Information/disturbance tradeoff in quantum measurements

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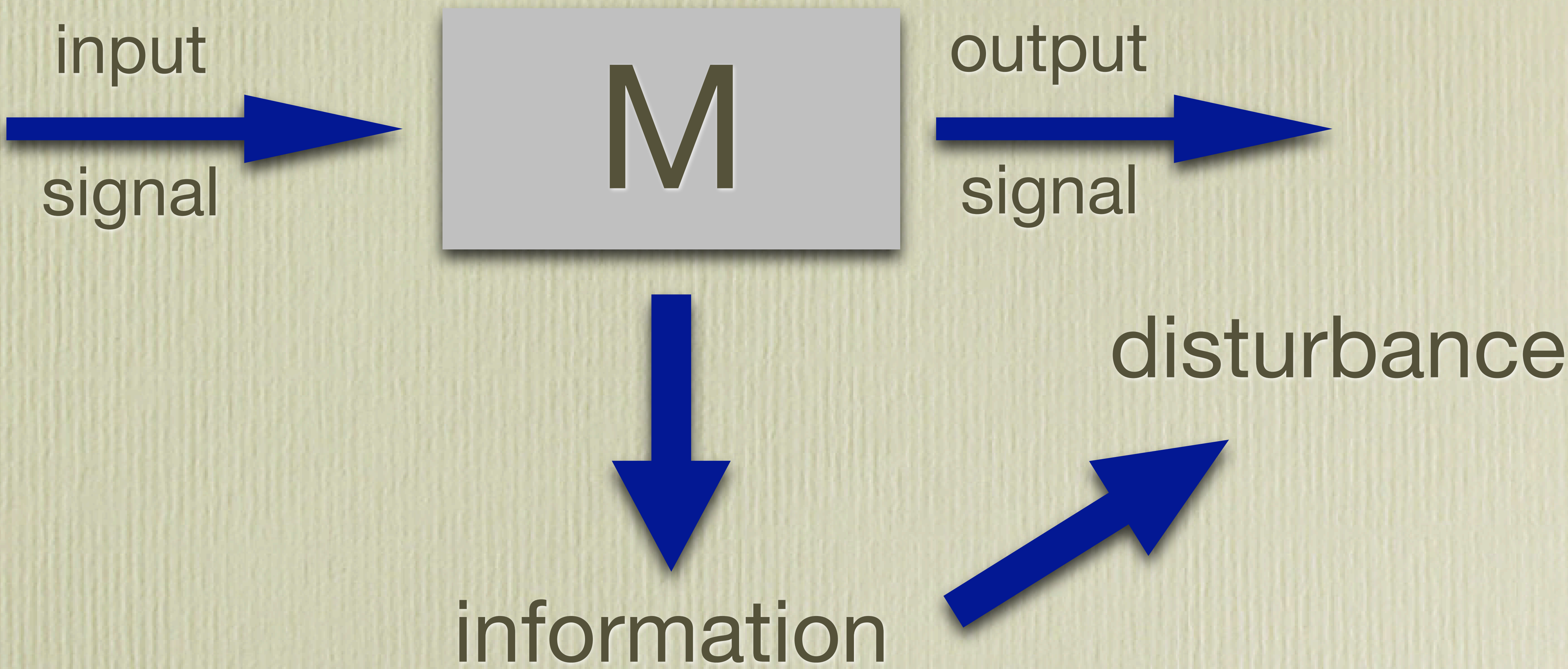
1. What are we speaking about ?

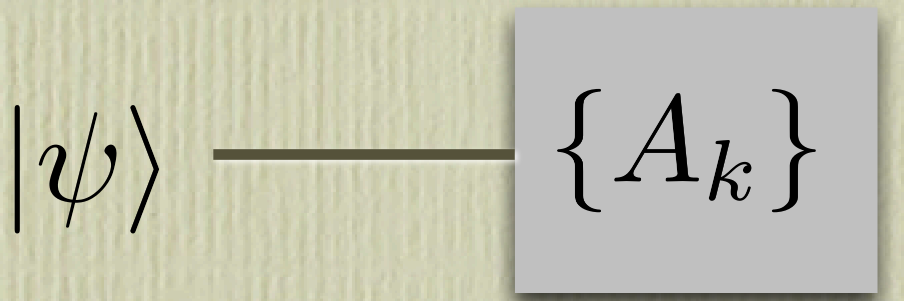


- fundamentals limits to precision
- quantum communication channels

2. Indirect measurement: abstract description

$\{A_k\}$ measurement operators

$$\sum_k A_k^\dagger A_k = \mathbb{I}$$



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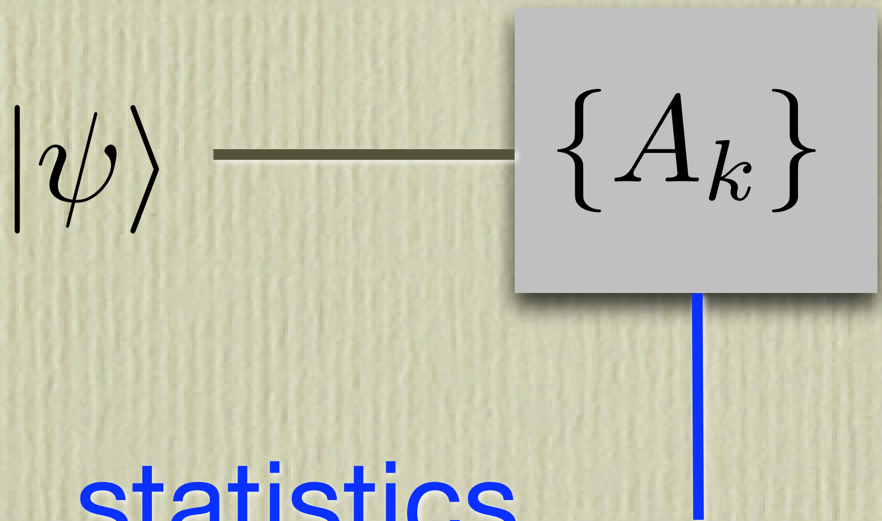

