On the perturbative formulation of quantum field theory with boundary

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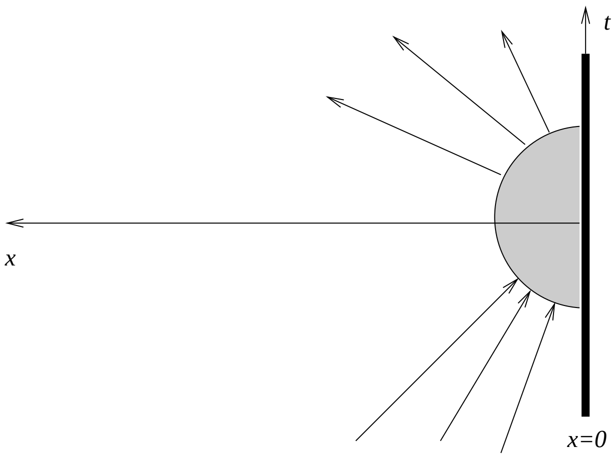
Florence, 2003

hep-th/0207079 (J. Phys. A), hep-th/0309119

Introduction

QFT with boundary



$$L = \int_{-\infty}^{0} dx \left\{ \frac{1}{2} \left[(\partial_t \Phi_\alpha)^2 - (\partial_x \Phi_\alpha)^2 - M_\alpha^2 \Phi_\alpha^2 \right] - V(\Phi_\alpha) \right\} + U(\Phi_\alpha(t, x = 0))$$

$\Phi_\alpha(t,x)$: bulk fields U: boundary potential

$$\partial_x \Phi_\alpha(t,x)|_{x=0} = - \frac{\partial U}{\partial \Phi_\alpha}\bigg|_{x=0}$$

What are we interested in?

Scattering amplitudes: reflection (R) matrix

$$R_{fi} =_{out} \langle f \mid i \rangle_{in}$$

Integrable bulk with integrable boundary condition:

Infinitely many local conserved charges

$\Downarrow$

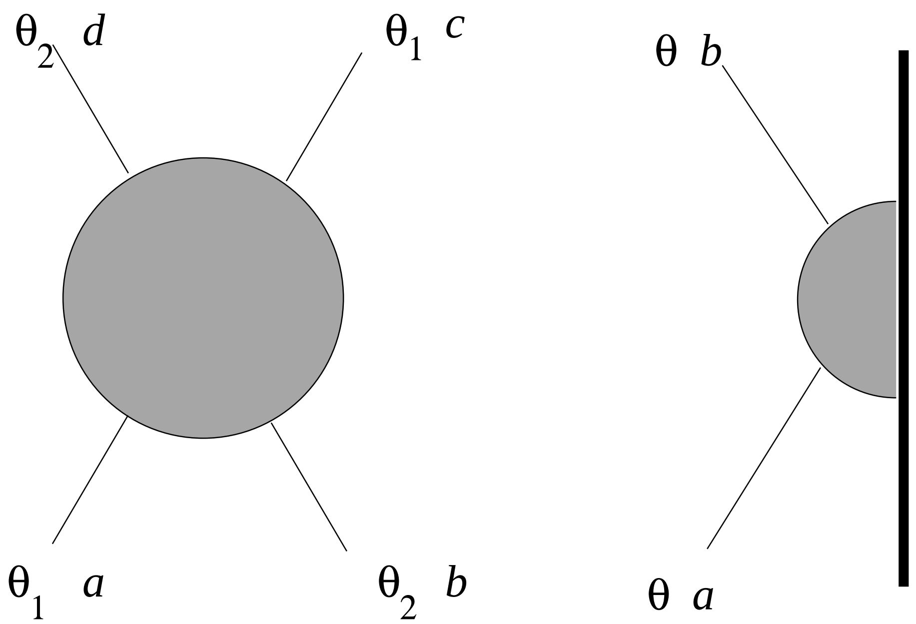
factorized scattering theory

(Ghoshal & Zamolodchikov 1994)

Multi-particle scattering amplitude

$\|$

Π (two-particle S matrices) (one-particle reflection factors)



$S_{ab}^{cd}(\theta_1 - \theta_2)$ $\qquad$ $R_a^b(\theta)$

Exact boundary R matrices

Input data: known bulk factorized S matrix

Principles to use:

1. Unitarity

$$\sum_c R_a^c(\theta)\, R_b^c(\theta)^* = \delta_{ab}$$

2. Analyticity: $R_a^b(\theta)$ is a meromorphic function and

$$R_a^b(\theta)^* = R_b^a(-\theta)$$

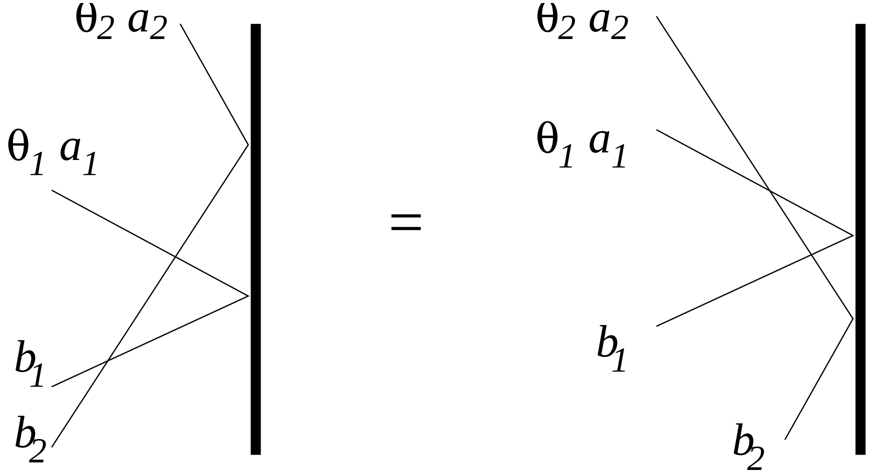
3. Crossing-unitarity (GZ!)

$$K^{ab}(\theta) := R_{\bar{a}}^b\left(\frac{i\pi}{2} - \theta\right)$$

$$K^{ab}(\theta) = S^{ab}_{cd}(2\theta)\, K^{dc}(-\theta)$$

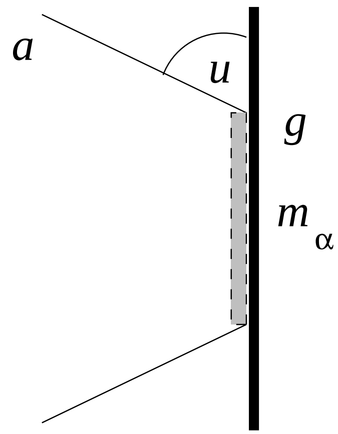
4. Factorization (boundary Yang-Baxter equation)

$$R^{c_2}_{a_2}(\theta_2)\, S^{c_1 d_2}_{a_1 c_2}(\theta_1+\theta_2)\, R^{d_1}_{c_1}(\theta_1)\, S^{b_2 b_1}_{d_2 d_1}(\theta_1-\theta_2)$$
$$\|$$
$$S^{c_1 c_2}_{a_1 a_2}(\theta_1-\theta_2)\, R^{d_1}_{c_1}(\theta_1)\, S^{d_2 b_1}_{c_2 d_1}(\theta_1+\theta_2)\, R^{b_2}_{d_2}(\theta_2)$$



5. Bootstrap (“nuclear democracy”)

Pole of reflection factor $\Rightarrow$boundary excited (bound) state

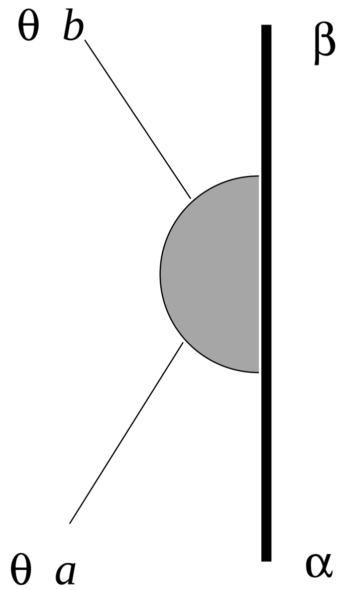


$$
\begin{aligned}
R_a(\theta) &\sim \frac{i}{2}\frac{gg^*}{\theta - iu} \\
m_\alpha &= M_a \cos u
\end{aligned}
$$

$\Rightarrow$ The boundary can exist in a number of different states α.

