Quantum Spin Chains, Loop Models and Alternating Sign Matrices

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Recent remarkable observations by Razumov & Stroganov, and Batchelor, de Gier & Nienhuis and others

Main idea:
The components of the ground state of some quantum systems are non negative integers. Combinatorial interpretation?

Plan

I A review of the main characters

XXZ Chain $\leftrightarrow$ $O(1)$ Dense Loop Model

Temperley-Lieb Algebra

Alternating Sign Matrices $\leftrightarrow$ Fully Packed Loops

II Some new conjectures

III Further Directions

A review of the main characters

XXZ antiferromagnetic chain

$$\mathcal{H} = -\frac{1}{2}\sum_{j=1}^{L} \sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z + \text{boundary terms}$$

For $\Delta = -\frac{1}{2}$, periodic chain and L odd,

Razumov & Stroganov : ground state $|\Psi\rangle$ has energy $= -3L/4$ and ...

components of $|\Psi\rangle$ (in basis $|\sigma_1^z \cdots \sigma_L^z\rangle$) are integers!

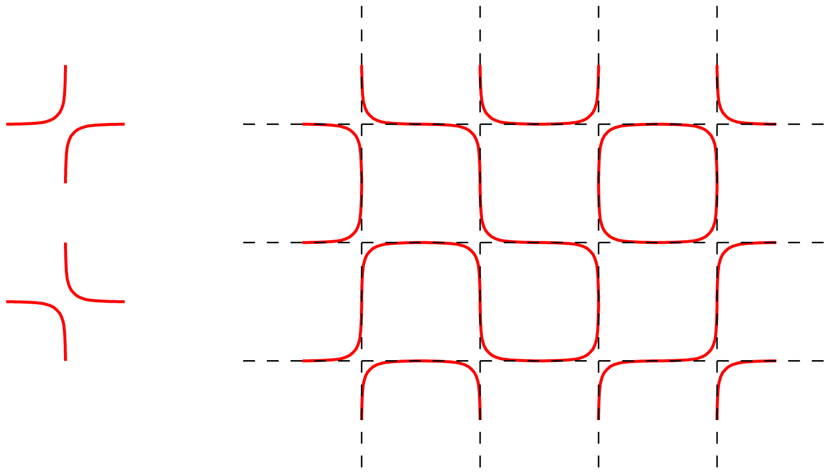
What are these integers?

Hamiltonian in terms of the Temperley-Lieb Algebra [Pasquier-Saleur 89]

Look at a related model...

Dense $O(1)$ Loop Model

Non intersecting closed loops on a square lattice, all edges occupied, each closed loop : weight 1, with suitable b.c.



On a "time" slice: states described by "connectivity patterns"

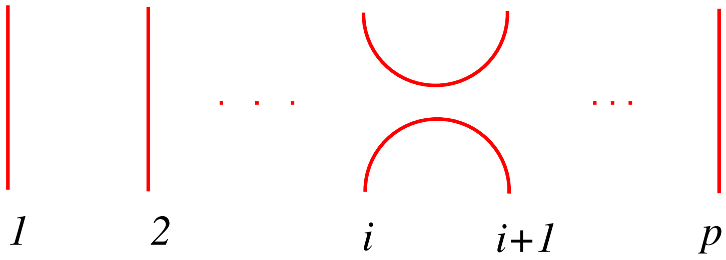
Hamiltonian (from transfer matrix) : Temperley-Lieb algebra $\mathcal{H} = -\sum e_i$

Temperley-Lieb Algebra

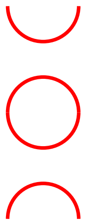
Generators e_i, $i = 1, \cdots, p$

$$e_i^2 = e_i; \qquad e_i e_j = e_j e_i \quad \text{if} \quad |i-j| > 1; \qquad e_i\, e_{i\pm1}\, e_i = e_i$$

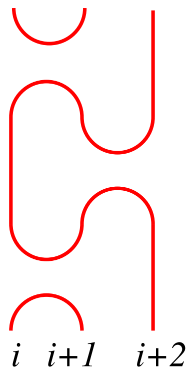
e_i =



e_i^2 =

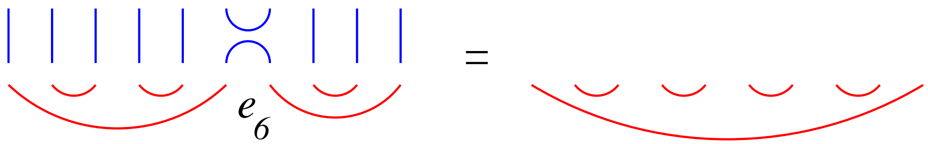


= e_i $\qquad$ $e_i\ e_{i+1}\ e_i$ =



= e_i

act on connectivity patterns



Alternating Sign Matrices (ASM)

