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Finite-Volume Form Factors in Semiclassical Approximation

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hep-th/0307125, to appear in Nucl.Phys.B

Correlation functions: infinite volume

$$\langle 0 \,|\, \phi(x)\, \phi(0) \,|\, 0 \rangle \;\equiv\; \int \frac{d^2 p}{(2\pi)^2}\, \rho(p^2)\, e^{ip\cdot x}$$

Form Factor representation:

$$\rho(p^2) = 2\pi \sum_{n=0}^{\infty} \frac{1}{n!} \int d\,\Omega_1 \dots d\,\Omega_n \times$$

$$\times\, \delta\,(p^0 - E_1 \dots - E_n)\, \delta\,(p^1 - p_1 \dots - p_n)\; |\langle 0 \,|\, \phi(0) \,|\, n \,\rangle|^2$$

Fast convergency:

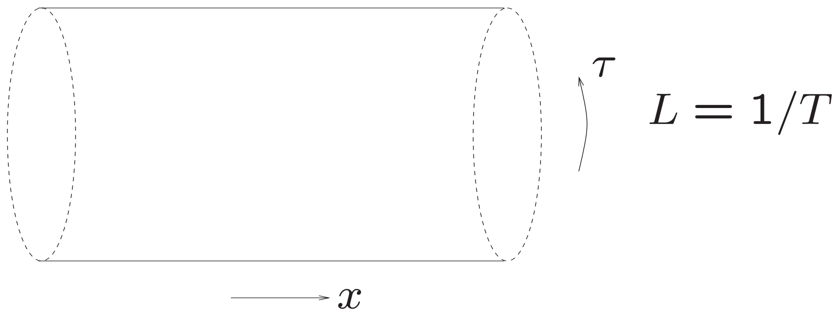
efficient approximation just with few terms

Al.Zamolodchikov	NPB 348 (91)
Yurov-Al.Zamolodchikov	IJMP A6 (91)
Cardy-Mussardo	NPB 340 (90), 410 (93)
Delfino-Mussardo	PLB 324 (94),
	NPB 455 (95)

$\longrightarrow$ expected also in finite volume

Correlation functions on the cylinder

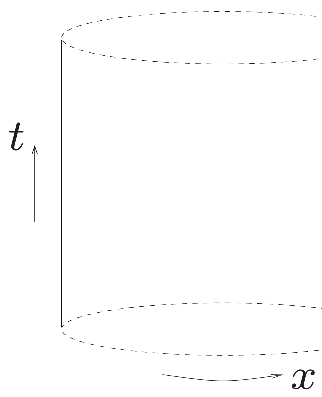
Finite temperature



Integrable theories: LeClair-Mussardo, NPB 522 (99)

infinite volume FF + TBA data

Finite volume



(Energy levels)

Matrix elements

Example: the free boson

$E_0(L)$ in the two channels

- Finite temperature: TBA $\qquad \ell = mL$

$$E_0(L) = \frac{1}{L}\int_0^\infty \frac{d\theta}{\pi}\,\ell\cosh\theta\,\ln\left(1 - e^{-\ell\cosh\theta}\right) =$$

$$= \frac{1}{L}\left\{ -\frac{\pi}{6} + \frac{\ell}{2} + \frac{\ell^2}{4}\left[\gamma_E + \ln\frac{\ell}{2\pi} - \boxed{\left(\ln 2 + \frac{1}{2}\right)}\right] + \right.$$

$$\left. + \sum_{n=1}^{\infty}\left[\sqrt{(2\pi n)^2 + \ell^2} - 2\pi n - \frac{\ell^2}{4\pi n}\right] \right\}$$

- Finite volume: UV regularization

$$E_0(L) =$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}\sqrt{\left(\frac{2\pi n}{L}\right)^2 + m^2}\,\Bigg|_{\text{f.p.}} - \frac{L}{4\pi}\boxed{\int_{-\infty}^{\infty} dp\,\sqrt{p^2 + m^2}\,\Bigg|_{\text{f.p.}}}$$

$\surd \quad$ with

$$\begin{cases} \displaystyle\sum_{n=1}^{\infty} n \,\Bigg|_{\text{reg}} = -\frac{1}{12} \\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\,\Bigg|_{\text{reg}} = \gamma_E + \ln\frac{\ell}{2\pi} \end{cases}$$

