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Finite lattice Bethe ansatz systems and the Heun equation

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Abstract

We study the Pöschl-Teller equation in complex domain and deduce infinite families of TQ and Bethe ansatz equations classified by four integers. In all these models the form of T is very simple, while Q can be explicitly written in terms of the Heun function. At particular values there is a interesting interpretation in terms of finite lattice spin (L-2)/2 XXZ quantum chain with $\Delta = \cos(\pi/L)$, or $\Delta = -\cos(\pi/L)$. [hep-th/0309054]

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OUTLINE

- Baxter's TQ relation
- The connection formula
- The ODE/IM correspondence
- The Pöschl-Teller equation
- The field theory limit
- Complex analysis of the equation
- Model identification and some numerics
- Open problems and conclusions

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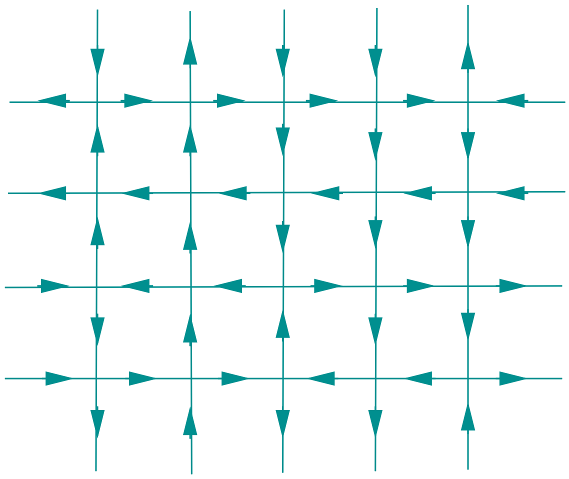
Baxter's TQ relation

(Baxter) Consider $N \times M$ lattice model with periodic BCs and $N/2$ even. On each link of the lattice, we place a spin



A) Only those configurations of spins which preserve the 'flux' of arrows through each vertex are allowed.

B) We shall only consider the zero field 6-V model which has an additional '4-spin reversal' symmetry. Locally this gives six options:

$$W\begin{bmatrix} & \uparrow & \\ \rightarrow & & \rightarrow \\ & \uparrow & \end{bmatrix} = W\begin{bmatrix} & \downarrow & \\ \leftarrow & & \leftarrow \\ & \downarrow & \end{bmatrix} = a$$

$$W\begin{bmatrix} & \downarrow & \\ \rightarrow & & \rightarrow \\ & \downarrow & \end{bmatrix} = W\begin{bmatrix} & \uparrow & \\ \leftarrow & & \leftarrow \\ & \uparrow & \end{bmatrix} = b$$

$$W\begin{bmatrix} & \uparrow & \\ \rightarrow & & \leftarrow \\ & \downarrow & \end{bmatrix} = W\begin{bmatrix} & \downarrow & \\ \leftarrow & & \rightarrow \\ & \uparrow & \end{bmatrix} = c$$

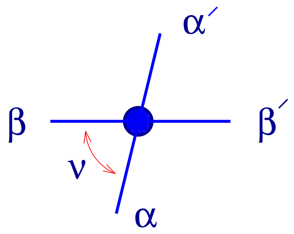
The overall normalisation factors out trivially from all calculations, and we can parametrise the remaining two degrees of freedom using:

ν	(the spectral parameter)
η	(the anisotropy)

as

$$a = \sinh(i\eta - \nu)\ , \quad b = \sinh(i\eta + \nu)\ , \quad c = \sinh(2i\eta)$$

$$\mathbf{W}\begin{bmatrix} & \alpha' & \\ \beta & & \beta' \\ & \alpha & \end{bmatrix}(\nu) \quad =$$