Square $n \times n$ matrices with entries $0, +1, -1$, such that

- signs $+1$ and -1 alternate along each row and each column
- the sum is $+1$ along each row and each column

Example There are seven 3×3 ASM :

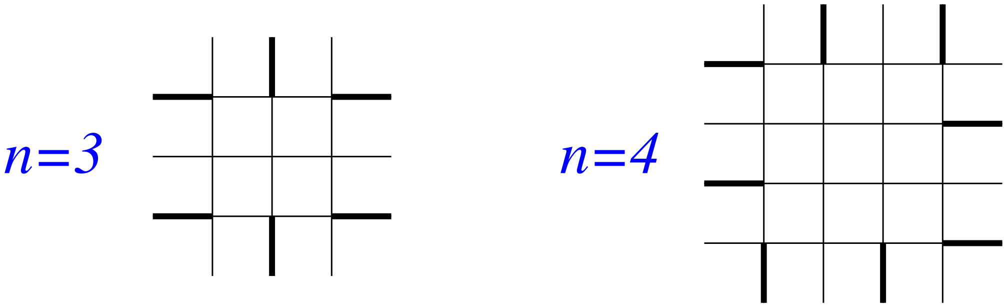
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Number of ASM of size n

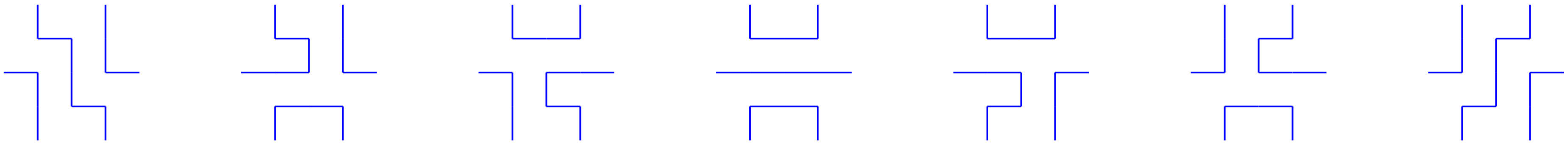
$$A_n = \prod_{j=1}^{n} \frac{(3j-2)!}{(n+j-1)!} = 1, 2, 7, 42, 429, \cdots$$

Fully Packed Loops (FPL)

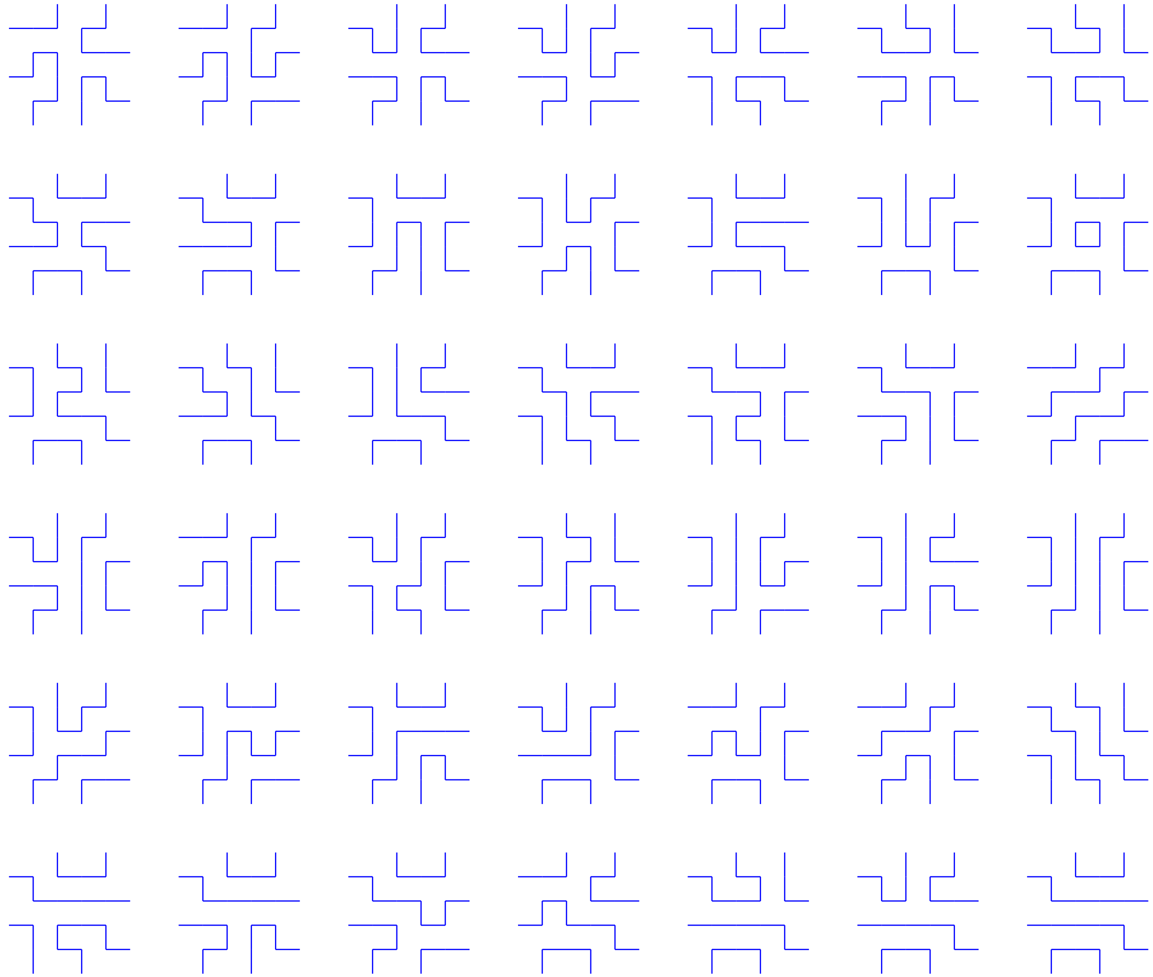
On a $n \times n$ grid, with $4n$ external links,
sets of disconnected paths passing through each of the n^2 vertices
and exiting through every second of the external links



