Entanglement in Strongly Correlated Systems: Scaling at Critical Points *vs* Long-distance Entanglement in Condensed Matter Physics

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Role of entanglement at QPT's

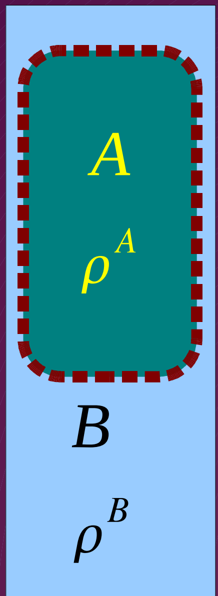
► Near criticality there are important fluctuations at all length scales;

► At very small temperatures there is a “domain of influence” of a QCP ($T = 0$) in which the correlations have a **genuine** quantum part (Sachdev's scenario);

► *Entanglement* measures in a many-body system: **Do they provide new insights into the quantum critical state?** [Osborne & Nielsen, PRA **66**, 032110 (2002)]

► XY spin-1/2 in a transverse field: The concurrence **decays very rapidly** but has diverging derivatives [Osterloh *et al.*, Nature **416**, 608 (2002)]

Sketch of entanglement indicators



ρ^{AB}

- A quantum state ρ^{AB} of $A+B$ is **entangled** if it **cannot** be written in this way $\rho^{AB}=\sum_i \lambda_i \rho_i^A \otimes \rho_i^B$

- **One site and the rest**: single site Entropy

$$S_1(|\Psi\rangle^{AB})=S_{vN}(\rho^A)=-Tr(\rho^A \ln \rho^A)$$

- **Two sites and the rest**

$$S_2(|\Psi\rangle^{AB})=S_{vN}(\rho^A)$$

pure state

- **Between two sites**: Concurrence (qubits only)

$$C(\rho^{AB}), \quad \rho^{AB}=Tr_{rest}|\Psi\rangle\langle\Psi|$$

Negativity: $N(\rho_{AB})=\left\|\rho_{AB}^{T_A}\right\|-1$

lack some requisites

Mutual info: $I(\rho_{AB})=S(\rho_A)+S(\rho_B)-S(\rho_{AB})$

Subsequent studies (a partial account)

- **Extended Hubbard model** [Gu *et al.*, PRL **93**, 86402 (2004)];

Maxima/singularities in (U,V) space at QPT's;

Role of symmetries? Limited to small sizes (no signatures of SC phase); At half-filling and $S^z = 0$, S_1 depends only on $\langle n_{i\uparrow} n_{i\downarrow} \rangle$

- **Hirsch model** at $x = 1$ (integrable)

[Anfossi *et al.*, PRL **95**, 056402 (2005)].

Analytical study of (mutipartite) entanglement in (U,n) space.

- **Hubbard +** $h\, S^z$ [Larsson & Johannesson PRL **95**, 196406 (2005)];
- Nontrivial spin systems: [Roscilde *et al.*, PRL **93**, 167203 (2004)]
- ...

However...

► The criterion to locate QCP's via (local) measures of entanglement seems to be system dependent: Some indicators are maximal at the transition point, other show a singularity in their derivative, other insensitive...

TABLE I: The basic features of typical spin models, such as the properties of the concurrence, level-crossing(LC) in the ground-state (GS) and the first excited-state(ES), symmetry at the transition point, and type of phase transition.

Gu *et al.* quant-ph/0511243

Model (QPT point)	GS LC	ES LC	concurrence	symmetry	transition type	type
XXZ chain($\Delta=-1$)	Yes		singular			I
J_1-J_2 model($J_2=0.5$)	Yes		singular			I
XXZ chain($\Delta=1$)	No	Yes	maximum, not singular	SU(2) point	order-to-order	II
spin ladder($J=0$)	No	Yes	maximum, not singular	SU(2)⊗SU(2)	disorder-to-disorder	II
XXZ 2&3D($\Delta=1$)	No	Yes	maximum, singular	SU(2)	order-to-order	II
J_1-J_2 model($J_2\simeq 0.241$)	No	Yes	not maximum	unknown	order-to-disorder	III
Ising model($\lambda=1$)	No	No	singular, not maximum	unknown	disorder-to-order	III

► Are we really getting insights into the genuine quantum nature of the ground states? Rather, can we exploit the classical/statistical counterpart to establish a connection with usual scaling theory?