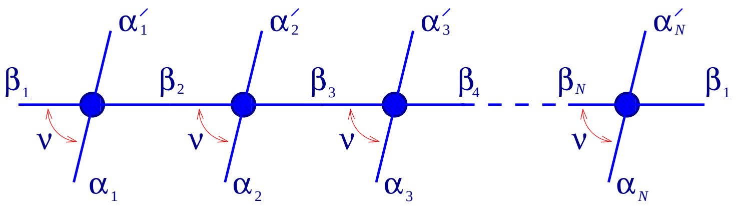


To calculate the partition function Z , define the transfer matrix, $\mathbf{T}$:

$$\mathbf{T}^{\{\alpha'\}}_{\{\alpha\}}(\nu) \ = \ \sum_{\{\beta_i\}} W\begin{bmatrix} & \alpha'_1 & \\ \beta_1 & & \beta_2 \\ & \alpha_1 & \end{bmatrix} W\begin{bmatrix} & \alpha'_2 & \\ \beta_2 & & \beta_3 \\ & \alpha_2 & \end{bmatrix} \dots W\begin{bmatrix} & \alpha'_N & \\ \beta_N & & \beta_1 \\ & \alpha_N & \end{bmatrix}$$

In pictures:

$$\mathbf{T}(\nu) \ = \ \sum_{\{\beta_i\}}$$



In terms of $\mathbf{T}$ the partition function is

$$Z = \text{Trace}\left[\mathbf{T}^M\right] \ .$$

The free energy per site in the limit $M \to \infty$ can be obtained as

$$f = \frac{1}{NM} \ln Z = \frac{1}{NM} \ln \text{Trace}\left[\mathbf{T}^M\right] \sim \frac{1}{N} \ln t_0 \ ,$$

where t_0 is the ground-state eigenvalue of $\mathbf{T}$. **Baxter's T-Q relation** : there exists an auxiliary function $q(\nu)$

$$q(\nu) = \prod_{n=0}^{N/2-1} \sinh(\nu - \nu_l) \ ,$$

such that

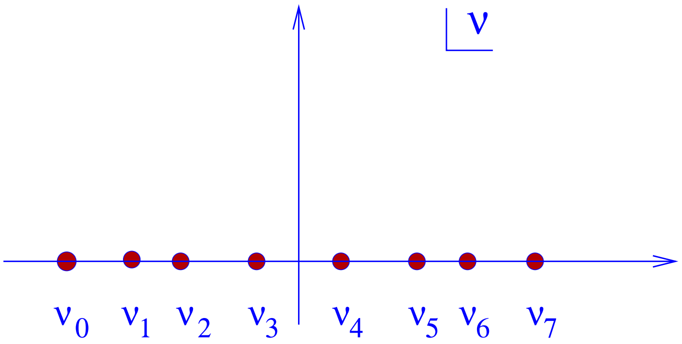
$$t_0(\nu) q(\nu) = a(\nu, \eta)^N q(\nu + 2i\eta) + b(\nu, \eta)^N q(\nu - 2i\eta)$$

BAE then emerge as a *consequence* of the fact that both t_0 and q are entire. Setting

$$q(\nu_i) = 0 \ ,$$

we find

$$-1 = \frac{a^N(\nu_i, \eta)}{b^N(\nu_i, \eta)} \frac{q(\nu_i + 2i\eta)}{q(\nu_i - 2i\eta)} \ .$$



Defining

$$\lambda_i = e^{2\nu_i} \quad , \quad \omega = -e^{i2\eta} \ ,$$

the BAE becomes

$$-1 = \left(\frac{1+\lambda_i/\omega}{1+\lambda_i\omega}\right)^N \prod_{n=0}^{r-1} \frac{(\lambda_n - \lambda_i\omega^2)}{(\lambda_n - \lambda_i\omega^{-2})} \ .$$

An interesting scaling limit is achieved by sending

$$N \to \infty \quad \text{and} \quad a = e^{\pi\nu/2\eta} \to 0 \ ,$$

with aN finite. Sending

$$a \to 0 \quad \text{as} \quad a \to a/N$$

one discovers that the λ_i 's for $i \ll \ln N$ rescale to zero as

$$\lambda_i \sim E_i a^{4\eta/\pi} \sim E_i N^{-4\eta/\pi} \ ,$$

for $\pi/4 < \eta < \pi/2$ one easily obtains

$$-1 = \prod_{n=0}^{\infty} \frac{(E_n - E_i\omega^2)}{(E_n - E_i\omega^{-2})} \ .$$

The 6V with twisted boundary conditions is also integrable. The twist is introduced by changing:

$$W\left[\beta_N \begin{matrix} \alpha'_N \\ \alpha_N \end{matrix} \beta_1\right] \to e^{i(\delta_{\beta_1,\uparrow} - \delta_{\beta_1,\downarrow})\phi} W\left[\beta_N \begin{matrix} \alpha'_N \\ \alpha_N \end{matrix} \beta_1\right]$$

in the original model. In the ‘scaling limit’ this leads to the more general BAE

$$-1 = e^{i2\phi} \prod_{n=0}^{\infty} \frac{(E_n - E_i\omega^2)}{(E_n - E_i\omega^{-2})} \ .$$