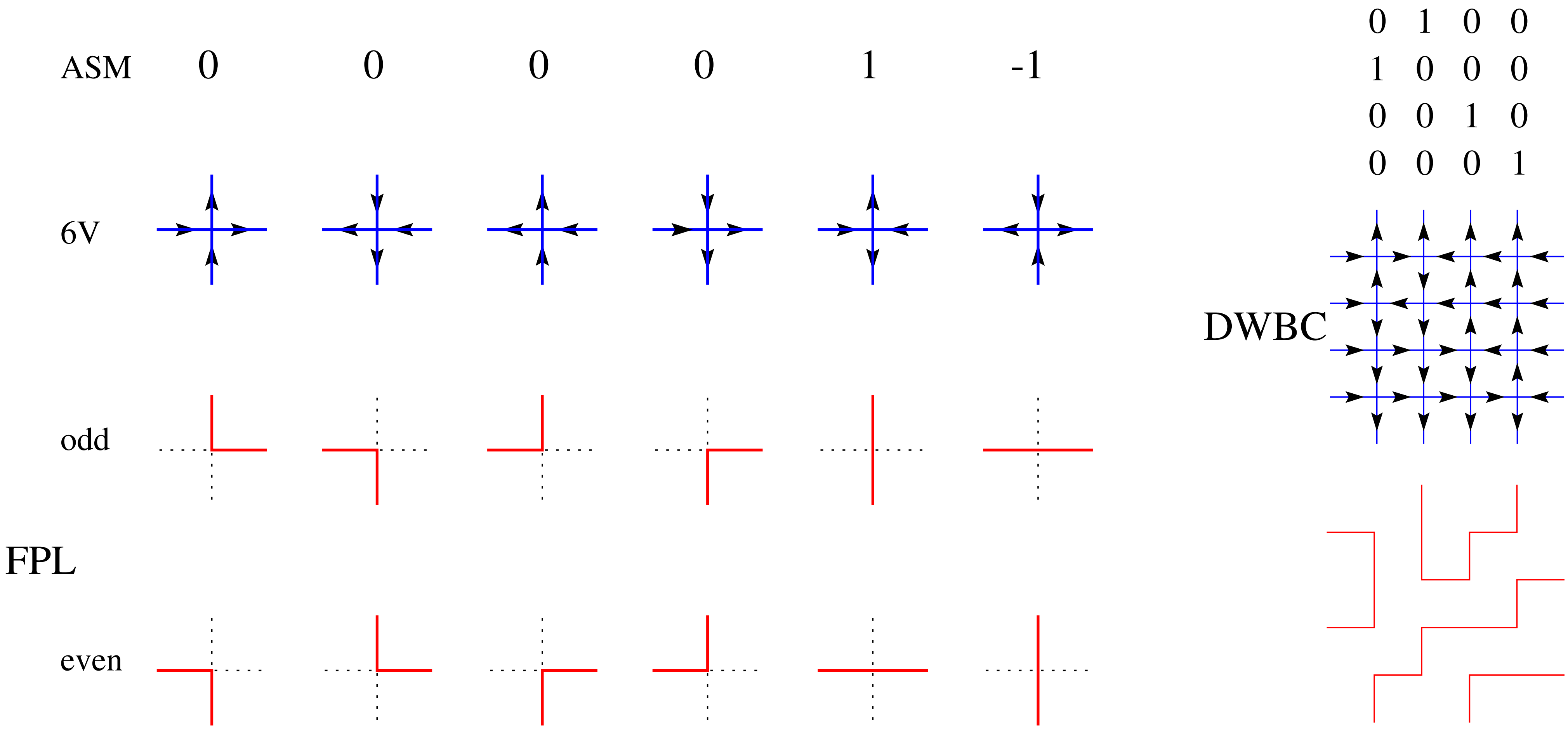
There are 7 FPL on a 3×3 grid



The 42 FPL on a 4×4 grid

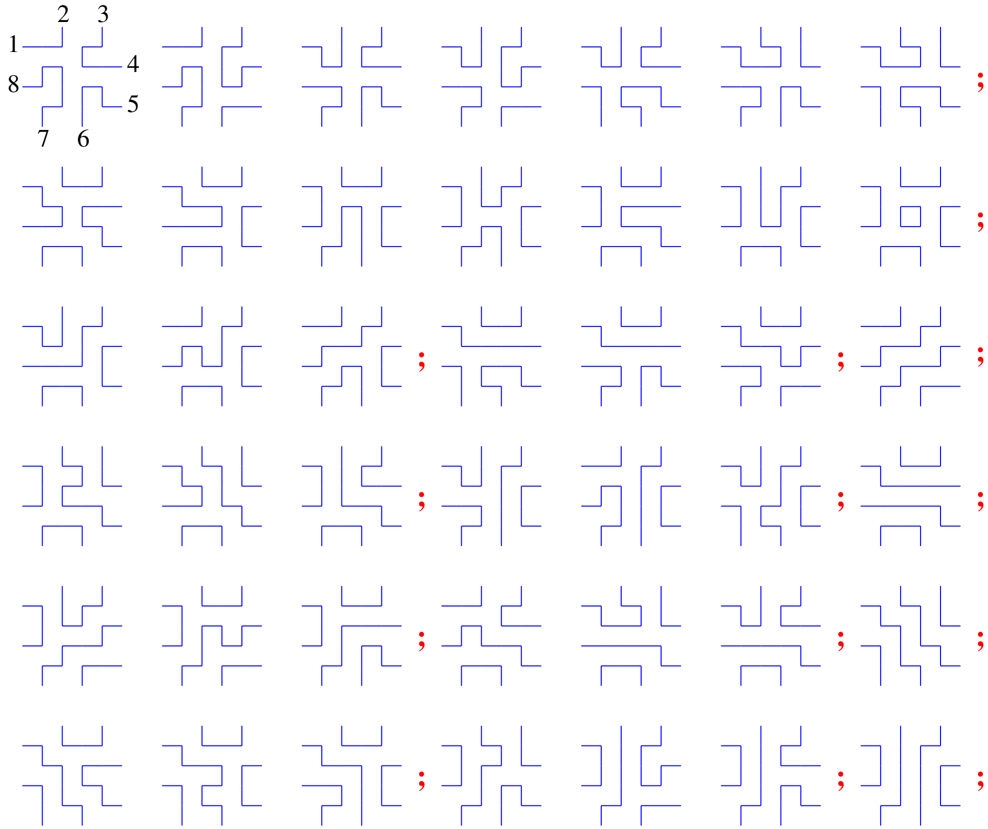


ASM ↔ 6 Vertex ↔ FPL



For a given size n, FPL configurations fall into Connectivity Classes (or "link patterns") π of their external links.

There are
$C_n = \frac{(2n)!}{(n+1)!n!}$
(Catalan number)
distinct link patterns π.
For example, for $n = 4$, 14 classes