${}_L\langle\, 0 \,|\, \phi^2(0) \,|\, 0 \,\rangle_L$ in the two channels

- **Finite temperature: TBA**

$$ {}_L\langle\, 0 \,|\, \phi^2(0) \,|\, 0 \,\rangle_L = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi}\, e^{-\ell \cosh\theta}\, \frac{1}{1-e^{-\ell \cosh\theta}} = $$

$$ = \frac{1}{2\ell} + \frac{1}{2\pi}\left[\gamma_E + \ln\frac{\ell}{2\pi} - \boxed{\ln 2}\right] + \sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{(2\pi n)^2+\ell^2}} - \frac{1}{2\pi n}\right] $$

- **Finite volume: UV regularization**

$$ {}_L\langle\, 0 \,|\, \phi^2(0) \,|\, 0 \,\rangle_L = $$

$$ = \frac{1}{L}\sum_{n=-\infty}^{\infty} \frac{1}{2\sqrt{\left(\frac{2\pi n}{L}\right)^2+m^2}}\Bigg|_{\text{f.p.}} - \boxed{\int_{-\infty}^{\infty}\frac{dp}{4\pi}\frac{1}{\sqrt{p^2+m^2}}\Bigg|_{\text{f.p.}}} $$

$\surd$ with $\displaystyle \sum_{n=1}^{\infty}\frac{1}{n}\Bigg|_{\text{reg}} = \gamma_E + \ln\frac{\ell}{2\pi}$

Form factors in finite volume

Previous studies:

- Smirnov hep-th/9802132
- Fonseca-Zamolodchikov hep-th/0112167
  Bugrij hep-th/0011104
- Korepin-Slavnov IJMP B 13 (99) 2933

Our approach:

- semiclassical method
- integrability not required
- kink states
- approximation of the spectral functions

Illustrative examples:

Sine-Gordon: $V(\phi) = \frac{m^2}{\beta^2}(1 - \cos\beta\phi)$

$\mathbb{Z}_2$ broken ϕ^4: $V(\phi) = \frac{\lambda}{4}\phi^4 - \frac{m^2}{2}\phi^2 + \frac{m^4}{4\lambda}$

Semiclassical form factors: infinite volume

E-L: $\partial_\mu \partial^\mu \phi_{cl} + V'(\phi_{cl}) = 0$

$\longrightarrow$ path integral, DHN,...

Goldstone and Jackiw: PRD 11 (75) 1486

$$\phi_{\text{kink}} \longleftrightarrow \begin{cases} M \sim O\left(\frac{1}{\lambda}\right) \\ p \ll M \end{cases}$$

$$\boxed{\langle P_2 \,|\, \phi(0) \,|\, P_1 \rangle \;=\; \int da\; e^{i\,(p_1 - p_2)\,a}\; \phi_{cl}(a)}$$

$$\uparrow$$

$$O\left(\frac{1}{\sqrt{\lambda}}\right)$$

Rapidity: $\theta \sim O(\lambda) \Rightarrow \begin{cases} E \equiv M\cosh\theta \simeq M \\ P \equiv M\sinh\theta \simeq M\theta \end{cases}$

$$\boxed{\langle\, P_2 \,|\, \phi(0) \,|\, P_1 \,\rangle \;\equiv\; f(\theta) \;\equiv\; M \int da\; e^{i\,M\theta a}\; \hat{f}(a)}$$

Heis.eq. for $\phi(x,t) \Rightarrow \dfrac{d^2}{da^2}\,\hat{f}(a) \overset{\overset{O\left(\frac{1}{\sqrt{\lambda}}\right)}{\downarrow}}{=} V'[\,\hat{f}(a)\,]$

$$\Rightarrow \boxed{\hat{f}(a) = \phi_{cl}(a)}$$

Crossed channel:

$$\boxed{F_2(\theta) \;\equiv\; \langle\, 0 \,|\, \phi(0) \,|\, P_1, \bar{P}_2 \,\rangle \;=\; f(i\pi - \theta)}$$

Dyn. poles at $\theta^* = i(\pi - u)$

$$\Downarrow$$

$$m_{(b)} = 2M \sin\frac{u}{2}$$

Examples

- Sine-Gordon:

Exact FF: $\begin{cases} \text{Karowski-Weisz} \\ \text{Smirnov} \end{cases}$ NBP 139 (78) 455 / *Form factors...*(92)

$$\phi_{cl}(x) = \frac{4}{\beta}\arctan(e^{mx}) \longrightarrow \begin{cases} M \sim \mathcal{E}_{cl} = \frac{8m}{\beta^2} \\ \theta \sim O(\beta^2) \end{cases}$$

Fundamental field ϕ : C - odd

$$F_2^{\phi}(\theta) = \langle 0|\,\phi(0)\,|\,P_1, \bar{P}_2\,\rangle_- = \frac{4\pi i}{\beta}\,\frac{1}{(i\pi-\theta)\,\cosh\left[\frac{4\pi}{\beta^2}\,(i\pi-\theta)\right]} \quad \surd$$

$$m_b^{(2n+1)} = 2M\sin\left[\frac{\beta^2}{16}\,(2n+1)\right] \quad \surd$$

Trace of the stress energy tensor ε : C - even

$$F_2^{\varepsilon}(\theta) = \langle 0|\,\varepsilon(0)\,|\,P_1, \bar{P}_2\,\rangle_+ = \frac{4\pi M^2}{\beta^2}\,\frac{(i\pi-\theta)}{\sinh\left[\frac{4\pi}{\beta^2}\,(i\pi-\theta)\right]} \quad \surd$$

$$m_b^{(2n)} = 2M\sin\left[\frac{\beta^2}{16}\,(2n)\right] \quad \surd$$

- ϕ^4-theory $(\mathbb{Z}_2$ broken phase):

$$\phi_{cl}(x) = \frac{m}{\sqrt{\lambda}} \tanh\left(\frac{mx}{\sqrt{2}}\right) \quad \longrightarrow \quad \begin{cases} M \sim \mathcal{E}_{cl} = \frac{2\sqrt{2}}{3}\frac{m^3}{\lambda} \\ \theta \sim O(\lambda) \end{cases}$$

Fundamental field ϕ :

$$F_2^{\phi}(\theta) = \frac{4}{3}\, i\pi \left(\frac{m}{\sqrt{\lambda}}\right)^3 \frac{1}{\sinh\left[\frac{2}{3}\,\pi\,\frac{m^2}{\lambda}\,(i\pi-\theta)\right]}$$

$$\boxed{m_b^{(n)} = 2M \sin\left[\frac{3}{4}\,\frac{\lambda}{m^2}\, n\right] \;=\; n\,\sqrt{2}m\,\left[1 - \frac{3\,n^2}{32}\frac{\lambda^2}{m^4} + \cdots\right]}$$

$\surd$ DHN

$$\text{Res}F_2(\theta)|_{\theta_1^*} \propto \; i\, g_{k\bar{k}\,b}\,\langle\, 0|\,\phi|\,b^{(1)}\,\rangle \;\Rightarrow\; \boxed{g_{k\bar{k}\,b} = \frac{32}{3}\,m^2\left(\frac{m}{\sqrt{\lambda}}\right)^3}$$

Trace of the stress energy tensor ε :

$$F_2^{\varepsilon}(\theta) = \frac{4}{27}\pi\, M^2\left(\frac{m^3}{\lambda}\right)^2\left[(i\pi-\theta)^2 + \frac{9}{4}\left(\frac{\lambda}{m^2}\right)^2\right]\frac{(i\pi-\theta)}{\sinh\left[\frac{2}{3}\pi\frac{m^2}{\lambda}\,(i\pi-\theta)\right]}$$

How good are the semiclassical formulas?