The connection formula

Consider an ODE of the form

$$\left(-\frac{d^2}{dx^2}+P(x,E)\right)\phi(x,E)=0\ ,$$

for x and E complex. Set

$$\omega=e^{\frac{i\pi}{M+1}}=-e^{-i2\eta}\ .$$

If

$$\omega^2 P(\omega x,\omega^{-2}E)=P(x,E)\ ,$$

then

$$\phi_k(x,E)=\phi(\omega^k x,\omega^{-2k}E)\qquad (k=1,2,\ldots)\ ,$$

are solutions of the same ODE, and

$$W_{-1,1}\ \phi_0(x,E)=W_{-1,0}\ \phi_1(x,E)+W_{0,1}\ \phi_{-1}(x,E)\ ,$$

where

$$W_{i,j}(E)=\phi_i(x,E)\phi'_j(x,E)-\phi'_i(x,E)\phi_j(x,E)\ .$$

Suppose that $\phi(0,E)=\text{FINITE}$ and set

$$E=e^{2\nu}\quad,\quad \phi(0,E)\to\phi(\nu)\quad,$$

$$W_{-\frac{1}{2},\frac{1}{2}}(E)\to W(\nu)\quad,\quad W_{-1,1}(E)\to W_{-1,1}(\nu)\ ,$$

then

$$W_{-1,1}(\nu)\phi(\nu)=W(\nu+i\eta)\phi(\nu-2i\eta)+W(\nu-i\eta)\phi(\nu+2i\eta)\ .$$

The ODE/IM correspondence

The Baxter's TQ relation is

$$t_0(\nu)Q(\nu) = \sinh^N(\nu+i\eta)Q(\nu-2i\eta)+\sinh^N(\nu-i\eta)Q(\nu+2i\eta) \ ,$$

so

$$W(\nu) = W_{-1/2,1/2}(\nu) \sim \sinh^N(\nu) \ , \ W_{-1,1}(\nu) \sim t_0(\nu) \ ,$$

and

$$\phi(\nu) \equiv \phi(x=0, E=e^{2\nu}) \sim Q(\nu)$$

$\Longrightarrow$ an ODE/IM correspondence (!!). Known cases:

$$A) \quad \left(-\frac{d^2}{dx^2} + (x^\alpha - E) + \frac{l(l+1)}{x^2}\right)\phi(x,E) = 0$$

DT,BLZ: twisted 6V (XXZ) model.

$$B) \ \left(-\frac{d^2}{dx^2} + (x^\alpha + a x^{\frac{\alpha}{2}-1} - E) + \frac{l(l+1)}{x^2}\right)\phi(x,E,a) = 0$$

JS,DDT: 3 state Perk-Schulz model.

$$C) \ \left((-1)^{n+1}\frac{d^n}{dx^n} + (x^\alpha + E) + \text{inv. terms}\right)\phi(x,E) = 0$$

DDT, JS, B-Hibberd, BL-Tsvelik: $SU(n)$, Coqblin-Schrieffer models.

$$D) \ \left(-\frac{d^2}{dx^2} + (x^\alpha - E)^{2S} + \frac{l(l+1)}{x^2}\right)\phi(x,E) = 0 \ ,$$

Lukyanov: spin S XXZ quantum chains.

Notice, at $l = 0$ $\alpha = 2$ and $\alpha = 1$ the solutions can be given explicitly, in terms of hypergeometric functions. For

$$\alpha = 1 : \qquad E \leftrightarrow x$$

setting $z = x - E$

$$-\frac{d^2}{dz^2}\psi(z) + z\psi(z) = 0 \qquad \text{(Airy)}$$

and therefore (Fendley, DT):

$$Q(E)|_{\alpha=1} = \mathrm{Ai}(-E) \ ,$$

and the function $Q(E)$ satisfies a differential equation!

Finite Lattice version: (E, x) -democracy??