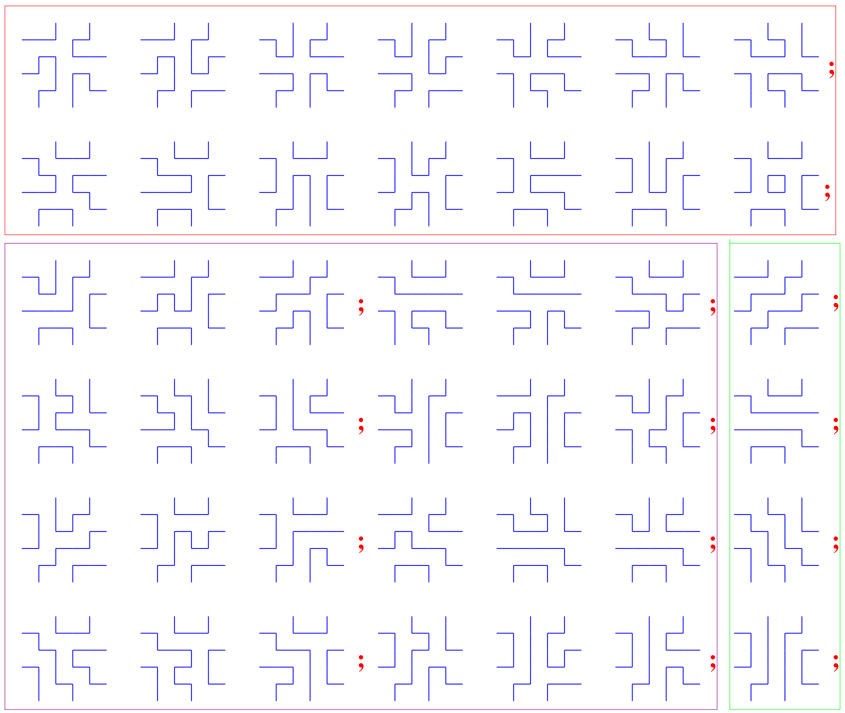


We want the numbers $A_n(\pi)$ of FPL configurations pertaining to π.

An unexpected dihedral symmetry of the $A_n(\pi)$

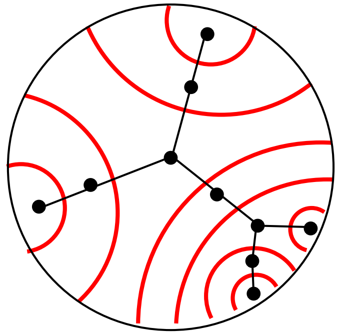
Theorem [Wieland 2000]
If π and π' are obtained from one another by $\sigma \in D_n$, the dihedral group, then $A_n(\pi) = A_n(\pi')$.

For $n = 4$, 3 independent $A_4(\pi)$



How many independent $A_n(\pi)$? i.e. how many orbits under the dihedral group?

Dual of a link pattern π is a Planar Projective Tree



Generating function of the numbers of PPTs computed by Stockmeyer

$$T(x) = x + x^2 + 2x^3 + 3x^4 + 6x^5 + 12x^6 + 27x^7 + 65x^8 + 175x^9 + 490x^{10} + 1473x^{11} + 4588x^{12} + \cdots$$

Reduced Hamiltonian on these orbits …

Razumov-Stroganov conjecture

Periodic boundary conditions, even number of sites

The Perron-Frobenius eigenvector of $\mathcal{H} = \sum_{i=1}^{2n} e_i$ is

$$|\Psi\rangle = \sum_a A_n(\pi_a)|\pi_a\rangle \qquad : \qquad \mathcal{H}|\Psi\rangle = 2n|\Psi\rangle$$

i.e. its components are the $A_n(\pi)$ (with proper normalization) [R&S 2001]

Other types of b.c. on TLA $\leftrightarrow$ different symmetry classes of ASM/FPL

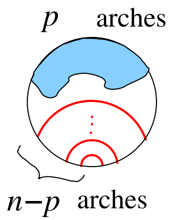
* periodic, odd number of sites : connection with half turn symm. ASM/FPL
* open, even number of sites : connection with vertically symm. ASM/FPL

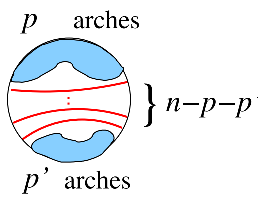
[Razumov-Stroganov 2001; Pearce, de Gier & Rittenberg 2001, …, Mitra et al. to appear 2003]

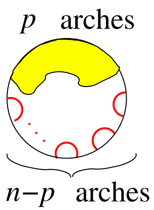
Some (mostly new) formulae

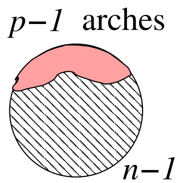
Beware! everything is just guessed!! from the data of Periodic (IC : "identified connectivities") TLA Hamiltonian, up to $L = 2n = 22$.