The reflection factor can be written properly as

$$R_{a,\alpha}^{b,\beta}(\theta)$$



Integrability: $M_a = M_b$, $m_\alpha = m_\beta$.

Successes in the above approach to boundary QFT

(i) Integrable boundary conditions classified for many field theories

(ii) Ground state reflection factors found for a lot of integrable BQFT

(iii) Bootstrap closed in some cases (sinh-Gordon theory, sine-Gordon theory, SUSY sG theory and some others)

(iv) Exact reflection factors checked against perturbation theory, TBA, TCSA etc.

and so on...

However...

1. Relation of R-matrix and Green functions

$$\left\langle \Phi^{a\dagger}(x,t)\Phi_b(x',t')\right\rangle \sim$$
$$\int \frac{d\omega dk}{(2\pi)^2} e^{-i\omega(t-t')} \frac{i}{\omega^2 - k^2 - m^2 + i\epsilon} \times$$
$$\left[e^{ik(x-x')} + \mathcal{R}^a_b\,(k)\, e^{-ik(x+x')}\right]$$
$$x, x' \;\rightarrow\; -\infty$$

Perturbation theory computes $\mathcal{R}^a_b\,(k)$.

$$R^a_b(\theta) \stackrel{?}{=} \mathcal{R}^a_b\,(k)$$

2. Not all the poles of the R-matrix correspond to bound states:

Coleman-Thun diagrams.

Dorey, Tateo, Watts 1998: conjectured rules using analogy with bulk.

How to derive these rules?

(Landau equations, Coleman-Norton interpretation, Cutkosky rules)

Boundary reduction formulae

In the bulk:

$$\begin{array}{c}\text{Green functions}\\ \Big| \\ \text{LSZ reduction formulae}\\ \downarrow \\ \text{scattering amplitudes}\end{array}$$

Boundary QFT: free fields

Eqs. of motion

$$\left(\partial_t^2 - \partial_x^2 + M_\alpha^2\right)\Phi_\alpha(t,x) = 0 \quad , \quad \partial_x \Phi_\alpha(t,x)|_{x=0} = 0$$

$$\begin{aligned}\Phi_\alpha(t,x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \cos(ky)\tilde{\Phi}_\alpha(k,t) \\ & \tilde{\Phi}_\alpha(k,t)^\dagger = \tilde{\Phi}_\alpha(k,t) \quad , \quad \tilde{\Phi}_\alpha(k,t) = \tilde{\Phi}_\alpha(-k,t)\end{aligned}$$

$$\Pi_\alpha = \partial_t \Phi_\alpha \quad , \quad \left[\Phi_\alpha(t,x), \Pi_\beta(t,x')\right] = i\delta_{\alpha\beta}\left(\delta(x-x') + \delta(x+x')\right)$$

Creation/annihilation operators

$$\left[A_\alpha(k,t),\, A_\beta(k',t)^\dagger\right] = (2\pi)\, 2\sqrt{k^2 + M_\alpha^2}\,\delta_{\alpha\beta}\left(\delta(k-k') + \delta(k+k')\right)$$

Free propagator

$$\begin{aligned} &G_{ab}(t,x\,;\,t',x') = \langle 0|\, T\Phi_a(t,x)\Phi_b(t',x')\, |0\rangle \\ &= i\delta_{ab} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\mathrm{e}^{-i\omega(t-t')}}{\omega^2 - k^2 - M_a^2 + i\varepsilon} \left(\mathrm{e}^{ik(x-x')} + \mathrm{e}^{ik(x+x')}\right)\end{aligned}$$

Asymptotic states

Assumption: the bulk and boundary (!) interactions are switched off adiabatically as $t \to \pm\infty$.

$$\lim_{t\to\mp\infty} \Phi_\alpha(t,x) - Z_\alpha \Phi_\alpha^{in/out}(t,x) = 0$$

$\Phi_\alpha^{as}(t,x)$, $as = in/out$ are free fields (Neumann BC!)

$$\begin{aligned} A_\alpha^{as}(k) &= \quad 2i \int_{-\infty}^{0} dx \cos(kx) \mathrm{e}^{i\Omega_\alpha(k)t} \overleftrightarrow{\partial_t} \Phi_\alpha^{as}(t,x) \\ A_\alpha^{as}(k)^\dagger &= - \quad 2i \int_{-\infty}^{0} dx \cos(kx) \mathrm{e}^{-i(\Omega_\alpha(k)t} \overleftrightarrow{\partial_t} \Phi_\alpha^{as}(t,x) \end{aligned}$$

Asymptotic states: $|k_1, \alpha_1; \ldots; k_M, \alpha_M\rangle_{as} = A_{\alpha_1}^{as}(k_1)^\dagger \ldots A_{\alpha_M}^{as}(k_M)^\dagger |0\rangle$

Asymptotic completeness: in (out) states form complete basis.