Summary of scaling theory for QCP at $T = 0$

$$H = H_0 + gV \qquad e(g) \equiv \frac{\langle H \rangle}{L^d} = e^{reg}(g) + e^{sing}[\xi(g)]$$

Near criticality the singular part of the (free) energy density depends on g through the correlation length ξ and the gap Δ:

$$e^{sing}[\xi(g)] \propto \Delta\, \xi^{-d} \propto |g - g_c|^{\nu(d+\zeta)} \qquad \xi \propto |g - g_c|^{-\nu},\ \Delta \propto \xi^{-\zeta}$$

Dynamic exp.

$$\langle V \rangle = \frac{\partial \langle H \rangle}{\partial g} \Longrightarrow \frac{\langle V \rangle^{sing}}{L^d} \propto sgn(g - g_c)|g - g_c|^{\rho}$$

$$\boxed{\rho = (d + \zeta)\nu - 1}$$

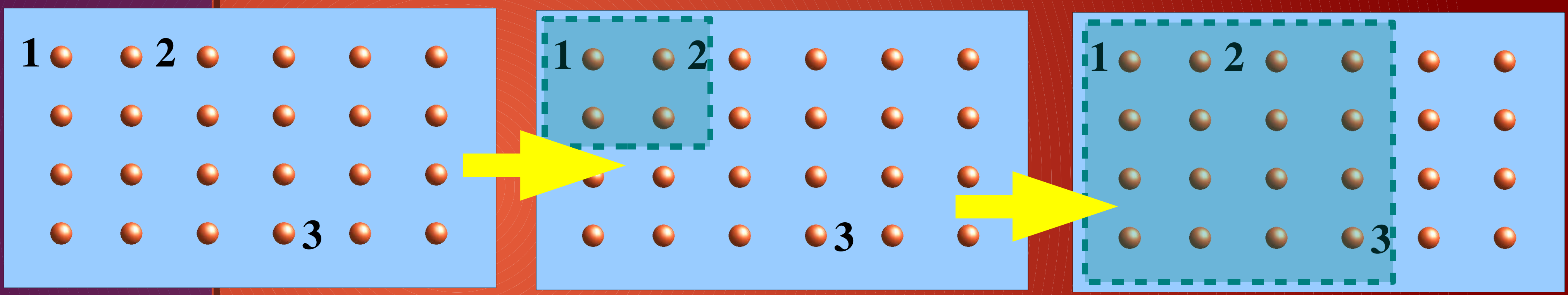
Local algebra hypothesis: every local operator can be expanded in terms of the scaling fields (including V itself)

$$\langle \sigma_a(x_1)\sigma_b(x_2)\sigma_c(x_3)\ldots \rangle = A_{rel}\langle \Phi_{rel} \rangle + \ldots$$

Most relevant in the RG sense

Behaviour of local measures of entang. (LME)

- This approach clarifies the meaning of '**local**': the set of points $X = \{x_1, x_2, x_3, \ldots\}$ is mapped onto a single point in the renormalisation-group (RG) flow



- In this sense, the block entropy for 'checkerboard'- or 'comb-like' partitions is nonlocal;
- Also the fidelity overlap of a many-body state is generally nonlocal $\langle \Psi(g) | \Psi(g+\delta g) \rangle$

[Zanardi & collabs., 2004-05]

► Since the LME's are functions of some density matrix elements, that are in turn combinations of correlation functions

$$\text{LME} = F\left(\langle \sigma_a(x_1)\sigma_b(x_2)\rangle \ldots\right)$$

if D = diameter(X) remains finite in the thermodynamic limit, after that the RG scale exceeds D we expect to see the singularity due to Φ_{rel} in **every LME**, unless $A_{rel} = 0$ (symmetry reasons) or $F()$ itself is such that the singularity is accidentally wiped out. [2-body inter. and measures: Wu *et al.*, PRL **93**, 250404 ('04)]

► For finite D $(\gg \xi)$, holds true also for the g-dependence of the block-entropy $S_A \sim b\, c/6 \log \xi(g)$

[Calabrese & Cardy, J. Stat. Mech. 0406 (2004) P002 and refs. therein]

► In order to locate the QCP why **not using simply** $O_g = \langle V \rangle / L^d$?