Three types of results

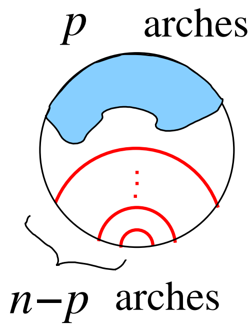
- Explicit formulae for simple link patterns of the form 

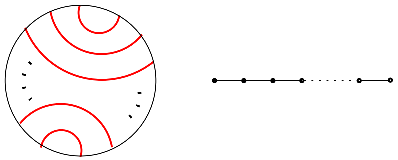
- General form and Asymptotic behavior for large n of 

- Explicit formulae for 

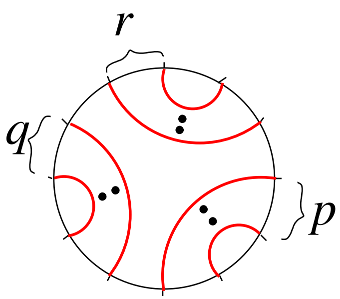
and relations with 

Expression of simple



Recall that  = 1

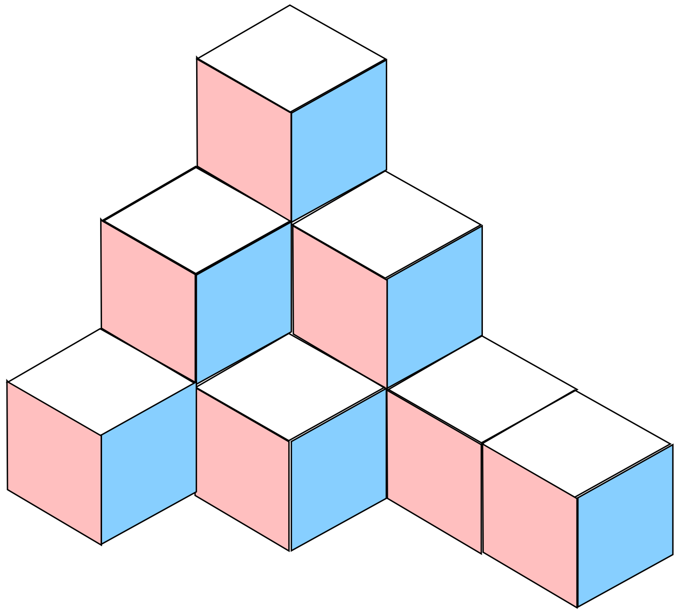
Introduce the "superfactorial" $m\overset{2}{!} := \prod_{r=1}^{m} r!\ , \quad (-1)\overset{2}{!} = 0\overset{2}{!} = 1.$



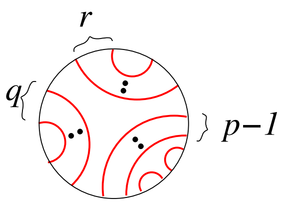
$$= \frac{(p+q+r-1)\overset{2}{!}\,(p-1)\overset{2}{!}\,(q-1)\overset{2}{!}\,(r-1)\overset{2}{!}}{(p+q-1)\overset{2}{!}\,(q+r-1)\overset{2}{!}\,(r+p-1)\overset{2}{!}} \qquad p,q,r, \geq 0$$

Unexpectedly, this is also the number of plane partitions in a box $p \times q \times r$ (MacMahon formula)

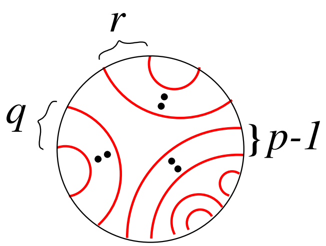
$$\prod_{i=1}^{p}\prod_{j=1}^{q}\prod_{k=1}^{r}\frac{i+j+k-1}{i+j+k-2}\,.$$



Unfortunately, formulae for more complicated link patterns are very messy!
For example



$$= \frac{(q-1)\overset{2}{!}\,(r-1)\overset{2}{!}}{(q+r-1)\overset{2}{!}}\;\frac{p\overset{2}{!}\,(p+q+r)\overset{2}{!}\,(p+q)!(p+r)!}{(p+q+1)\overset{2}{!}\,(p+r+1)\overset{2}{!}} \times$$
$$\times\left[p^3+2p^2(q+r+1)+p(q^2+qr+r^2+3(q+r)+1)+q(q+1)+r(r+1)\right]$$