R–matrix: unitary change of basis

$$|final\rangle_{out} = R|initial\rangle_{in}$$

Boundary LSZ formulae

$$
{}_{out}\langle A|\,\mathcal{O}|k,\alpha;B\rangle_{in} = {}_{out}\langle A|\,\mathcal{O}A^{in}_{\alpha}(k)^{\dagger}|B\rangle_{in} =
$$

$$
= \text{disconnected part} + 2iZ_{\alpha}^{-1/2}\int dt \int_{-\infty}^{0} dx\, \cos(kx) e^{-i\Omega_{\alpha}(k)t} \times
$$

$$
\left\{\partial_t^2 - \partial_x^2 + M_{\alpha}^2 + \delta(x)\partial_x\right\} {}_{out}\langle A|\,T\mathcal{O}\Phi_{\alpha}(t,x)|B\rangle_{in}
$$

(similarly for an outgoing particle).

Reflection of a particle

$$
{}_{out}\left\langle k'\right|\left. k\right\rangle_{in} - {}_{out}\left\langle k'\right|\left. k\right\rangle_{out} =
$$

$$
-4Z^{-1}\int d^2x' \int d^2x\, \cos(kx')\cos(kx) e^{i\Omega_{\alpha}(k')t' - i\Omega_{\alpha}(k)t} \times
$$

$$
\left\{\partial_t^2 - \partial_x^2 + M^2 + \delta(x)\partial_x\right\}\left\{\partial_{t'}^2 - \partial_{x'}^2 + M^2 + \delta(x')\partial_{x'}\right\} {}_{out}\langle A|\,T\Phi(t',x')\Phi(t,x)|B\rangle_{in}
$$

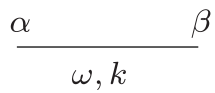
From this we can prove

$$
R_b^a(\theta) = \mathcal{R}_b^a(k)
$$

Feynman rules in momentum space

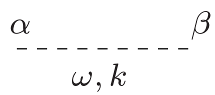
$$G^{(m)}_{\alpha_1 \ldots \alpha_m}(k_1, \omega_1, \ldots, k_m, \omega_m) =$$
$$\prod_{i=1}^{m} \left(\int_{-\infty}^{+\infty} dx_i \int dt_i \mathrm{e}^{i(\omega_i t_i - k_i x_i)} \right) \langle 0| T \prod_{i=1}^{m} \Phi_{\alpha_m}(t_m, x_m) |0\rangle$$

1. Direct bulk propagator



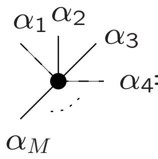
$$= \frac{i\delta_{\alpha\beta}}{\omega^2 - k^2 - M_\alpha^2 + i\varepsilon}$$

2. Reflected bulk propagator



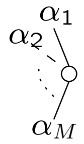
$$= \frac{i\delta_{\alpha\beta}}{\omega^2 - k^2 - M_\alpha^2 + i\varepsilon}$$

3. Bulk vertex



$$= -iv_{\alpha_1\ldots\alpha_M} 2\pi\delta\left(\sum \omega\right)\pi\delta\left({\sum}' k\right)$$

4. Boundary vertex



$$= -iu_{\alpha_1\alpha_2\ldots\alpha_M} 2\pi\delta\left(\sum \omega\right)$$

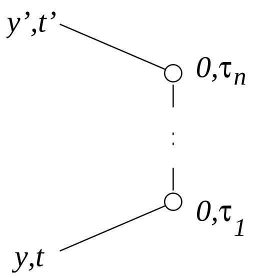
5. Integration over internal momenta + symmetry factors

Toy model

$$L = \int_{-\infty}^{0} dx \frac{1}{2}\left[(\partial\Phi)^2 - m^2\Phi^2\right] - \frac{\lambda}{2}\Phi(x=0)^2$$

