1}^{M+1}g(\nu,\lambda_j)\prod_{i=1}^{M}g^{-1}(\nu,\mu_i).$$

Here

$$S_{kl}=\frac{b^2(\lambda_k,\mu_l)}{c(\lambda_k,\mu_l)}\frac{\prod_{m=1}^{M+1}b(\lambda_k,\lambda_m)}{\prod_{m=1}^{M}b(\lambda_k,\mu_m)}-\frac{b^2(\mu_l,\lambda_k)}{c(\mu_l,\lambda_k)}\frac{\prod_{m=1}^{M+1}b(\lambda_m,\lambda_k)}{\prod_{m=1}^{M}b(\mu_m,\lambda_k)},$$

$$D(\lambda)|\Omega\rangle=d(\lambda)|\Omega\rangle,\qquad g(\lambda,\mu)=b(\lambda,\mu)/c(\lambda,\mu).$$

5. Local operators for the LSG model

- The reference state of the LSG model:

No λ-independent state which satisfies: $(L_n(\lambda))_{21}|\Omega\rangle_n = 0$.

Composite L-operators:

$$\mathcal{L}_k(\lambda) = L_{2k}(\lambda)L_{2k-1}(\lambda), \qquad k = 1, 2, \ldots, N/2.$$

$$(\mathcal{L}_k(\lambda))_{21}|\Omega\rangle_k = 0, \qquad |\Omega\rangle = |\Omega\rangle_{N/2} \otimes \cdots \otimes |\Omega\rangle_1.$$

- The shift operator $U = U_0^2$:

$$U_0|\Omega\rangle \neq |\Omega\rangle, \qquad U|\Omega\rangle = U_0^2|\Omega\rangle = |\Omega\rangle, \qquad U\mathcal{L}_k(\lambda)U^{-1} = \mathcal{L}_{k+1}(\lambda).$$

- Factorization: At

$$\lambda = \nu^{(\pm)} := (1/2)(i\gamma \pm \Lambda + i\pi), \qquad 2r\cosh\Lambda = 1,$$

$$(L_n(\lambda))_{ij}\Big|_{\lambda=\nu^{(\pm)}} = P_i^{(\pm)}(n)Q_j^{(\pm)}(n).$$

Local operators

$$P_n^{(\pm)} := P_1^{(\pm)}(n)\left(P_2^{(\pm)}(n)\right)^{-1}, \qquad Q_n^{(\pm)} := \left(Q_2^{(\pm)}(n)\right)^{-1} Q_1^{(\pm)}(n).$$

By explicit calculations, $P_n^{(\pm)} = i\mathcal{O}_n^{(\mp)}$, $Q_n^{(\pm)} = -i\mathcal{O}_n^{(\pm)}$, where

$$\mathcal{O}_n^{(\pm)} := \pi_n^{-1/2}\left[\frac{\cos(1/2)(\beta u_n \mp i\Lambda)}{\cos(1/2)(\beta u_n \pm i\Lambda)}\right]^{1/2}\pi_n^{-1/2}.$$

- The continuum limit :

$$\lim_{\Delta\to 0}\mathcal{O}_n^{(\pm)} = e^{\pm(1/2)i(\beta u(x)-\gamma)}.$$

- Formally,

$$\mathcal{O}_{2k}^{(\pm)} = -iU^k B(\nu^{(\mp)})D^{-1}(\nu^{(\mp)})U^{-k}, \qquad \mathcal{O}_{2k+1}^{(\pm)} = iU^k D^{-1}(\nu^{(\pm)})C(\nu^{(\pm)})U^{-k}.$$

6. Discussion

- Local operators $\mathcal{O}_n^{(\pm)}$ are constructed
  - $*$ Lattice analog of the exponential operators $e^{\pm(1/2)i\beta u(x)}$
- Form factors
  - $*$ Subtlety due to zero eigenvalue of $D(\nu^{(\pm)})$:

$$\text{“}D^{-1}(\nu^{(\pm)})C(\nu^{(\pm)})\text{”}|\Omega\rangle = -iQ_1^{(\pm)}|\Omega\rangle \neq 0.$$

  - $*$ “Anomalous term” for the action of $Q_1^{(\pm)}$ on the Bethe state
  - $*$ The “anomalous term” gives no contribution to the form factors
- To be done : Thermodynamic limit