$$F(\theta) = N\langle P_2 \,|\, \epsilon(0) \,|\, P_1 \rangle$$

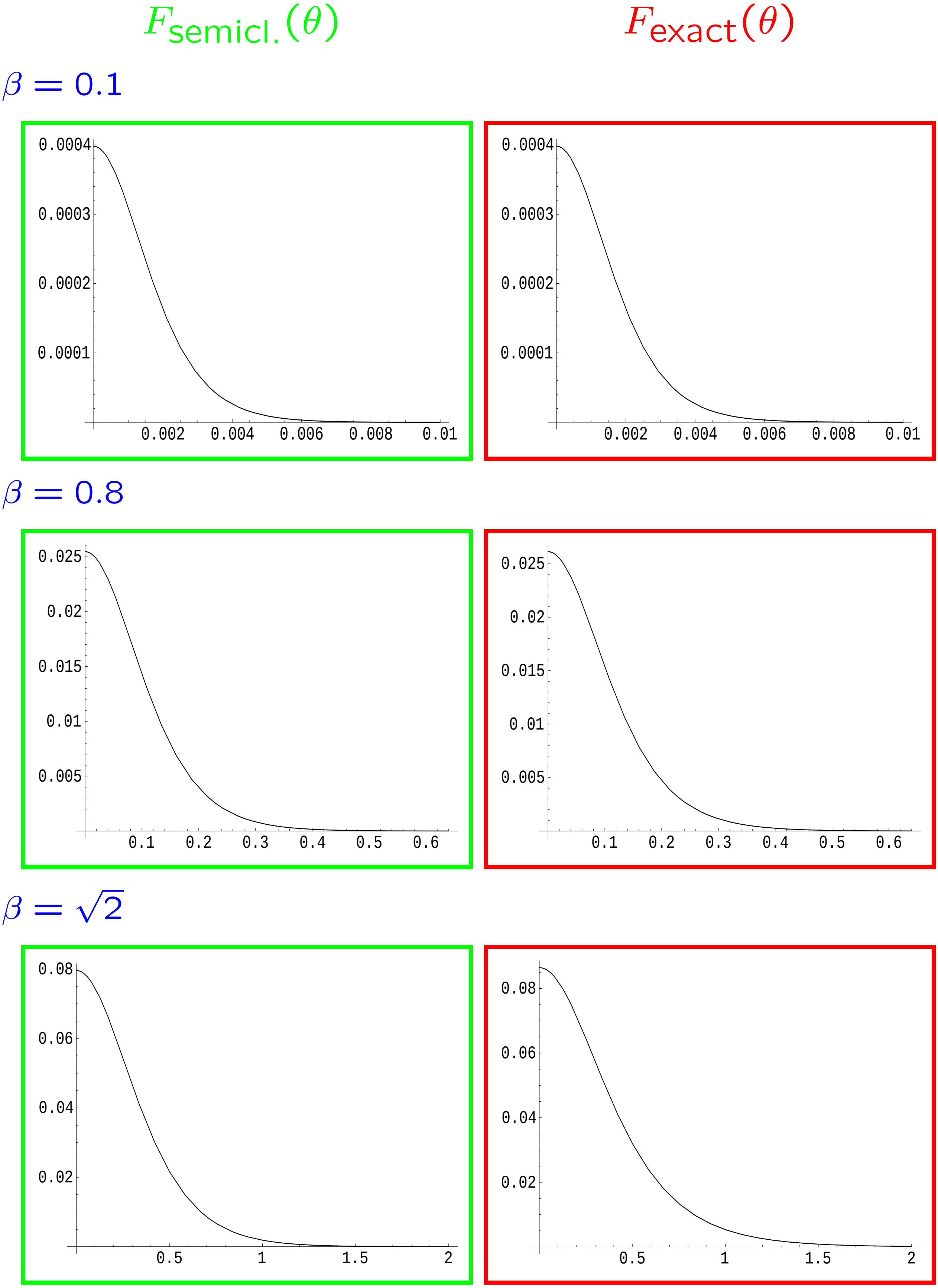
- $F_{\text{semicl.}}(\theta) = \frac{\theta}{2} \cdot \frac{1}{\sinh\left[\frac{4\pi}{\beta^2}\,\theta\right]}$

- $F_{\text{exact}}(\theta) = \sinh\frac{\theta}{2} \cdot \frac{1}{\sinh\frac{\theta}{2\xi}} \cdot G(\theta)$