Two-point function

$$\begin{aligned} G(y,\, t;\, y',\, t') &= \sum_{n=0}^{\infty} G^{(n)}(y,\, t;\, y',\, t') \\ G^{(n)}(y,\, t;\, y',\, t') &= \frac{1}{n!}\left(-\frac{i\lambda}{2}\right)^n \langle 0|\, T\left(\Phi(y,\, t)\Phi(y',\, t')\right) \prod_{i=1}^{n} \int_{-\infty}^{\infty} d\tau_i \,:\, \Phi\left(0, \tau_i\right)^2 \,:\, |0\rangle \end{aligned}$$



$+$ other graphs with reflected propagators (2^n)

$$G(y,\,t;\,y',\,t') = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega dk}{(2\pi)^2}\frac{i}{\omega^2-k^2-m^2+i\epsilon}\left(e^{ik(y-y')-i\omega(t-t')}+R(k)\,e^{ik(y+y')-i\omega(t-t')}\right)$$

$$R(k) = 1+2\sum_{n=1}^{\infty}\left(-\frac{i\lambda}{k}\right)^n = \frac{k-i\lambda}{k+i\lambda} \quad \surd\,\mathrm{OK}$$

Boundary bound state: normalizable solution of eqs. of motion.

Exists when $-M<\lambda<0 \quad , \quad m^2 = M^2-\lambda^2$

$$\begin{aligned} G(y,\,t;\,y',\,t') \ &= \ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\omega dk}{(2\pi)^2}\frac{i}{\omega^2-k^2-M^2+i\epsilon}\left(e^{ik(y-y')-i\omega(t-t')}+R(k)\,e^{ik(y+y')-i\omega(t-t')}\right) \\ &+ \ \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{i}{\omega^2-m^2+i\epsilon}e^{-i\omega(t-t')}\,f_B(y)f_B(y') \qquad , \qquad f_B(y)=\sqrt{2|\lambda|}e^{-\lambda y}\end{aligned}$$

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{i}{\omega^2-m^2+i\epsilon}e^{-i\omega(t-t')}$$

Propagator of 0+1 dimensional scalar field!

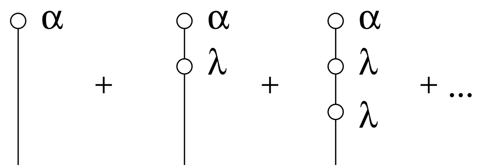
Tadpoles

$$L = \int_{-\infty}^{0} dx \frac{1}{2}\left[(\partial\Phi)^2 - M^2\Phi^2\right] - \frac{\lambda}{2}\Phi(x=0)^2 - \alpha\Phi(x=0)$$

Nontrivial field background

$$\Phi_{vac}(x,t) = -\frac{\alpha}{M+\lambda}e^{Mx}$$

$$\langle\Phi(x,t)\rangle =$$

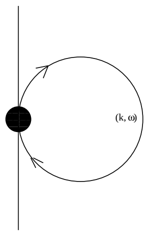


$$\langle\Phi(x,t)\rangle = -i\alpha\int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\frac{i}{\omega^2 - k^2 - M^2 + i\epsilon}e^{-i\omega t}\left[e^{ikx} + R(k)e^{-ikx}\right]$$

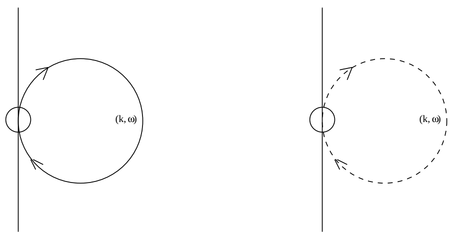
$\Rightarrow \langle\Phi(x,t)\rangle = -\frac{\alpha}{M+\lambda}e^{Mx}$: correct! (same for n-point functions)

One-loop renormalization in boundary SG

$$L=\int_{-\infty}^{0}dx\left[\frac{1}{2}(\partial\Phi)^2-\frac{m^2}{\beta^2}(1-\cos\beta\Phi)\right]+M_0\left(1-\cos\frac{\beta}{2}(\Phi(x=0)-\varphi_0)\right)$$



$$\Rightarrow\ \delta m^2=-\frac{m^2\beta^2}{4\pi}\int_0^{\Lambda}\frac{dk}{\sqrt{k^2+m^2}}$$



$$\Rightarrow\ \delta M_0=-\frac{M_0\beta^2}{8\pi}\int_0^{\Lambda}\frac{dk}{\sqrt{k^2+m^2}}$$