Byproduct: Finite-size crossing method

► O_g has scaling dimension of the most relevant operator. The corresponding FSS *ansatz* reads ($\varphi(0)=0$ for continuity):

$$O_g \propto \mathrm{sgn}(g-g_c)\,L^{-r}\varphi(z), \quad z \equiv (L/\xi)^{1/\nu}$$

$$r = \frac{\rho}{\nu} = d + \zeta - \frac{1}{\nu}$$

► Due to the *sign*, two curves at successive values of L **cross**, as functions of g, at a single point g^*_L. The sequence g^*_L converges to the critical point g_c.

$$g^*_L = g_c + \frac{A}{L^\theta}$$

► Once the critical point is located, the exponent ν can be computed from the first diverging derivative, say the n-th

$$\left|\frac{\partial^n O_g}{\partial g^n}\right| \propto L^{-r_n}\varphi^{(n)}(z), \qquad r_n = d + \zeta - \frac{1+n}{\nu}$$

Finite-size crossing method (II)

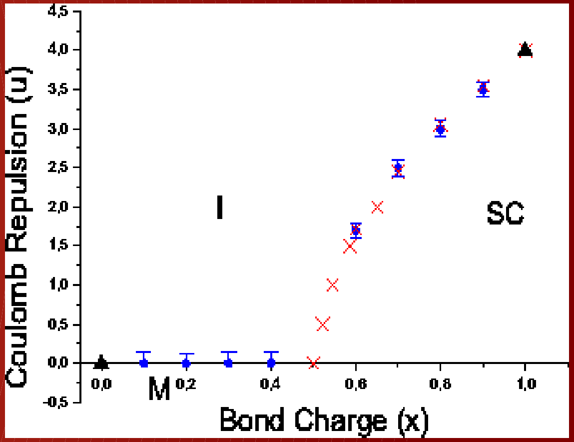
[Campos Venuti *et al.*, PRA **73** 010303(R) (2006)]

- Successfully tested on the Ising model with transverse field and on a nontrivial spin-1 model (see **Roncaglia's poster**):

$$H=\sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \lambda S_i^z S_{i+1}^z + D\left(S_i^z\right)^2$$

- Used also for 2-leg spin-1/2 ladders

[Hung *et al.*, PRB **73**, 224433 (2006)]



- Phase diagram of the Hirsch model *via* $S_1(n_d)$

[Anfossi *et al.*, PRB **73**, 085113 (2006)]

Study of the phases (unpubl.)

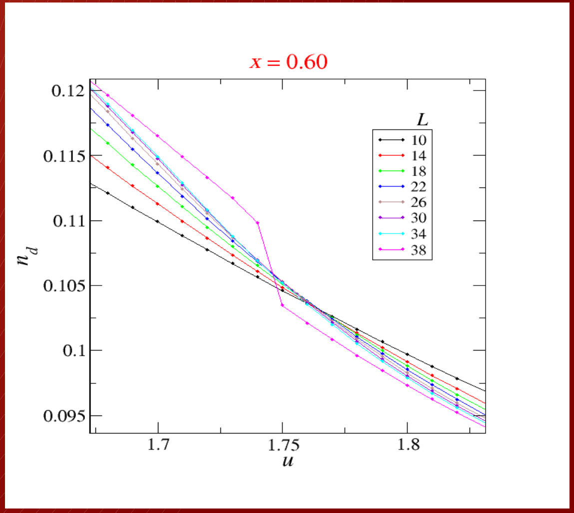
x=0.6 $\quad 0.55 < \nu < 0.59$

x=0.7 $\quad 0.53 < \nu < 0.57$

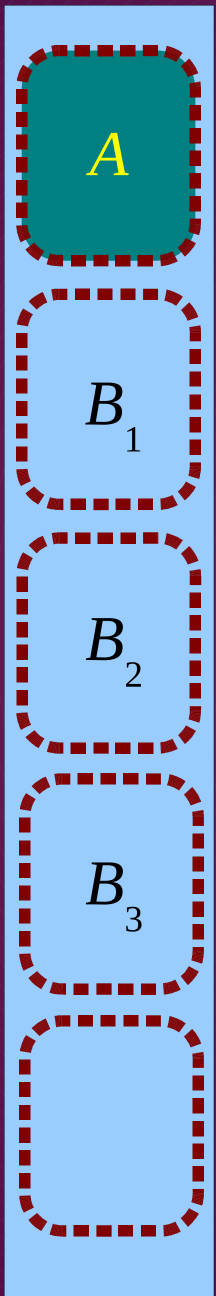
x=0.8 $\quad 0.38 < \nu < 0.47$

$\boldsymbol{x = 1} \quad \boldsymbol{\nu = 1/2}$

$$n_d \equiv \langle n_{i\uparrow} n_{i\downarrow} \rangle = \partial \langle H \rangle / \partial u$$



Recognising *purely* quantum correlations



From a physical point of view:

► At variance with classical correlations, they exhibit nonlocal behaviour; for example some entangled states of qubits violate Bell-like inequalities;

► At variance with usual correlation functions, they have to be shared between the various subsystems (“monogamy”). For qubits: Coffman-Kundu-Wootters bound $C^2(A|B_1)+C^2(A|B_2)+C^2(A|B_3)+\ldots\leq 4\,det\,\rho^A\leq 1$ no matter how many subsystems B one has.