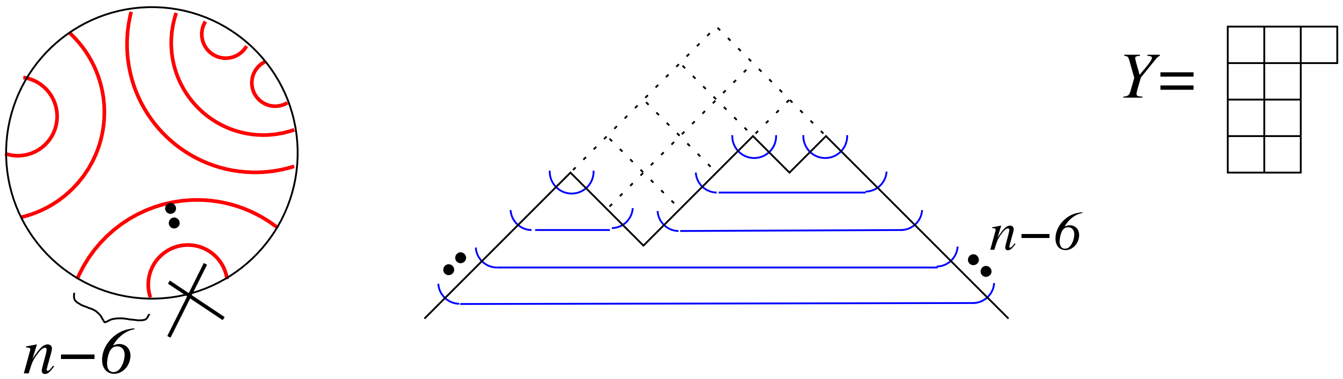


$$= \frac{(q-1)\overset{2}{!}\,(r-1)\overset{2}{!}}{2(q+r-1)\overset{2}{!}}\;\frac{(p+1)\overset{2}{!}\,(p+q+r+1)\overset{2}{!}}{(p+q+3)\overset{2}{!}\,(p+r+3)\overset{2}{!}}$$
$$\times(p+q+2)!(p+q+1)!(p+r+3)!(p+r)!$$

$$\times\Big[p^5+p^4(7+4q+4r)+p^3(17+22q+6q^2+24r+10qr+6r^2)$$
$$+p^2(17+40q+24q^2+4q^3+46r+42qr+8q^2r+30r^2+8qr^2+4r^3)$$
$$+p(6+28q+29q^2+10q^3+q^4+32r+49qr+17q^2r+2q^3r+41r^2+23qr^2+3q^2r^2+16r^3+2qr^3+r^4)$$
$$+6q+11q^2+6q^3+q^4+6r+13qr+3q^2r+15r^2+15qr^2+3q^2r^2+12r^3+2qr^3+3r^4\Big]$$

Is there some pattern ?. . .

Represent an arch system by a Dyck path or by the complementary Young tableau:

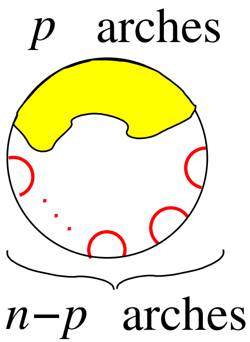


Conjecture $A_n(\pi) = \frac{1}{|Y|!} P_Y(n)$, P_Y polynomial of degree $|Y|$ with coeffts in $\mathbb{Z}$

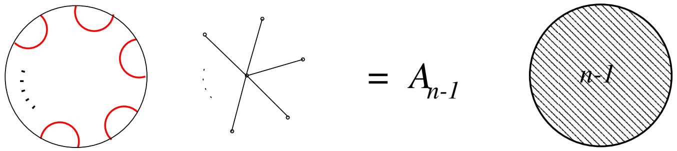
and for n large $A_n(\pi) \approx \frac{\dim Y}{|Y|!} n^{|Y|}$

Consistent with eigenvector equation $\sum_i e_i |\Psi\rangle = 2n|\Psi\rangle$ [P. Di Francesco]