statistics

$$\Pi_k = A_k^\dagger A_k$$

$$P_k = \langle\psi|\Pi_k|\psi\rangle$$

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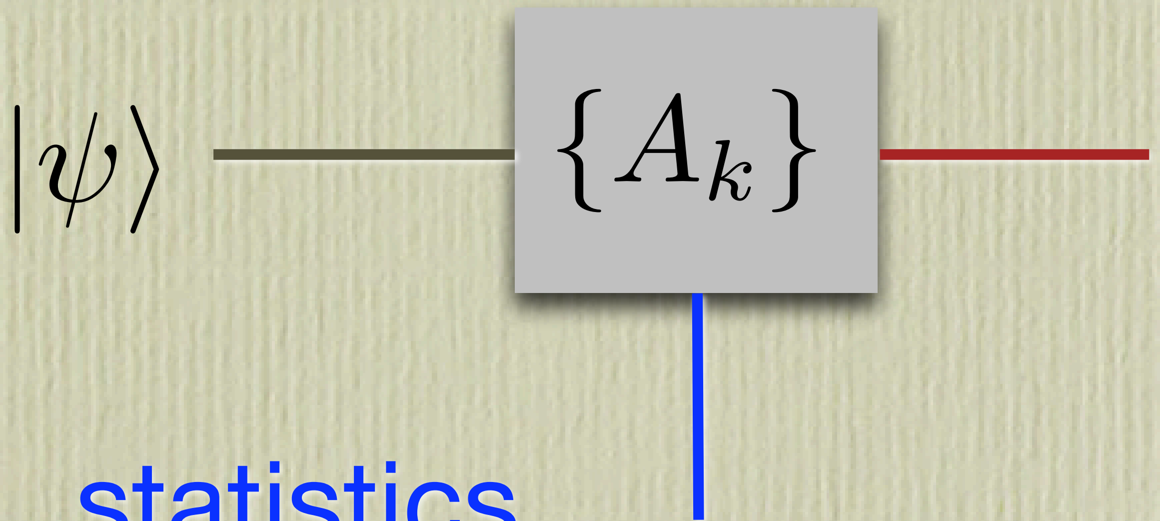

state reduction

$$|\psi_k\rangle = \frac{1}{\sqrt{P_k}} A_k |\psi\rangle$$

$$\varrho = \sum_k A_k |\psi\rangle\langle\psi| A_k^\dagger$$

statistics

$$\Pi_k = A_k^\dagger A_k$$

$$P_k = \langle\psi|\Pi_k|\psi\rangle$$

3. Inference and fidelities

- disturbance $|\psi\rangle \to |\psi_k\rangle$

$$F_\psi = \sum_k P_k |\langle\psi|\psi_k\rangle|^2 = \sum_k |\langle\psi|A_k|\psi\rangle|^2$$

$$F = \int d\psi\, F_\psi$$

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$$F_\psi = \sum_k P_k |\langle\psi|\psi_k\rangle|^2 = \sum_k |\langle\psi|A_k|\psi\rangle|^2$$

- information $k \longrightarrow |\phi_k\rangle$

$$G_\psi = \sum_k P_k \, |\langle\psi|\phi_k\rangle|^2$$

$$F = \int d\psi \, F_\psi \qquad G = \int d\psi \, G_\psi$$

4. Inf/dist tradeoff: extreme cases (finite dimension)

- blind repeater = nothing is done

$$F = 1 \qquad\qquad G = 1/d$$

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- blind repeater = nothing is done

$$F = 1 \qquad\qquad G = 1/d$$

- optimal state estimation

$$G = \frac{2}{1+d} \qquad\qquad F = \frac{2}{1+d}$$

(Massar et al, 1995)
(Bruss et al, 1999)

5. Information/disturbance tradeoff
(d-dimensional random signals)

$F = \int d\psi\, F_\psi \quad G = \int d\psi\, G_\psi$

- $$F = \frac{1}{d(d+1)}\left[d + \sum_k |\mathrm{Tr}[A_k]|^2\right]$$

- $$G = \frac{1}{d(d+1)}\left[d + \sum_k \langle\phi_k|\Pi_k|\phi_k\rangle\right]$$