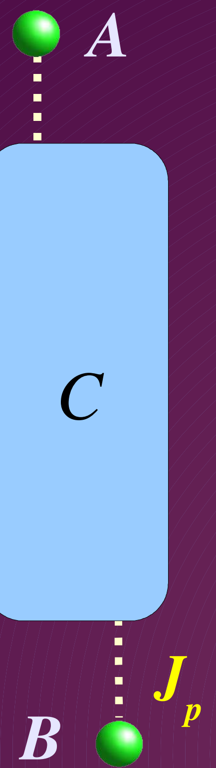
[Osborne & Verstraete, PRL **96**, 220503 (2006); for continuous variables: Hiroshima *et al.*, quant-ph/0605021]

Long-distance entanglement (LDE)

► In order to have high entanglement in a many-body environment the two subsytems must be somehow different from those in the bulk (boundaries, probes..);

► Typically, entanglement created by means of a switchable, **direct interaction**; in most systems with short-range couplings the entanglement decays very rapidly with the distance. For example, $C(j|j+r) = 0$ for $r > a$ in the Heisenberg model;

► The idea is to concentrate a sizable quantity of entanglement between two distant regions by means of a **mediated interaction**...



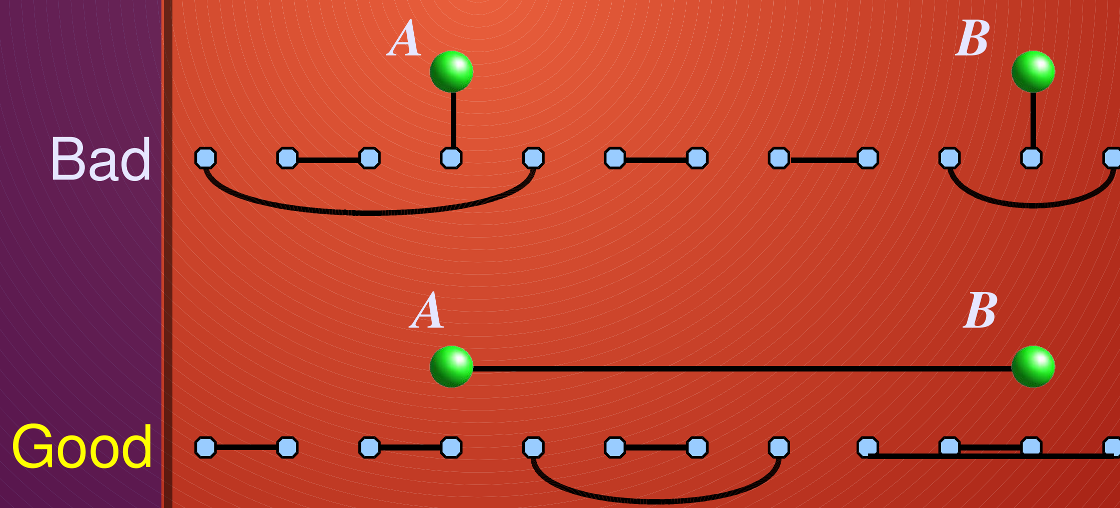
Formally, we have **LDE** if

$$E\left(\rho_{AB}\right) \xrightarrow{d\to\infty} E_{\infty} \neq 0 \quad \text{with} \quad \rho_{AB} = Tr_C \left|\Psi\right\rangle\left\langle\Psi\right|$$

where $E()$ is some measure of bipartite entanglement and the distance d is set by the number of single short-range interactions within C.

Physical idea: **Valence-bond states**

Maximally ent. singlet



SU(2) spin Hamiltonians:
2nd order pert. theory

$$H_{eff}^{AB} = J_{eff}\vec{S}_A \cdot \vec{S}_B$$

$$J_{eff} \simeq \frac{J_p^2}{J} f\left(L\right)$$

Good configurations are obtained when the interactions within C (J) are **stronger** than those with A and B (J_p)

LDE in models of solid-state physics

- Entanglement needs strong correlations:

low-dimensional systems are preferred.

$$H = \sum_i \left\{ \left[1 + \delta (-1)^i \right] \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} + \alpha \vec{\sigma}_i \cdot \vec{\sigma}_{i+2} \right\} \text{ (OBC)}$$

For ex.