Agrees with counterterms used in semiclassical (DHN) path integral calculations by Corrigan & Taormina (1999), Corrigan & Delius (2000), Kormos & Palla (2002) $\surd$ OK

Generalizing the action: explicit boundary fields

$$L = \int_{-\infty}^{0} dx \left\{ \frac{1}{2}\left[(\partial_t \Phi_\alpha)^2 - (\partial_x \Phi_\alpha)^2 - M_\alpha^2 \Phi_\alpha^2 \right] - V(\Phi_\alpha) \right\} + \frac{1}{2}(\partial_t \phi_a)^2 - \frac{m_a^2}{2}\phi_a^2 - U(\phi_a, \Phi_\alpha(x=0))$$

New in perturbation theory: boundary propagator

$$\underset{\omega}{a \cdots\cdots\cdots b} = \frac{i\delta_{ab}}{\omega^2 - m_a^2 + i\varepsilon}$$

General Feynman integral

$$G = \int \prod_{i=1}^{L} \frac{d\chi_i}{2\pi} \prod_{j=1}^{K} \frac{dq_j}{2\pi} \prod_{r=1}^{I} (p_r^2 - M_r^2 + i\epsilon)^{-1} \prod_{s=I+1}^{I+J} (\omega_s^2 - m_s^2 + i\epsilon)^{-1}$$

χ_i: loop energies q_j: loop momenta
$p_r = (\omega_r, \pi_r)$: 2-momenta of $\# I$ bulk internal lines
ω_s: energy of $\# J$ boundary internal lines

Feynman's parameterization

$$G = \int \prod_{i=1}^{L} \frac{d\chi_i}{2\pi} \prod_{j=1}^{K} \frac{dq_j}{2\pi} \prod_{r=1}^{I+J} \int_0^1 d\alpha_i \delta\left(\sum \alpha_i - 1\right) \psi(k,\chi,\alpha)^{I+J}$$

$$\psi(k,\chi,\alpha)^{-1} = \sum_{r=1}^{I} \alpha_r (p_r^2 - M_r^2 + i\epsilon) + \sum_{s=I+1}^{I+J} \alpha_s (\omega_s^2 - m_s^2 + i\epsilon)$$

Landau equations

L1 : $\alpha_r = 0$ or $p_r^2 - M_r^2 = 0$, $r = 1, \ldots, I$

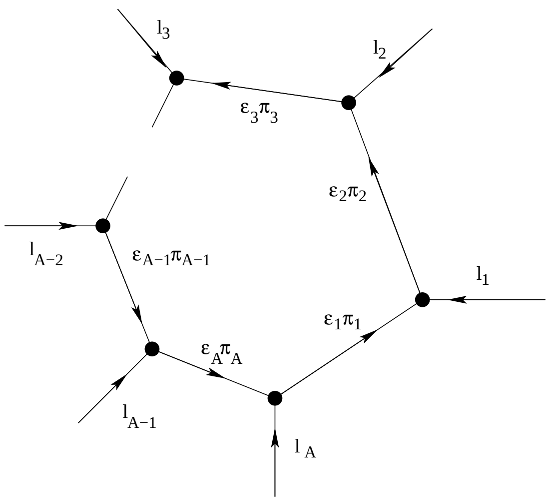
L2 : $\alpha_s = 0$ or $\omega_s^2 - m_s^2 = 0$, $s = I+1, \ldots, I+J$

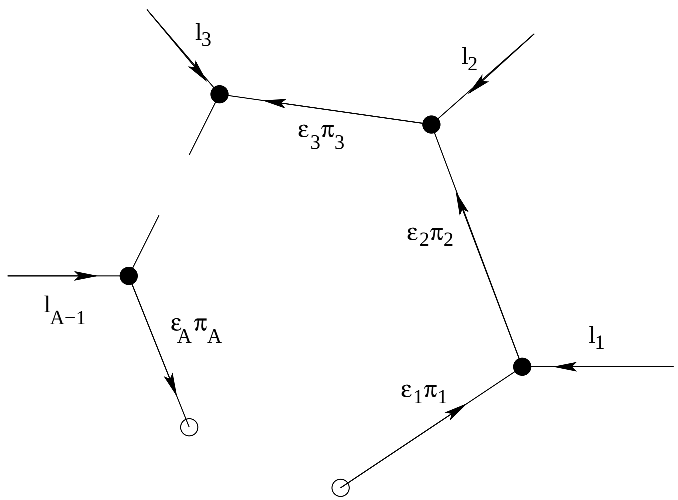
L3 : $$\frac{\partial}{\partial \chi_i}\left(\sum_{r=1}^{I} \alpha_r (p_r^2 - M_r^2 + i\epsilon) + \sum_{s=I+1}^{I+J} \alpha_s (\omega_s^2 - m_s^2 + i\epsilon)\right) = 0$$