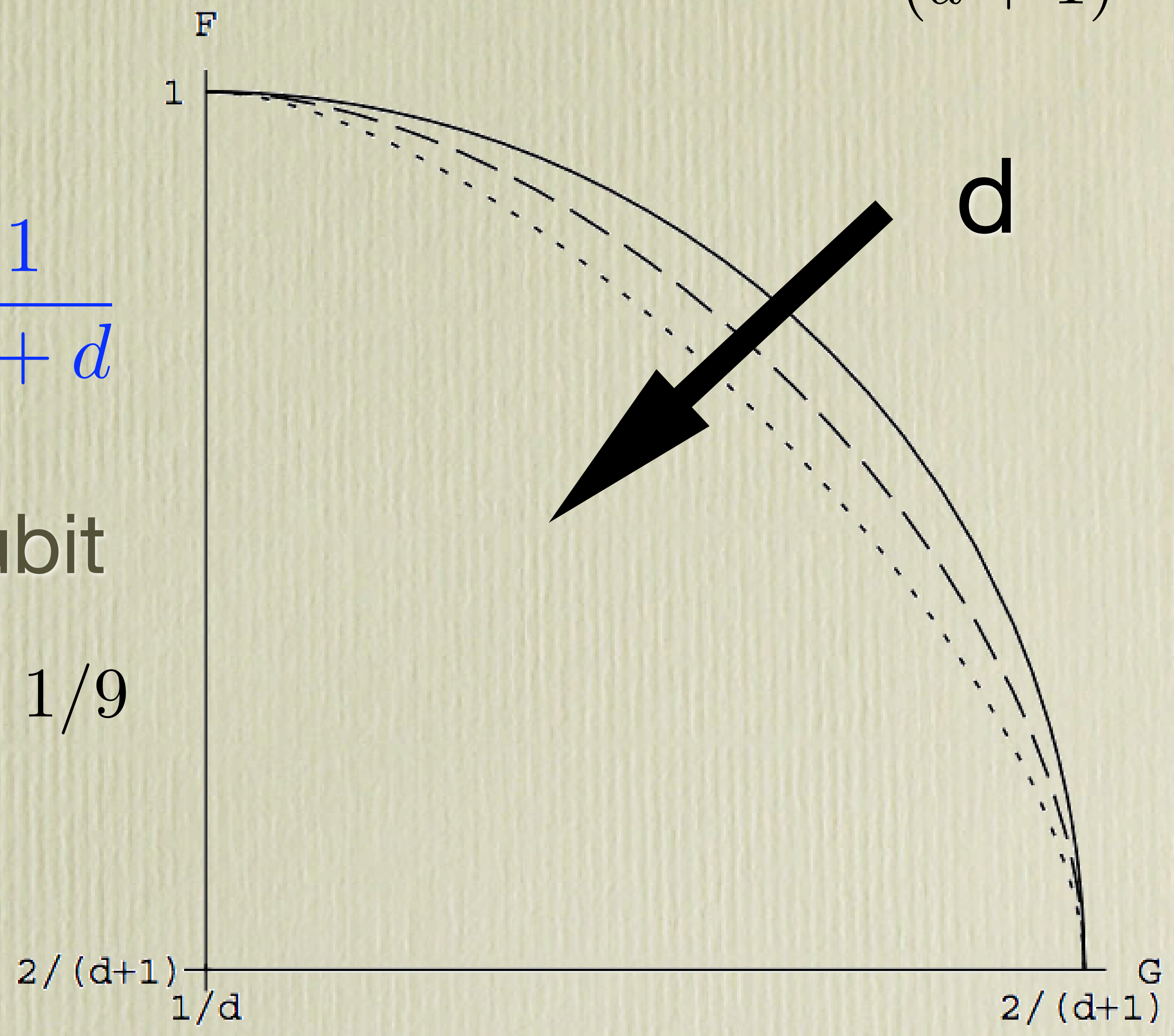
$|\phi_k\rangle \longrightarrow$ eigenvector of Π_k with max eigenvalue

6. Ultimate bound on fidelities

(K. Banaszek, 2001)

$$(F - F_0)^2 + d^2 (G - G_0)^2 + 2(d-2)(F - F_0)(G - G_0) \leq \frac{d-1}{(d+1)^2}$$