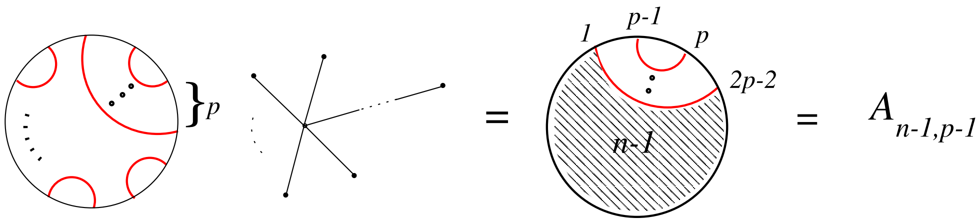
Expression of simple



Batchelor, de Gier, Nienhuis noticed that



Generalized to

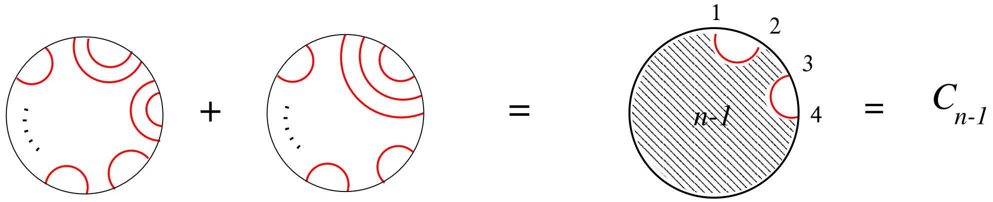


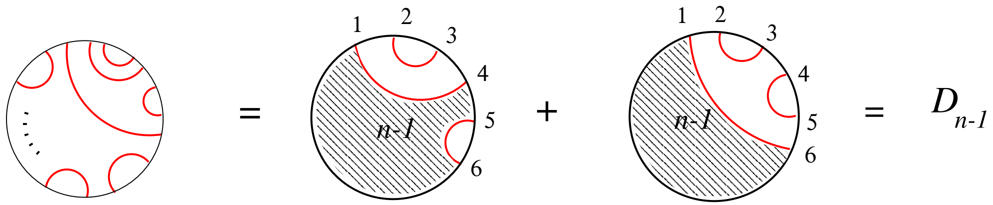
[Wieland 2000]

$$A_{n,p} = \frac{P_{p(p+1)}(n^2)}{\prod_{\ell=1}^{p}(4n^2-(2\ell-1)^2)^{p+1-\ell}} A_n,$$

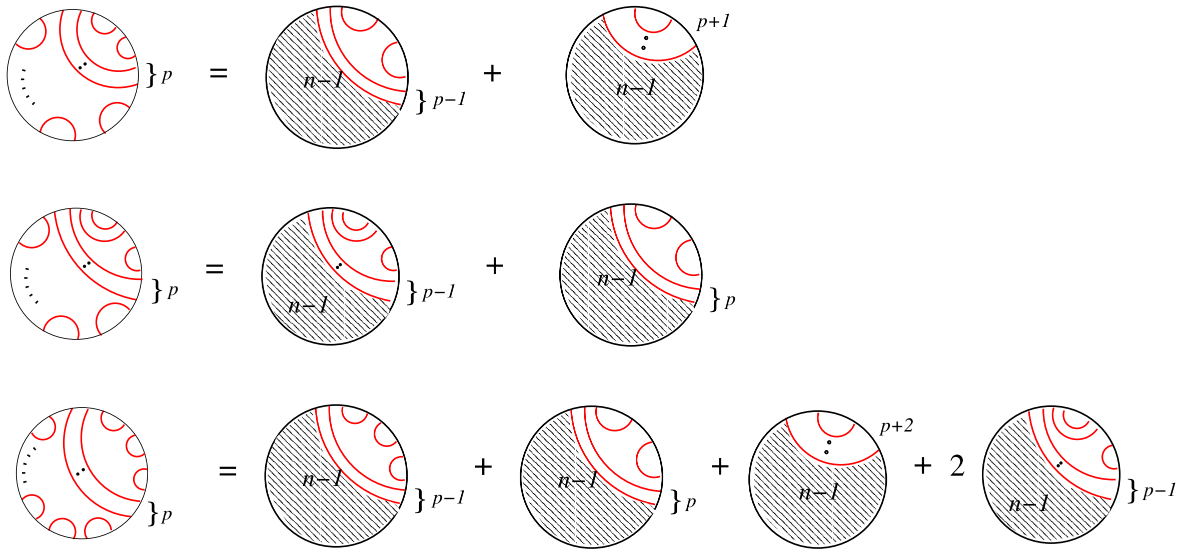
for example, $A_{n,1} = \frac{3}{2}\frac{n^2+1}{4n^2-1}$ [Wilson], $A_{n,2}, A_{n,3}$ also known.

But also





Expression of C_n, D_n also known : again rational fractions in n^2



Lessons and questions:

(-) wide set of "recursion formulae"

(-) what is their origin?

(-) significance of their pattern (evenness in n, etc)?

(-) simplifying role of periodic b.c. (compare [Mitra, Nienhuis, de Gier & Batchelor])

Further Directions

Other types of b.c., other quantities [Mitra, Nienhuis, de Gier & Batchelor]

Conformal limit : Logarithmic CFT ?. . .[Read & Saleur 2001, Pearce, Rittenberg & de Gier 2001, Pearce & Z 200x]

A Quiz !

Who wrote

Florence est ville et fleur et femme,
elle est ville-fleur et ville-femme et fille-fleur tout à la fois.

???