L4 : $$\frac{\partial}{\partial q_j}\left(\sum_{r=1}^{I} \alpha_r (p_r^2 - M_r^2 + i\epsilon) + \sum_{s=I+1}^{I+J} \alpha_s (\omega_s^2 - m_s^2 + i\epsilon)\right) = -\frac{\partial}{\partial q_j}\left(\sum_{r=1}^{I} \alpha_r \pi_r^2\right) = 0$$

Rewriting Landau equations

$$\begin{aligned}
&L1 \quad : \quad \alpha_r = 0 \quad \text{or} \quad p_r^2 - \pi_r^2 - M_r^2 = 0 \quad , \quad r = 1, \ldots, I \\
&L2 \quad : \quad \alpha_s = 0 \quad \text{or} \quad \omega_s^2 - m_s^2 = 0 \quad , \quad s = I+1, \ldots, I+J \\
&L3 \quad : \quad \sum_{\text{any loop}} \alpha_q \omega_q = 0 \\
&L4 \quad : \quad \sum_{\text{any loop}} \mu_q \alpha_q \pi_q = 0 \quad , \quad \mu_q = \prod_{r=1}^{q} \epsilon_r
\end{aligned}$$





Bulk loop Boundary loop

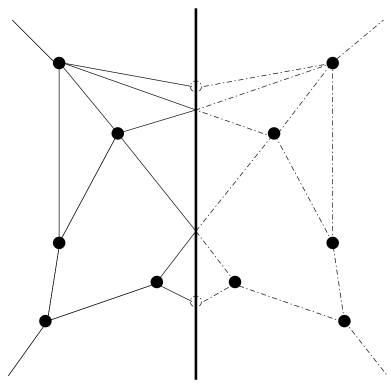
$\epsilon_r = \pm 1$: bulk direct/reflected line

Coleman-Norton type interpretation

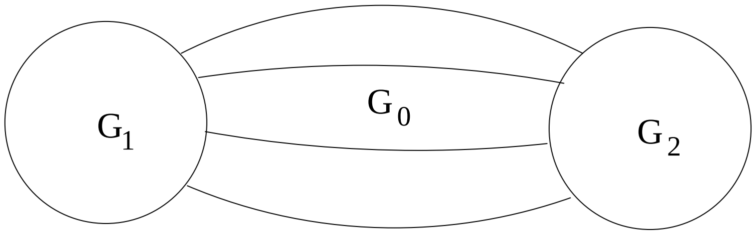
Solution of boundary Landau equations

$\Updownarrow$

existence of a real space-time process involving particles on the mass shell, all moving forward in time, and interacting only through energy and momentum conserving interactions localized at bulk space-time points + particles can scatter on the boundary (reversing the sign of their momenta).



Cutkosky rules



1. Substitute the propagators of the internal lines for which $\alpha_r \neq 0$ as follows:

$$\frac{1}{p_r^2 - M_r^2 + i\epsilon} \to -2\pi i \delta^{(+)}\left(p_r^2 - M_r^2\right) \quad , \quad \frac{1}{\omega_r^2 - m_r^2 + i\epsilon} \to -2\pi i \delta^{(+)}\left(\omega_r^2 - m_r^2\right)$$

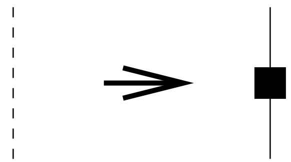
2. change $\epsilon \to -\epsilon$ for internal lines in one of the components (say G_2, the other choice just gives the jump in the opposite direction)

3. perform the integrations.