Fridkin, Stroganov and Zagier:

the Q function for the $\Delta = 1/2$, spin $\frac{1}{2}$ XXZ quantum chain with free-free BCs is related to a specialisation of the Pöschl-Teller equation

$$\left(-\frac{d^2}{ds^2} - \frac{9n(2n+1)}{2\cosh^2 3s/2}\right)\chi(s) = -\chi(s) \ .$$

(**Stroganov:** $\Delta = -1/2$, periodic BCs also related to Pöschl-Teller with $2n =$ odd)

The Pöschl-Teller equation

Consider the equation

$$\left(-\frac{d^2}{ds^2}-\frac{N(N+1)}{\cosh^2 s}+\frac{M(M+1)}{\sinh^2 s}\right)\chi(s)=-\sigma^2\chi(s).$$

4 regular singularities $\to$ **Heun** equation.

In **QM**, $\chi(s)$ is usually defined on the real axis, and the requirement of L^2 constrains σ to $\mathbb{Z}$

(Sol: ${}_2F_1$).

Our case: we demand that $\chi(s)$ be single-valued.

Around the singularities of $\cosh^{-2} s$ and $\sinh^{-2} s$ restricts N and M to integer values.

Define $x^L = -\exp(2s)$, and set

$$m=\sigma L-1\ , \qquad \psi_{M,N,m}(x)=x^{(m+1)/2}\chi(\ln(\sqrt{-x^L}))\ .$$

Then

$$\left[\frac{d^2}{dx^2}-\frac{m}{x}\frac{d}{dx}-\frac{L^2N(N+1)x^{L-2}}{(x^L-1)^2}+\frac{L^2M(M+1)x^{L-2}}{(x^L+1)^2}\right]\psi=0$$

The points $x=0$ and $x=\infty$ are, in general, singular points, and

$$\psi_{M,N,m}(x)\sim\alpha+\beta x^{m+1}+\dots\quad ,$$

therefore single-valuedness constrains $m\in\mathbb{Z}$:

$$N,M,L,m\in\mathbb{Z}\qquad(\geq 0)$$

To single out one particular solution, impose

$$\psi_{M,N,m}(x)|_{x\sim 1} \sim (1-x)^{N+1} \ .$$

Near the regular singularities at

$$(x^L \mp 1)|_{x=x_{i,\pm}} = 0$$

in general

$$\begin{aligned} \psi_{M,N.m}(x) &\sim c(x-x_{i-})^{-N} \ , \quad (x^L \sim 1) \ , \\ \psi_{M,N,m}(x) &\sim \tilde{c}(x-x_{i+})^{-M} \ , \quad (x^L \sim -1) \ . \end{aligned}$$

Exceptionally for $c = 0$

$$\psi_{M,N,m}(x) \sim d(x-x_{i-})^{N+1} \ ,$$

which, up to the (arbitrary) normalisation, is exactly the condition imposed at $x = 1$.

Notice that equation is invariant under the transformation

$$(x, M, N) \to (x\omega^{1/2}, N, M) \ ,$$

where $\omega = \exp(2i\pi/L)$, and

$$(x, M, N) \to (x\omega, M, N) \ .$$

These symmetries force further zeroes in $\psi(x)$.

Being images of $x = 1$, they are located on the unit circle, and are, in some sense, trivial.

They will, however, contribute non-trivially to the BAE which will fix the nontrivial zeroes. The determination of these zeroes is a bit technical...

The field theory limit

It is interesting to find the field theory limit of the equation. Send

$$N \to \infty \ , \quad \text{keeping} \quad L, M \text{ and } m \quad \text{finite}$$

At the same time we focus on the region near $x = 0$ by introducing a new variable z

$$x^L = \frac{z^L}{L^2 N(N+1)}$$

and confine our analysis to the region

$$|z^L| \ll L^2 N(N+1).$$

Taking $N \to \infty$ limit,

$$\psi_{M,N,m}(z) \to \psi_m(z)$$

with

$$\frac{d^2\psi_m(z)}{dz^2} - \frac{m}{z}\frac{d\psi_m(z)}{dz} - z^{L-2}\psi_m(z) = 0 \ .$$

At $m = 0$ and $L = 3$, this equation coincides with the Airy equation.

For $m = 0$ and L general it coincides with the

$$\alpha = 1 \ , \quad l = 0 \ , \quad S = (L-2)/2$$

case of the equation identified by **Lukyanov** with the scaling limit of the spin S XXZ models.