$$F_0 = \frac{1}{2}\frac{2+d}{1+d} \qquad G_0 = \frac{3}{2}\frac{1}{1+d}$$



Example: trade-off for qubit

$$(F - 2/3)^2 + 4\ (G - 1/2)^2 \leq 1/9$$

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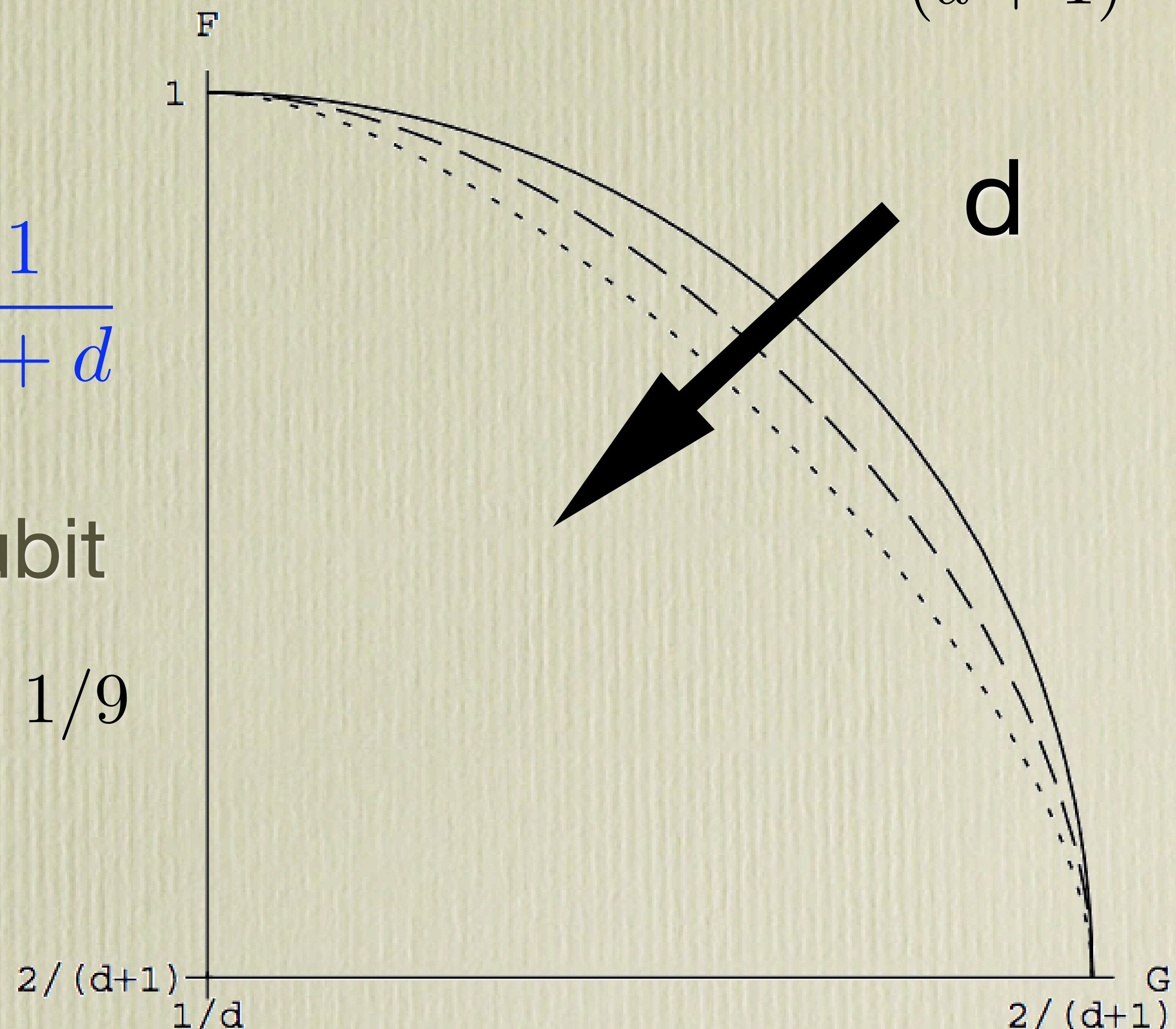


Example: trade-off for qubit

$$(F - 2/3)^2 + 4\,(G - 1/2)^2 \le 1/9$$

- No $d \longrightarrow \infty$ limit

7. A couple of questions

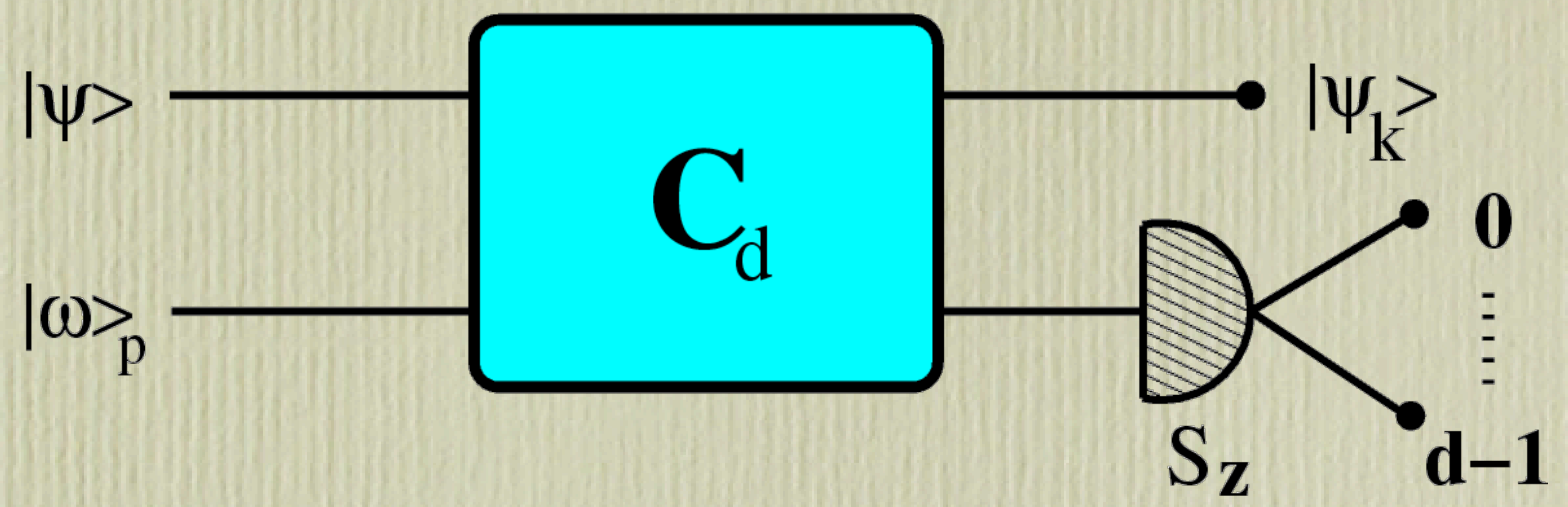
- is it possible to saturate the bound for finite dimensional systems ?
  - unitary (feasible) realization

    (MG and M. Paris, PRA 2005)
  - N-user measurement scheme

    (MG and M. Paris, JOP submitted)
- what about infinite dimensional (continuous variable) systems ?
  - constraint of energy

    (MG and M. Paris, PRA 2006)

8. Optimal (and minimal) scheme for qudit (1)



- standard basis measurement

- d-dimensional C-not gate

$$C_d\,|p\rangle|q\rangle = |p\rangle|q \oplus p\rangle$$

- one-parameter probe

$$|\omega\rangle_p = \cos\theta|0\rangle_p + \gamma\sin\theta\frac{1}{\sqrt{d}}\sum_{s=0}^{d-1}|s\rangle_p$$