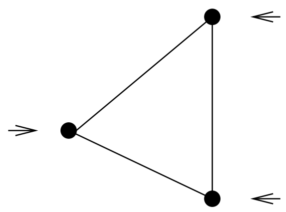
Proof: following Nakanishi [Itzykson-Zuber].

Reduced diagrams

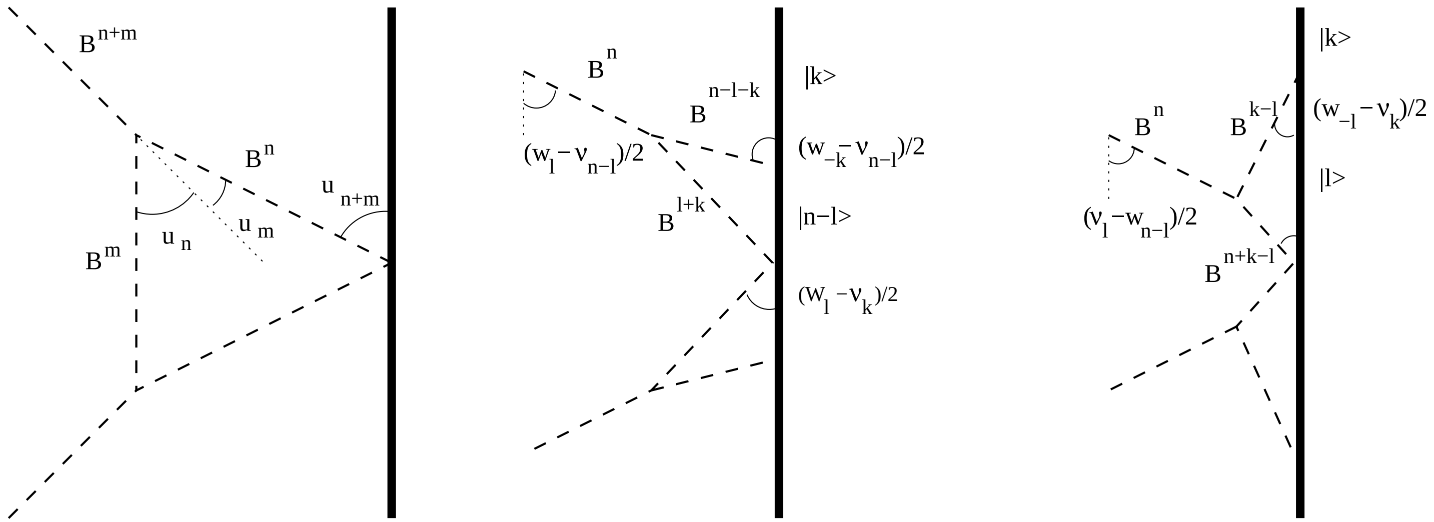
Shrink all off-shell internal lines ($\alpha \neq 0$) to a point.
Get rid of reflected propagators by



$\blacksquare$: reverses momentum, conserves energy (“Neumann vertex”)
Summing up all contributions to a given reduced diagram:
exact on-shell vertex function appear



Coleman-Thun diagrams



1st order pole

2nd order pole
(BSG: 1st order)

2nd order pole
(BSG: 1st order)

Also: residues match perfectly!

Nontrivial: dependence on bulk coupling + 2 boundary parameters.

Conclusions

1. Boundary LSZ formulae derived: connection between Green´s functions and reflection amplitudes established.

2. Perturbation theory around Neumann BC developed: allows to derive Landau equations, Coleman-Norton picture and Cutkosky rules, placing boundary Coleman-Thun mechanism on safe grounds.

3. Perturbation theory passes basic “sanity” checks: toy model works (including resummation), consistently includes nontrivial background fields (tadpoles), one-loop renormalization in BSG OK.

Remark: Dirichlet BC can be incorporated: sign factor in reflected propagator ($\blacksquare = -1$) + asymptotic field with DBC.

Open problems

1. All-order renormalization theory: cannot be too difficult. Power counting is different (bulk vs. boundary loops).

2. General discussion of background field dependence of perturbation theory (probably using functional methods: generating functionals for Green´s functions).

3. RG flows in Lagrangian formalism (so far mainly discussed in perturbed CFT framework).

4. Nonperturbative, axiomatic R matrix theory a la [Eden et al.: The analytic S matrix]: nonperturbative approach to analiticity (GZ!).