- $\delta = 0, \alpha = 0$ (Heisenberg) $\langle \sigma_1^z \sigma_L^z \rangle \sim L^{-2}$ → No LDE!

- $\delta > 0$ Gapped bulk spectrum $\Delta \sim \delta^{2/3}$; **Midgap** $\Delta_L \sim e^{-\Delta L}$

Limiting cases: $\delta = -1, \alpha = 0$ $\qquad$ $\delta = 1, \alpha = 0$

LDE in models of solid-state physics

Spin-Peierls materials:

$CuGeO_3$

$\alpha\sim 0.35$

$\delta\sim 0.01$

NaV_2O_5

$\alpha\sim 0$

$\delta\sim 0.05$

- Entanglement needs strong correlations:

low-dimensional systems are preferred.

$$H=\sum_i\left\{\left[1+\delta(-1)^i\right]\vec{\sigma}_i\cdot\vec{\sigma}_{i+1}+\alpha\,\vec{\sigma}_i\cdot\vec{\sigma}_{i+2}\right\}$$ (OBC)

For ex.

- $\delta=0,\ \alpha=0$ (Heisenberg) $\langle\sigma_1^z\sigma_L^z\rangle\sim L^{-2}$ → No LDE!

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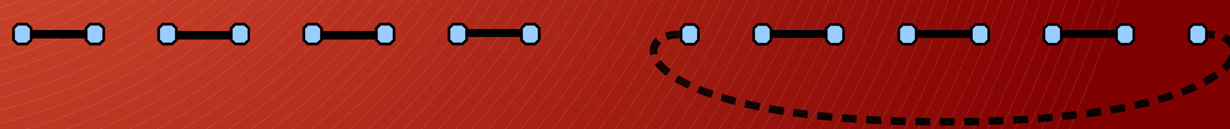


For $\delta<1$ the end spins tend to form a **maximally entangled singlet**

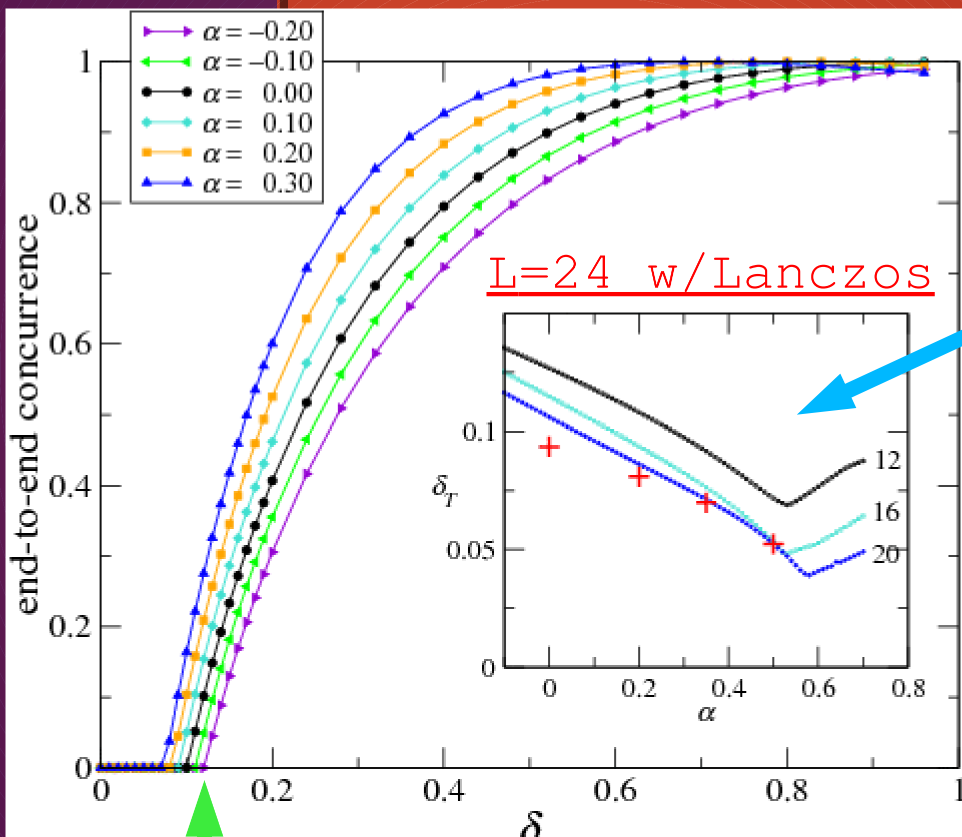
How to quantify the deviation from the (singlet) pure state? For SU(2)-symmetric qubits:

$$C(A|B) = \frac{1}{2} max\left\{0, -\langle\vec{\sigma}_A\cdot\vec{\sigma}_B\rangle - 1\right\}$$