Complex analysis of the equation

Set $g(x) = \psi_{M,N,m}(x)$ and define

$$g_k(x) = g(\omega^k x) \ .$$

Then $g_1(x)$ and $g_{-1}(x)$ are also solutions of the equation. Near $x = 1$:

$$g_1(x) \sim c_+(x-1)^{-N} \quad , \quad g_{-1}(x) \sim c_-(x-1)^{-N} \ ,$$

and so $\{g_0(x), g_1(x)\}$ is (apart from $m = L{-}1 \bmod L$) a basis of solutions. Expanding g_{-1} :

$$W_{-1,1}\ g_0(x) = W_{-1,0}\ g_1(x) + W_{0,1}\ g_{-1}(x) \ .$$

Because of $-\frac{m}{x}\frac{dg(x)}{dx}$:

$$W[g,f] = cst\ x^m \ ,$$

and one can factorise x^m out.

Use the large x asymptotic

$$g(x) \sim a + bx^{m+1} + \ldots$$

to get

$$\begin{aligned} W[-1,0] &= 2iab(m+1)\omega^{(m+1)/2}\sin(\tfrac{m+1}{L}\pi)x^m \quad , \\ W[0,1] &= 2iab(m+1)\omega^{-(m+1)/2}\sin(\tfrac{m+1}{L}\pi)x^m \quad , \\ W[-1,1] &= 4iab(m+1)\sin(\tfrac{m+1}{L}\pi)\cos(\tfrac{m+1}{L}\pi)x^m \ , \end{aligned}$$

and then

$$2\cos(\tfrac{m+1}{L}\pi)g(x) = \omega^{\frac{m+1}{2}}g(\omega^{-1}x) + \omega^{-\frac{m+1}{2}}g(\omega x) \ .$$

which is identical to a TQ relation, save for the fact that $Q(x)$ is, by definition, entire while $g(x)$ is not.

Define $q(x)$ as:

$$(x^L-1)^N(x^L+1)^M g(x) \quad = \quad \prod_{j=0}^{\ell-1}(x-(\omega')^j)^{2N+1} \quad \prod_{i=1}^{N_k}(x-(\omega)^{\frac{2k_i+1}{2}})^{2M+1} q(x)$$

where

$$\ell = GCD(m+1, L)\ , \quad \omega = e^{2i\pi/L}\ , \quad \omega' = e^{2i\pi/l}\ ,$$

the k_i 's are positive integers satisfying

$$(2k_i+1)(N+M-\frac{m+1}{L}) \in 2\mathbb{Z}\ , \quad (1 \le k_i \le L-1)$$

and N_k ($= 0$ or l) is the number of k_i 's.

Then

$$q(x) = \prod_{i=1}^{K}\left(1-\frac{x}{x_j}\right),$$

with $x_j = 1/x_{K+1-j}$. The function $q(x)$ also satisfies a TQ-type relation, and it is entire (!)

Notice that

$$\psi_{M,N,m}(x \to \infty) \sim x^{m+1} + \dots \qquad ((m+1) > 0)\ ,$$

is a meromorphic function of x. Then, $m+1$ should be equal to the total number of zeroes minus the total number of poles existing at finite x :

$$K = (m+1)+N(L-\ell)+M(L-N_k)-\ell(N+1)-N_k(M+1)\ .$$

To match the standard notation, set $x = -\exp(u)$

$$\tau(u)Q(u) = \phi(u-2i\eta)Q(u-2i\eta)+\phi(u+2i\eta)Q(u+2i\eta) ,$$

where

$$\begin{aligned}
Q(u) &= \prod_{j=1}^{K} \sinh(\frac{u-u_j}{2}) , \\
\phi(u) &= \prod_{j=0}^{\ell-1} \cosh^{2N+1}(\frac{u}{2} - i\eta' j) \prod_{i=1}^{N_k} \cosh^{2M+1}(\frac{u}{2} - i\frac{2k_i+1}{2}\eta) , \\
\tau(u) &= (-1)^{N+M} 2\cos(\tfrac{m+1}{L}\pi)\phi(u) ,
\end{aligned}$$

with $\eta = \pi/L$ and $\eta' = \pi/\ell$.