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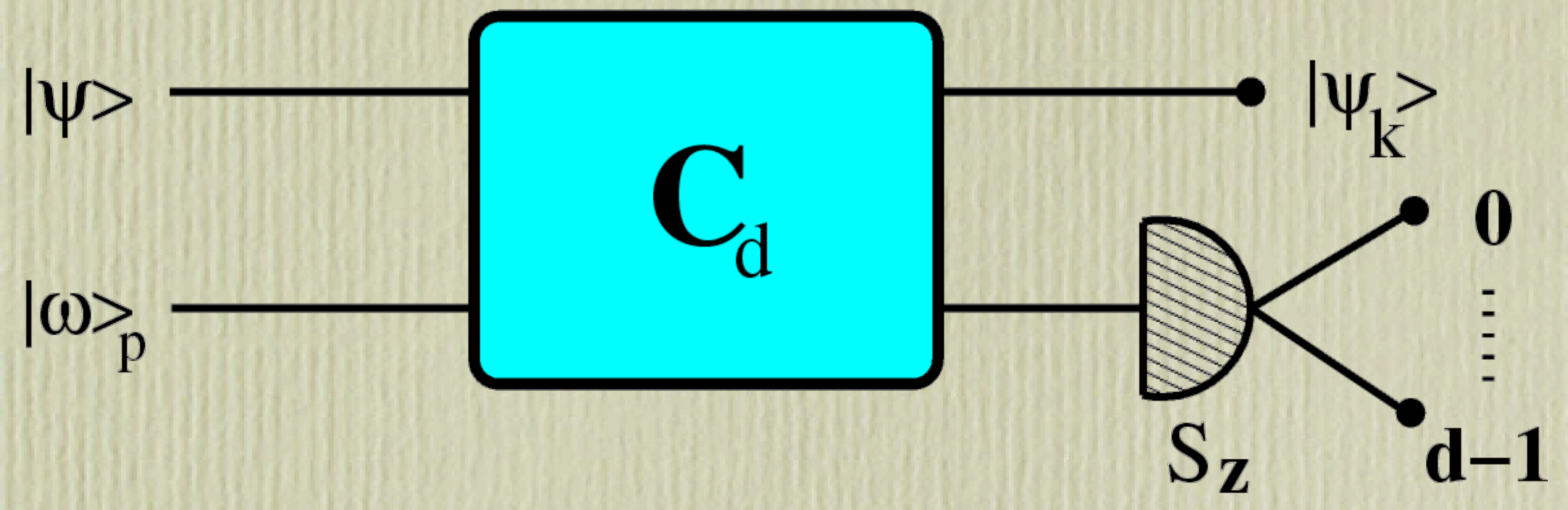

- standard basis measurement

single-measure inference rule

$$k \longrightarrow |k\rangle$$

- d-dimensional C-not gate

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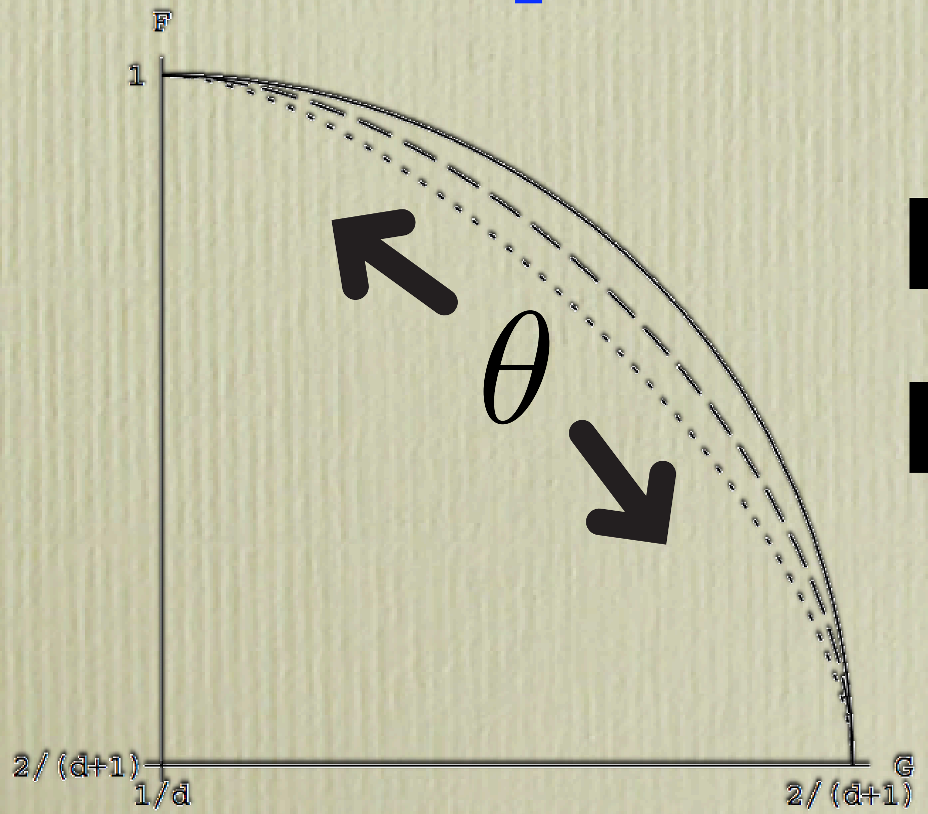
8. Optimal (and minimal) scheme for qudit (2)

$$A_k = {}_p\langle k|C_d|\omega\rangle_p \qquad (A_k)_{ij} = \delta_{ij}\left[\delta_{kj}\cos\theta + \frac{\gamma}{\sqrt{d}}\sin\theta\right]$$