with

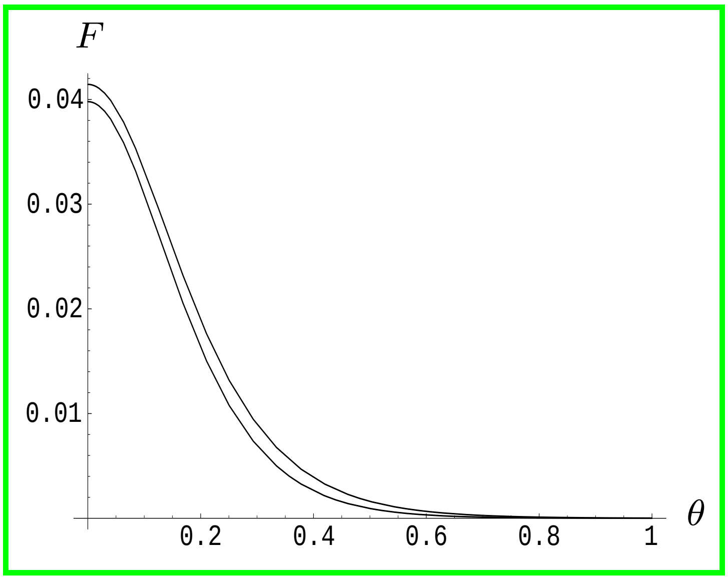
$$\xi = \frac{\beta^2/8\pi}{1-\beta^2/8\pi}$$

$$G(\theta) = \exp\left[\int_0^\infty \frac{dt}{t}\, \frac{\sinh\frac{t}{2}(1-\xi)}{\sinh\frac{t}{2}\xi\ \cosh\frac{t}{2}}\, \frac{\sin^2\frac{\theta\, t}{2\pi}}{\sinh t}\right]$$

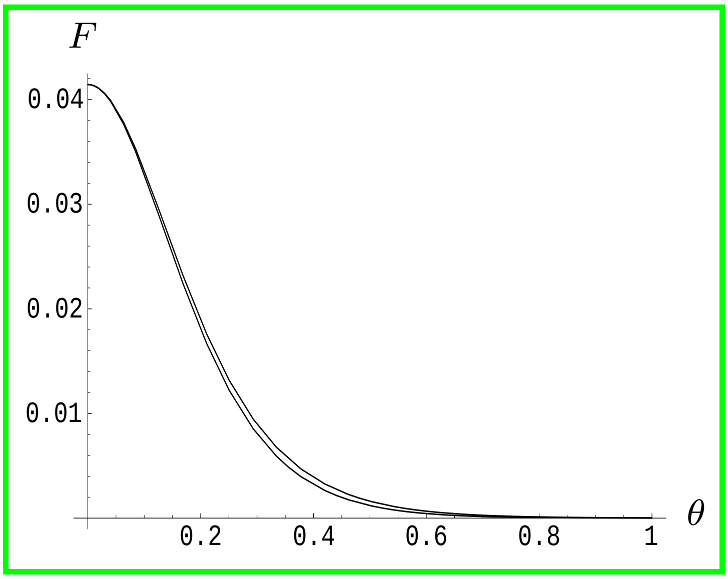

$F_{\text{semicl.}}(\theta)$
$F_{\text{exact}}(\theta)$
$\beta = 0.1$
0.0004
0.0003
0.0002
0.0001
0.002 0.004 0.006 0.008 0.01
$\beta = 0.8$
0.025
0.02
0.015
0.01
0.005
0.1 0.2 0.3 0.4 0.5 0.6
$\beta = \sqrt{2}$
0.08
0.06
0.04
0.02
0.5 1 1.5 2

$$F_{\text{exact}}(0) = \xi = \frac{\beta^2/8\pi}{1-\beta^2/8\pi}$$

$\beta = 1$



"Dressing" $F_{\text{semicl.}}(\theta)$ with $\frac{\beta^2}{8\pi} \to \xi$:



Classical solutions in finite volume

$$\partial_\mu \partial^\mu \phi_{cl} + \frac{m^2}{\beta} \sin \beta\phi_{cl} = 0 \quad + \quad \text{PBC}$$

$\longrightarrow$ *algebro-geometrical* solutions

STATIC:

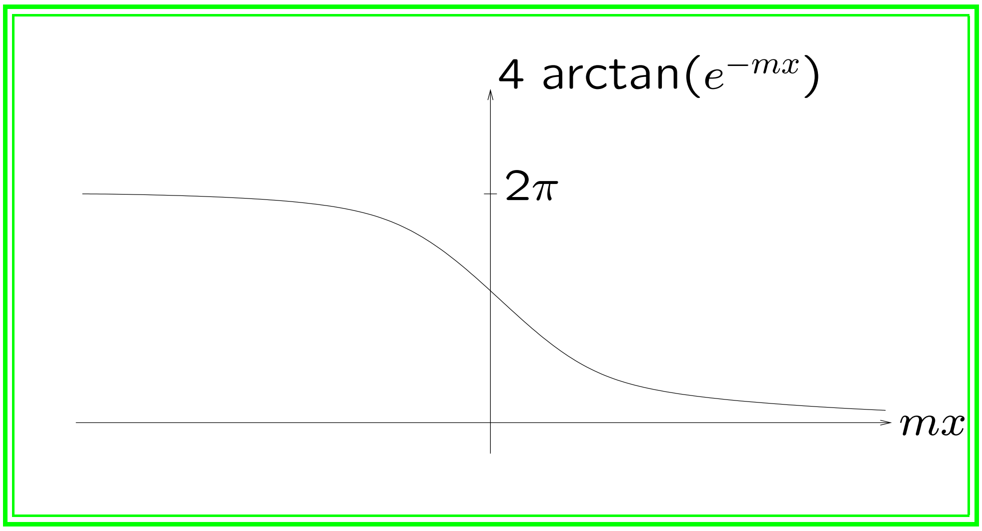
$$\frac{1}{2}\left(\frac{d\phi_{cl}}{dx}\right)^2 = \frac{m^2}{\beta^2}\left(1 - \cos\beta\phi_{cl} + \boxed{C}\,\right)$$

Antiperiodic boundary conditions:

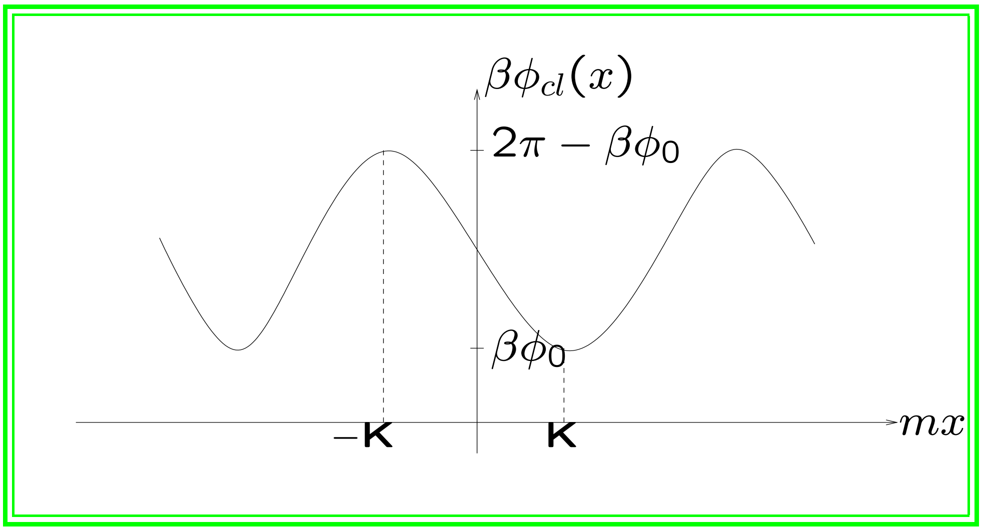
Sine-Gordon: $\phi_{cl}(x+L) = 2\pi - \phi_{cl}(x)$

Broken ϕ^4: $\phi_{cl}(x+L) = -\phi_{cl}(x)$

- $C = 0:$ infinite volume



- $C = -V(\phi_0) \in [-2, 0]:$ $\begin{cases} \ell = mL = 2\mathbf{K}(k) \\ k^2 = 1 + \frac{C}{2} \end{cases}$



$$\phi_{cl}(x) = \frac{2}{\beta} \arccos\left[k\ \mathrm{sn}\,(mx, k)\right]^*$$

*Takayama-Oka, NPA 551 (93) 637

Why this solution?