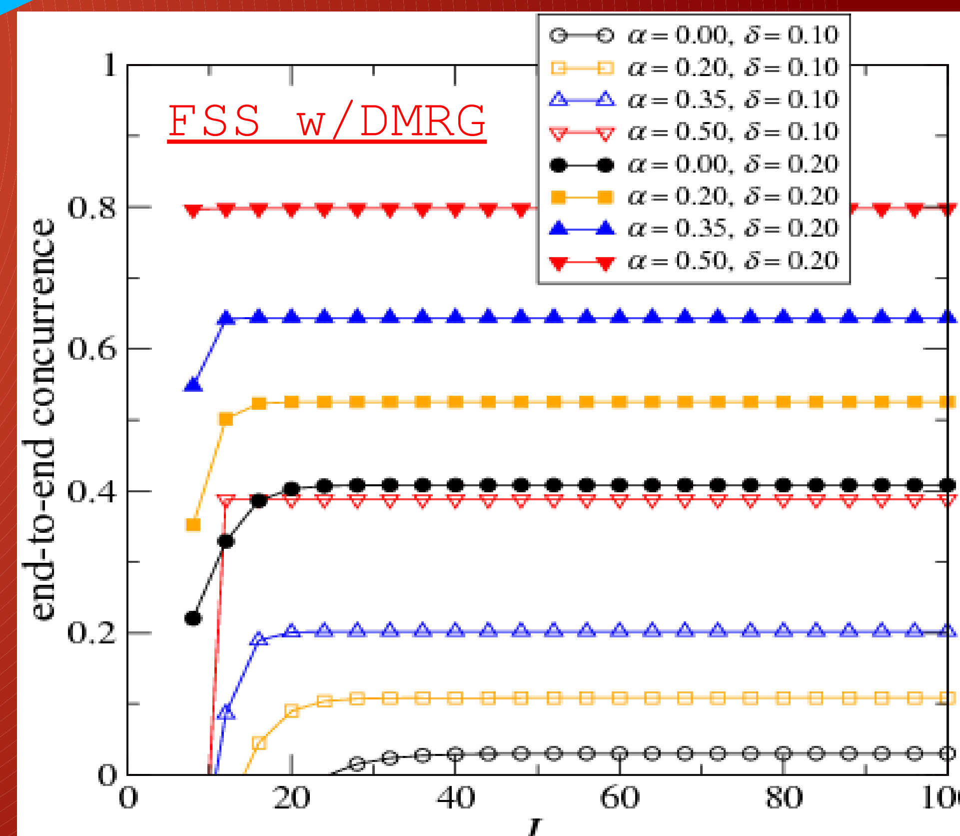
classical regime

$$-3 \le {}^{-1\le} \langle\vec{\sigma}_A\cdot\vec{\sigma}_B\rangle \le 1$$

quantum entangled

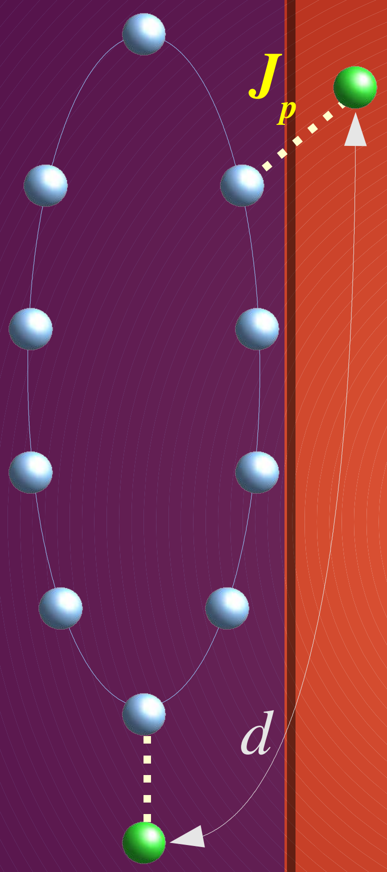


Enhancement due to frustration



LDE appears by increasing the dimerization above a threshold value (**not the critical one!**)