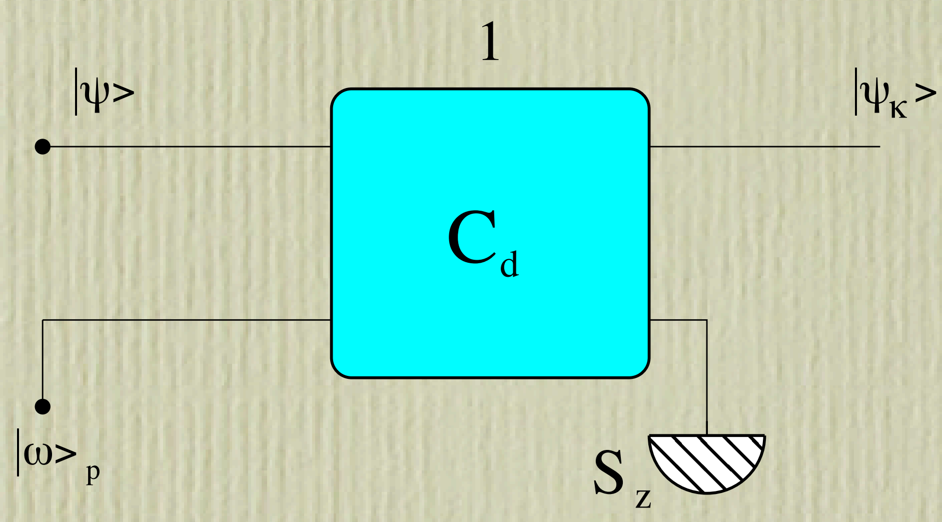
$$F = \frac{1}{1+d}\left[1 + \left(\cos\theta + \gamma\sqrt{d}\sin\theta\right)^2\right]$$

$$G = \frac{1}{1+d}\left[1 + \left(\cos\theta + \frac{\gamma}{\sqrt{d}}\sin\theta\right)^2\right]$$



- F and G saturate the QM bound
- a single d-dim probe is needed

9. N-user measurement scheme (1)



- Sequence of N optimal schemes for qudits
- one-parameter probes for all users

$$|\omega\rangle_p = \cos\theta|0\rangle_p + \gamma\sin\theta\frac{1}{\sqrt{d}}\sum_{s=0}^{d-1}|s\rangle_p$$

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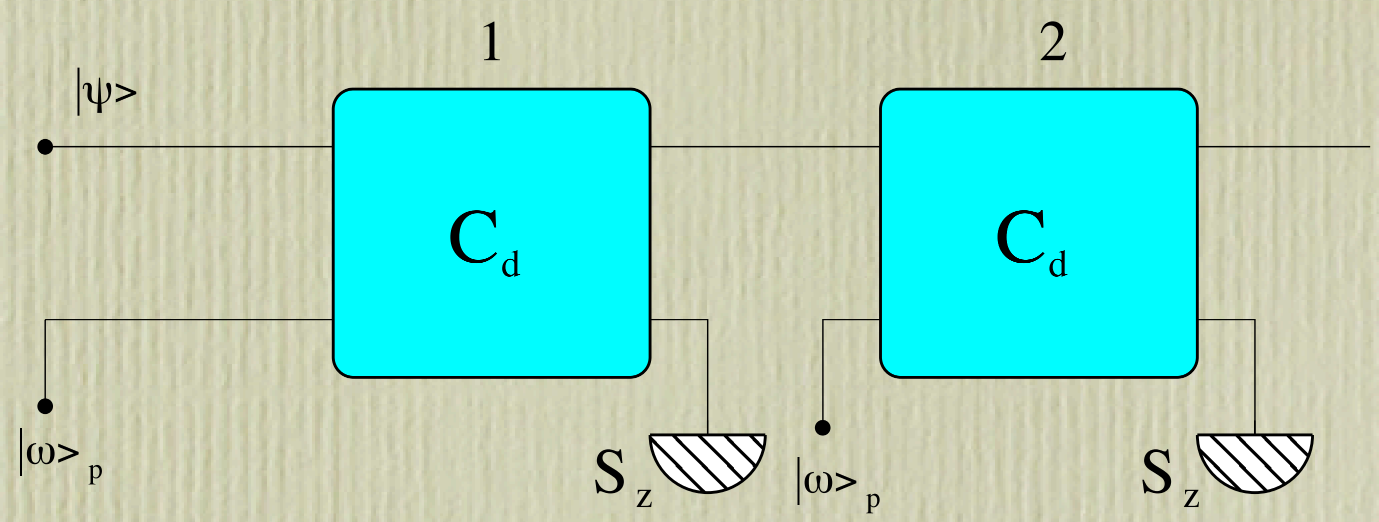