Model identification

The XXZ Hamiltonian for $S = \frac{1}{2}$, and periodic boundary conditions:

$$\mathcal{H} = \sum_{n=1}^{\mathcal{N}_s} (\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{\Delta}{2} \sigma_n^z \sigma_{n+1}^z) .$$

Then the following relation holds at general spin S

$$T_s(u)Q_s(u) = \phi_s(u-2iS\lambda)Q_s(u+2i\lambda)+\phi_s(u+2iS\lambda)Q_s(u-2i\lambda) ,$$

with

$$Q_s(u) = \prod_j \sinh\frac{u-v_j}{2} ,$$

$$\phi_s(u) = \prod_{\alpha=1}^{\mathcal{N}_s} \sinh \frac{u - \omega_\alpha}{2} \ ,$$

where ω_α stands for the inhomogeneity. Simple example ($M = 0$ and $m = 0$ and L odd):

$$\phi(u) = \cosh^{2N+1}(\frac{u}{2}) \ ,$$

$$\cosh(\frac{u}{2} \pm i\frac{\pi}{L}) = \pm i \sinh(\frac{u}{2} \mp i\frac{L-2}{2L}\pi) = \pm i \sinh(\frac{u}{2} \mp i(L-2)\frac{\eta}{2}) \ ,$$

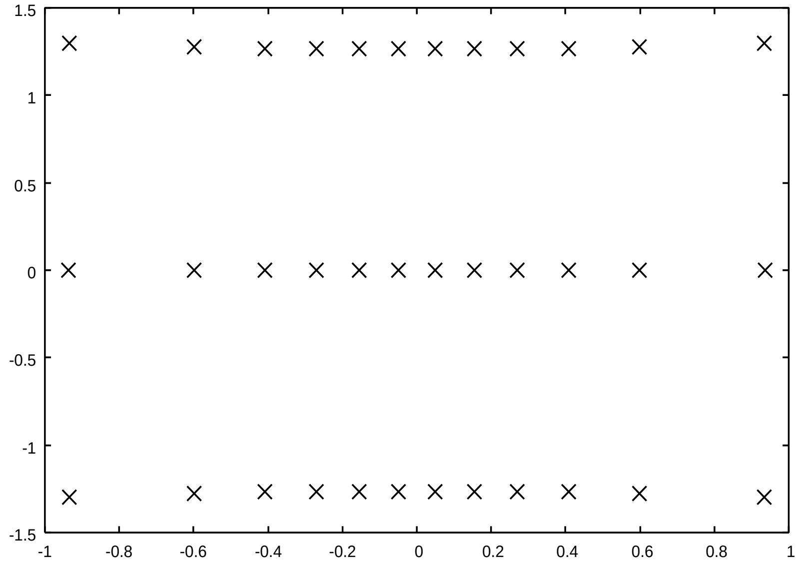
and the connection rule coincides with the TQ for the spin $\frac{L-2}{2}$ with periodic BCs and an odd number of sites $\mathcal{N}_s = 2N + 1$:

$$\lambda = \pi - \eta = \frac{(L-1)\pi}{L} \ , \quad u_j = v_j + \pi i \ ,$$

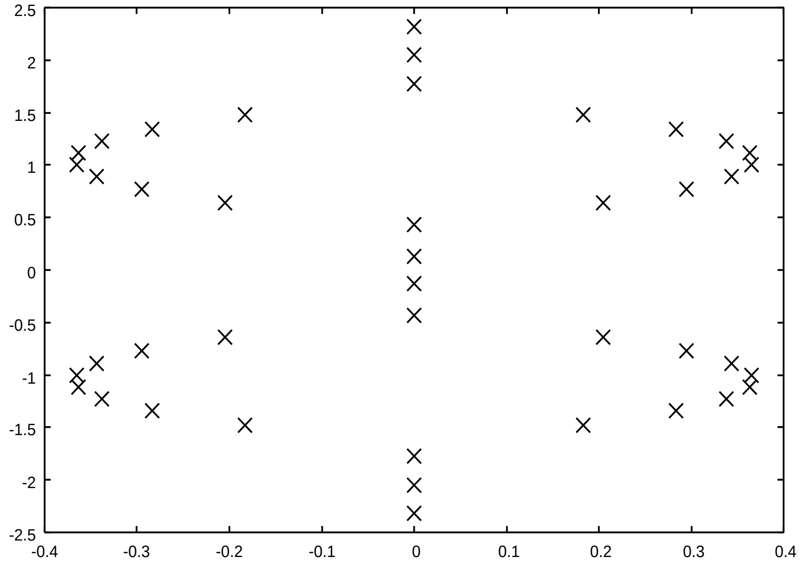
$$\tau(u) = (-1)^{\frac{L}{2}} T_s(u + \pi i) \ , \quad \omega_\alpha = 0$$

At $L = 3$, the findings of Stroganov are recovered.