- $\phi_{cl}(x+L) = 2\pi - \phi_{cl}(x)$

- correct IR behavior: $C \to 0$ $\begin{cases} L \to \infty \\ \phi_{cl}(x) \to \text{kink} \end{cases}$

- large L correction to the mass:

$$\mathcal{E}_{cl}(L) = \frac{8m}{\beta^2}\left\{\mathbf{E}(k) - \frac{1}{2}(1-k^2)\,\mathbf{K}(k)\right\}$$

$$\xrightarrow[C\to 0]{} M_\infty - \boxed{32}\,\frac{m}{\beta^2}\,e^{-mL} + O(e^{-2mL})$$

Lüscher formula: Lüscher, CMP 104 (86)

$$\boxed{M(L) - M_\infty = -\frac{1}{8\,m\,M_\infty^2}\;g_{k\bar{k}\,b}^2\;e^{-mL} + \ldots}$$

exact S matrix $\longrightarrow$ $g_{k\bar{k}\,b}$ $\longrightarrow$ $\boxed{32}$

$*$ Broken ϕ^4:

$$\mathcal{E}_{cl}(L) \to M_\infty - \boxed{8}\sqrt{2}\,m\,\frac{m^2}{\lambda}\,e^{-\sqrt{2}mL} + \ldots$$

ResF_2 $\longrightarrow$ $g_{k\bar{k}\,b}$ $\longrightarrow$ $\boxed{8}$

$\boxed{\text{Semiclassical form factors:}}$
$\boxed{\text{finite volume}}$

$f(\theta_n) \equiv \langle\, P_{n_2} | \phi(0) | P_{n_1}\, \rangle \longleftrightarrow$ eigenstates of H_L

$M(L)\,\theta_{n_i} \equiv P_{n_i} = \frac{(2n_i - 1)\pi}{L}$ $\begin{cases} \text{"quasi-momentum"} \\ \text{"quasi-rapidity"} \end{cases}$

$$\boxed{\begin{cases} f(\theta_n) \;\equiv\; M(L) \displaystyle\int_{-L/2}^{L/2} da\, e^{i\,\frac{2n\pi}{L}\,a}\, \hat{f}(a) \\[2ex] \hat{f}(a) \;\equiv\; \dfrac{1}{L\,M(L)} \displaystyle\sum_{n=-\infty}^{\infty} e^{-i\,\frac{2n\pi}{L}\,a}\, f(\theta_n) \end{cases}}$$

Heis.eq. for $\phi(x,t)$ $\Rightarrow$ $\frac{d^2}{da^2}\hat{f}(a) = V'[\hat{f}(a)]$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{d\hat{f}}{da}\right)^2 = V[\hat{f}(a)] + \boxed{C}$$

Crossed channel: $\boxed{F_2(\theta_n) = f(i\pi - \theta_n)}$

$\uparrow$

$O\left(\frac{1}{\sqrt{\lambda}}\right)$

Sine-Gordon example:

$$F_2(\theta_n) = \frac{4\pi i}{\beta(i\pi-\theta_n)} \left\{ \frac{1}{\cosh\left[\frac{1}{m}\, M(L)\, \mathbf{K}'\, (i\pi-\theta_n)\right]} + \right.$$

$$\left. + \left(-1 + \frac{2}{\pi} \arctan \frac{k'}{k}\right) \;\cos\left[M(L)\,(i\pi - \theta_n)\, \frac{L}{2}\right] \right\}$$

Spectral function for small p:

$$\rho(E_k, p_k) = \sum_n \frac{2\pi}{n!\,(2L)^n} \sum_{k_1,\ldots,k_n} \boxed{\frac{1}{E_{k_1} E_{k_2} \ldots E_{k_n}}} \times$$

$$\times\, \delta(E_k - E_{k_1} \ldots - E_{k_n})\, \delta(p_k - p_{k_1} \ldots - p_{k_n})\, |\langle\, 0\,|\, \phi(0)\,|\, n\,\rangle|^2$$

At leading order: vacuum + kink-$\overline{\text{kink}}$

$$\widehat{\rho}(E_k, p_k) = 2\pi\, \delta(E_k)\, \delta(p_k)\, |\langle\, 0\,|\, \phi(0)\,|\, 0\,\rangle|^2 +$$

$$+ \frac{\pi}{4} \boxed{\frac{1}{M^2}}\, \delta\left(\frac{E_k}{M} - 2\right) \sum_{\theta_{k_1}} \left| F_2\left(2\theta_{k_1} + i\pi - \frac{p_k}{M}\right)\right|^2$$

Conclusions and outlook

- The semiclassical approximation seems to be an efficient tool even for
  - finite values of the coupling β^2
  - non-integrable theories
- It permits an estimate of the spectral function in finite volume

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- Analysis of higher soliton form factors
- Extend DHN quantization to finite volume kinks
- Analysis of the vacuum sector
- Higher loop corrections