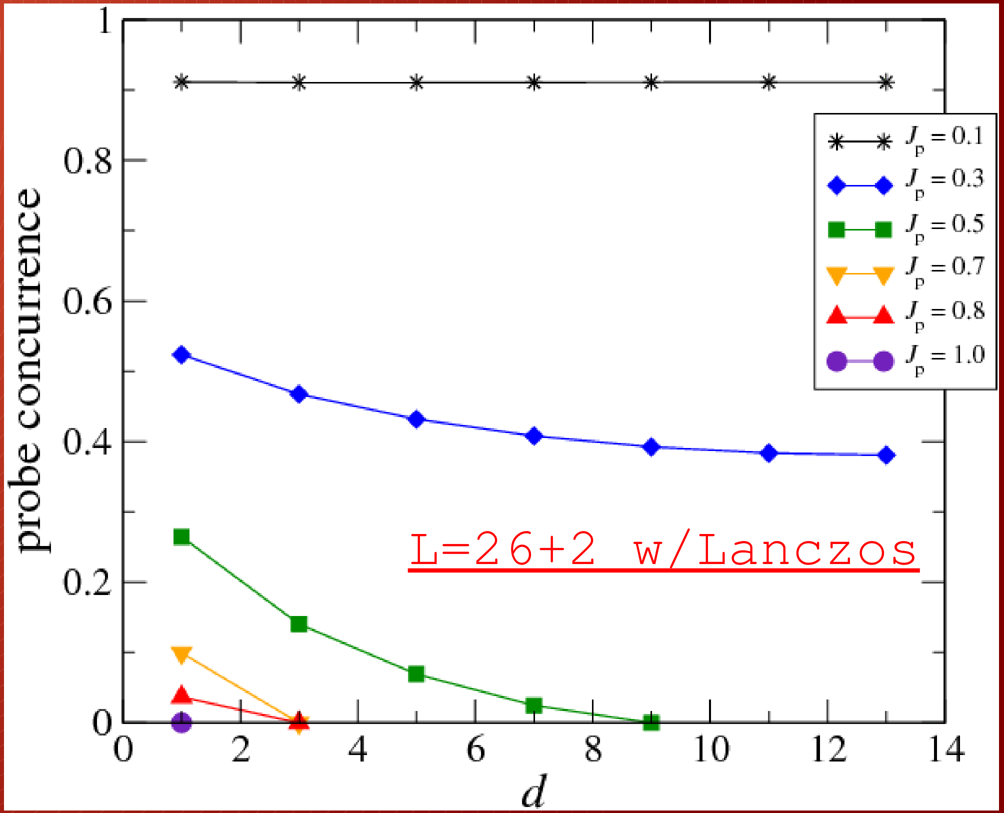
Gapless bulk: weakly coupled probes



- $J_p = 1$: separable state $\forall\, d$. Sizable entanglement between the probes is created by **lowering** J_p;
- The range of nonzero concurrence grows;
- Similar behaviour is observed when the probes are placed at the ends of an **open** chain.

Open questions

- When do we have true LDE?
- Surface order is a necessary condition: Does it occur, at least for $J_p < 0$ (ferro)?



Analytic studies of LDE

- **Partial concurrence** and **negativity** for spin-1 RVB states (AKLT - see **Campos Venuti's poster**)

- Onset of LDE/surface order in XY(Z) spin-1/2 chains and feasibility in experiments with optical lattices (with F. Illuminati and collabs.)

Exploting LDE for QI tasks: the example of <u>teleportation</u> in solid-state systems

► Many quantum information protocols, like teleportation and quantum cryptography, require entanglement generation and distribution.

► For SU(2) invariant states of qbits the best teleportation protocol is actually the celebrated one:

[Bennett *et al.*, PRL **70**, 1895 (1993)]

<u>0</u>. Collect a state $|S\rangle$ to send and an **entangled pair;**

<u>1</u>. Perform a Bell measurement on S and A;

<u>2</u>. Send the result r to B using a normal channel;

<u>3</u>. The observer in B performs a unitary transf. related r to on the other element of the pair and has a copy (not a clone!) of $|S\rangle$.