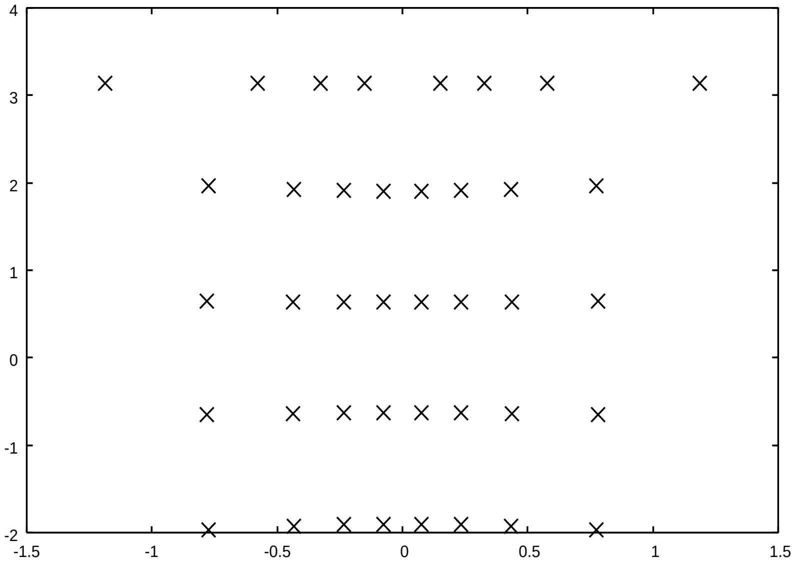
PS: there are ambiguities in the choice of functions in the lattice model and the ODE..



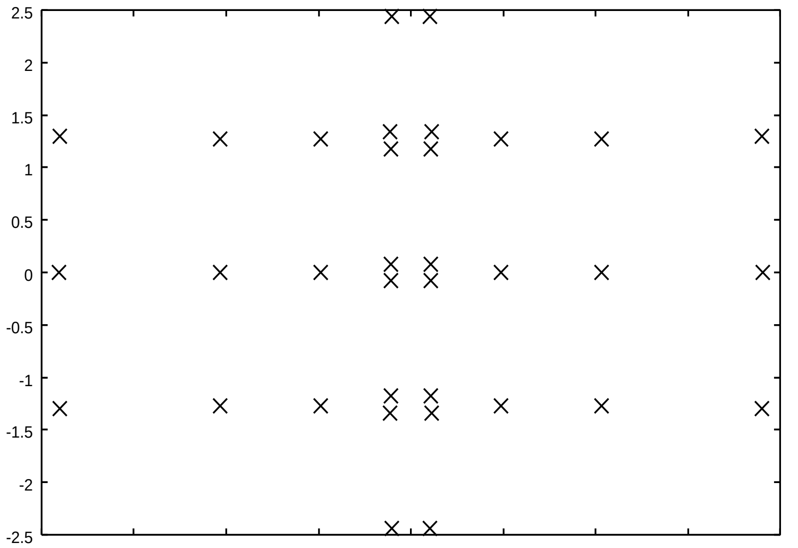
Zeroes of Q at $(M, N, L, m) = (0, 12, 5, 0)$, which illustrates the 3 string solution.



$(M, N, L, m) = (0, 8, 3, 34)$.



$(M, N, L, m) = (8, 0, 5, 0)$: we have $M-$ string in the vertical direction.



The $(M, N, L, m) = (2, 8, 5, 0)$ case.

Open problems and conclusions

- We have shown that there exist non-trivial examples of the ODE/IM correspondence involving finite lattice models

- A main open problem is the generalisation outside this set of values and to excited states.

- The correspondence at the level of QFT has been extended in to higher order ODEs. These more complicated families of systems also possess exactly solvable points. At least for these cases, the finite lattice extension of the correspondence should be straightforward.

- The correspondence remains at a rather formal and mathematical level. At some stage one should ask what physics lies behind all of this.