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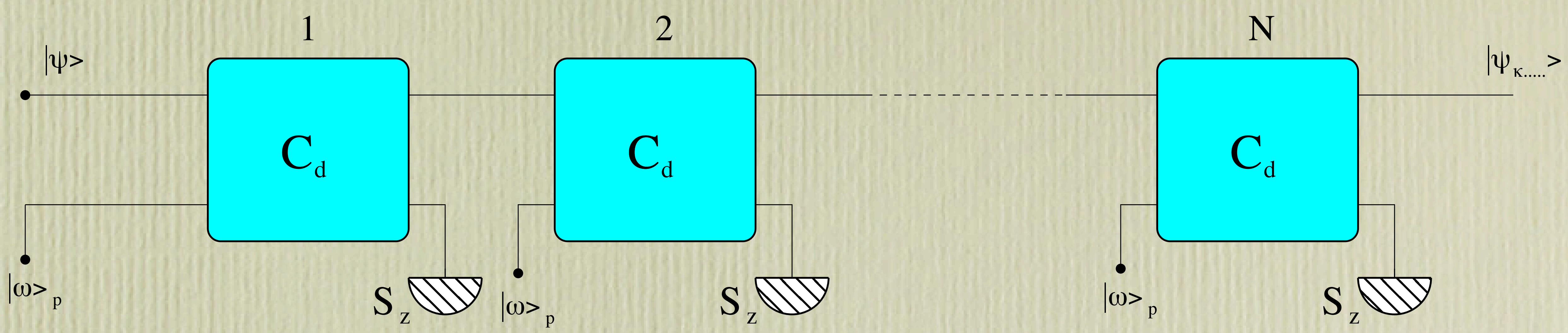

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9. N-user measurement scheme (2) - Disturbance

- N-th step conditional state

$$\varrho_k^N = \frac{1}{P_k} A_k \left(\sum_{k_1, k_2, \ldots k_{N-1}} A_{k_{N-1}} A_{k_{N-2}} \ldots A_{k_1} |\psi\rangle\langle\psi| \left(A_{k_{N-1}} A_{k_{N-2}} \ldots A_{k_1} \right)^\dagger \right) A_k^\dagger$$

$$F_{\psi,N} = \sum_k P_k \mathrm{Tr}\left[|\psi\rangle\langle\psi| \varrho_k^N \right] = \sum_{\{k_i\}} |\langle\psi| A_{\{k_i\}} |\psi\rangle|^2$$

$A_{\{k_i\}} = A_k A_{k_{N-1}} \ldots A_{k_1}$ *overall* measurement operator

$$F_N = \frac{1}{d(d+1)} \left(d + \sum_{\{k_i\}} |\mathrm{Tr}[A_{\{k_i\}}]|^2 \right)$$

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Qubit (d=2)

$$F_N = \frac{1}{3}\left(2 + (\sin\theta)^{2N}\right)$$

Qutrit (d=3)

$$F_N = \frac{1}{2}\left(1 + (\sin\theta)^{2N}\right)$$

9. N-user measurement scheme (3) - Information

- statistics

$$P_k = \mathrm{Tr}\left[A_k\left(\sum_{k_1,k_2,\ldots,k_{N-1}} A_{k_{N-1}}A_{k_{N-2}}\ldots A_{k_1}|\psi\rangle\langle\psi|\left(A_{k_{N-1}}A_{k_{N-2}}\ldots A_{k_1}\right)^\dagger\right)A_k^\dagger\right]$$

$$= \mathrm{Tr}\left[|\psi\rangle\langle\psi|A_k^\dagger A_k\right]$$

→ POVM $\Pi_k = A_k^\dagger A_k$

- inference

$k \rightarrow |k\rangle\ eigenstate\ of\ S_z$ *Single-measure inference rule*

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$$= \mathrm{Tr}\left[|\psi\rangle\langle\psi|A_k^\dagger A_k\right]$$

→ POVM $\Pi_k = A_k^\dagger A_k$

- inference

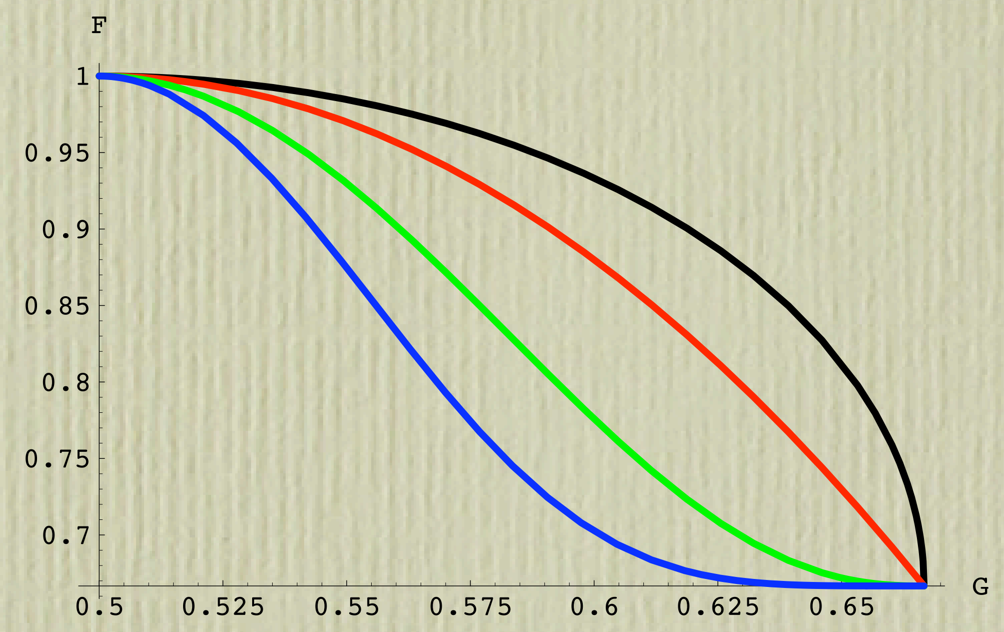
$k \rightarrow |k\rangle\ eigenstate\ of\ S_z$ *Single-measure inference rule*