Teleporting in a thermalised channel

- At low temperatures the Gibbs state is approximated as

$$e^{-H/\kappa T} \simeq |GS\rangle\langle GS| + e^{-\Delta_L/\kappa T} \sum_{m=-1}^{1} |m\rangle\langle m|$$

Lowest triplet excitation

- For nonzero T the SU(2) is mantained and the fidelity of teleportation does not depend on $|S\rangle$ and r :

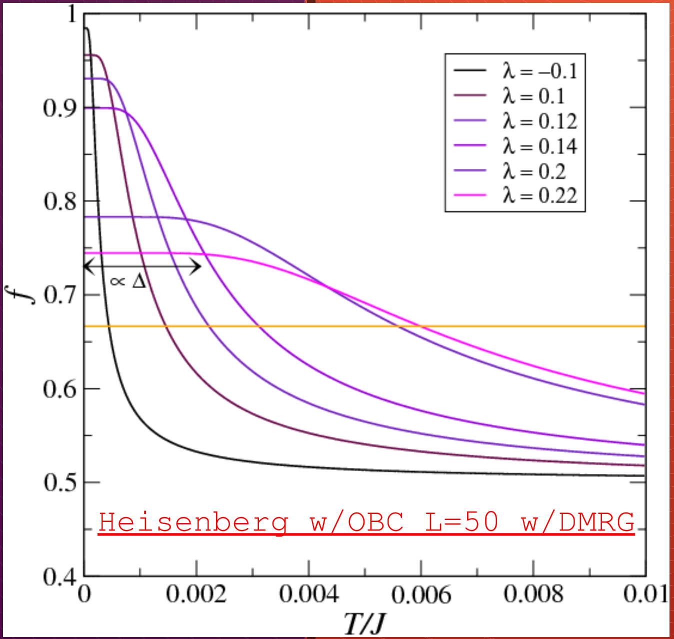
$$f = \overline{f} = \frac{1}{2} - 2\left\langle S_A^z S_B^z \right\rangle_T$$

Nonclassical for $\left\langle S_A^z S_B^z \right\rangle_T < -1/12$

The midgap decreases as:

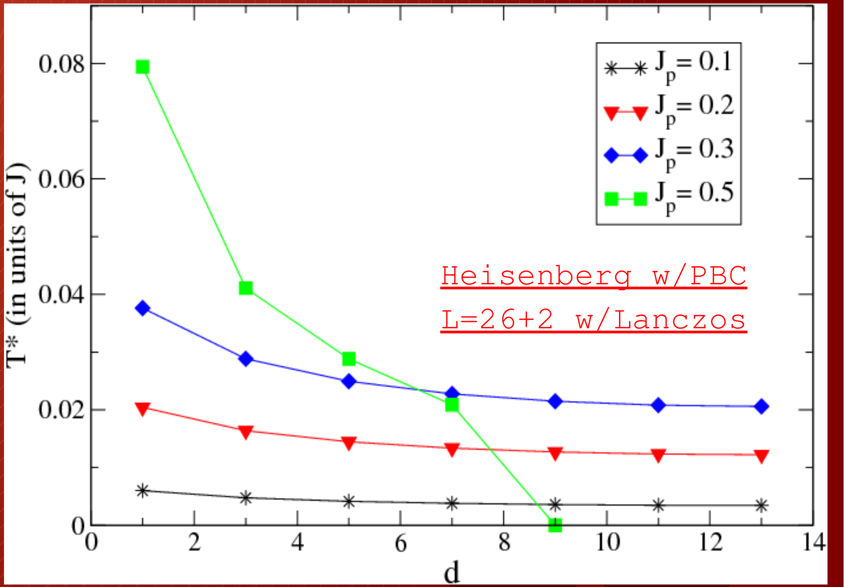
- d increases
- J_p decreases

(**tradeoff**)



For $T > T^* \sim \Delta_L$ the ent. and the fidelity deteriorate

$$T^* = \Delta_L \left[\log\left(\frac{\langle 1|S_A^z S_B^z|1\rangle + 2\langle 1|S_A^x S_B^x|1\rangle + 1/4}{-\langle GS|S_A^z S_B^z|GS\rangle - 1/12} \right) \right]^{-1}$$



Summary

- Apart from accidental cancellations the scaling properties of every local measure of entanglement come from the most relevant scaling operator.
- **LDE**: Onset of genuine quantum correlations between arbitrarily far apart (selected) sites that does **not** generally coincide with known quantum phase transitions.
- Possibility to engineer quantum information devices (entanglers, quantum channels) using the low-energy states of strongly correlated condensed-matter systems.