→ *Single-user scheme* estimation fidelity

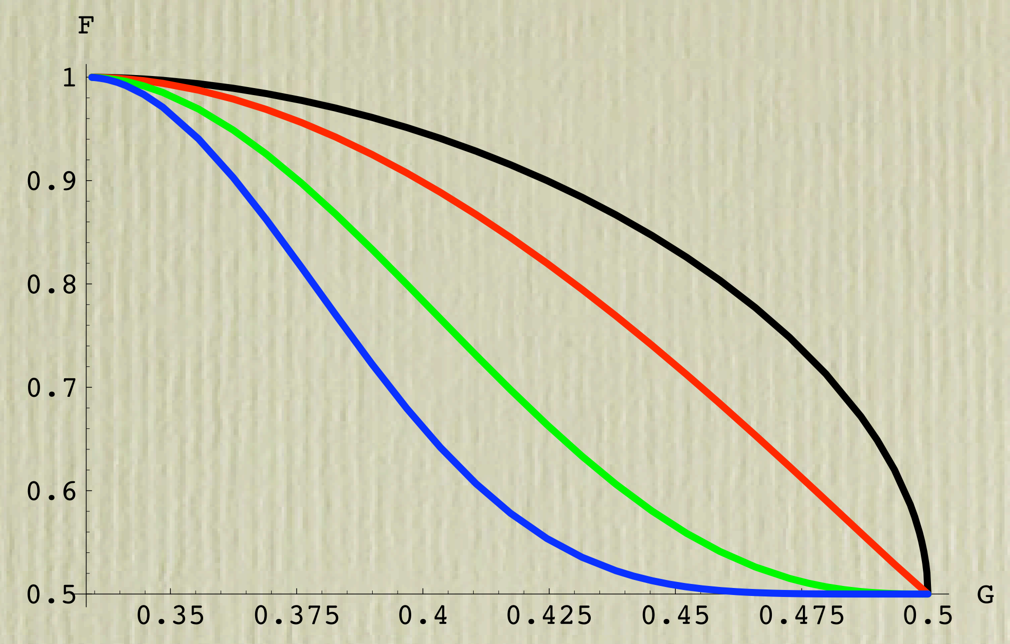
$$G = \frac{1}{1+d}\left[1+\left(\cos\theta+\frac{\gamma}{\sqrt{d}}\sin\theta\right)^2\right] \qquad \forall\, N$$

9. N-user measurement scheme (4) - Trade-off

Qubit



Qutrit



N=1 optimal

N=2

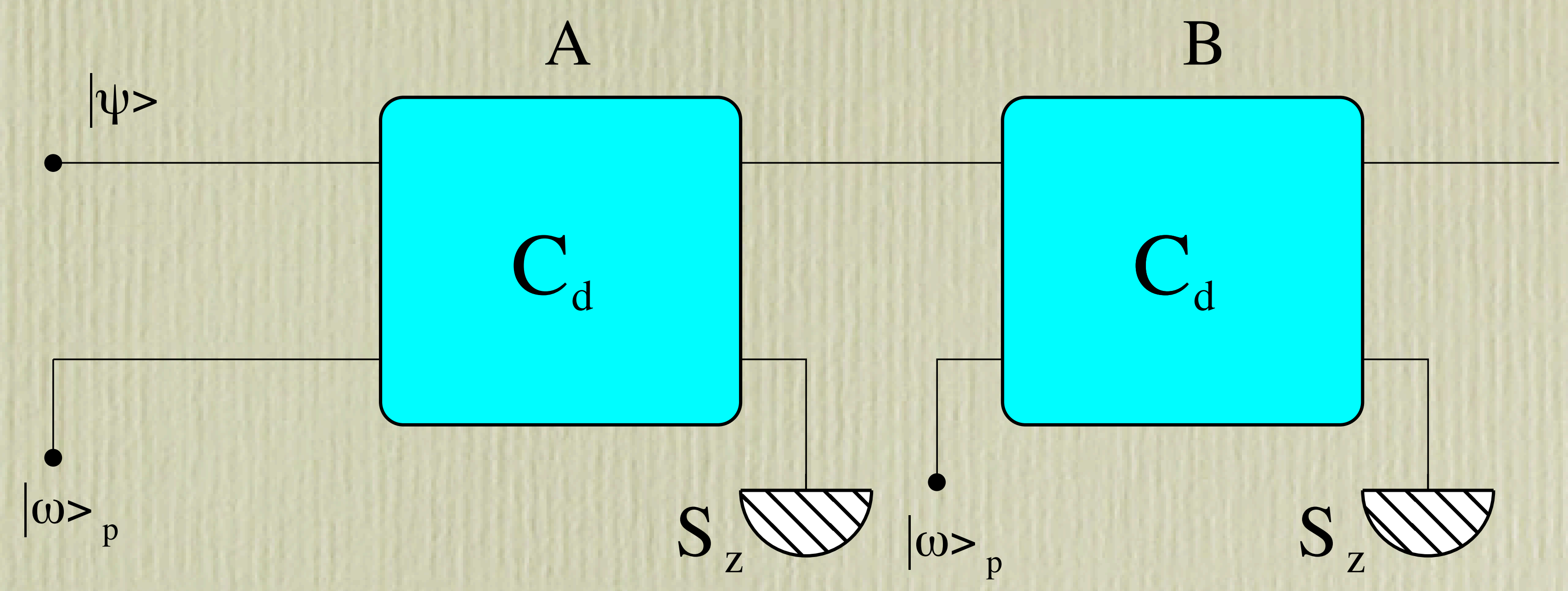
N=5

N=10

- Trade-off gets worse for increasing N
- Empty region between optimality and N=2

Is this region achievable for the Nth user?

10. 2-user measurement scheme for qubit (1)



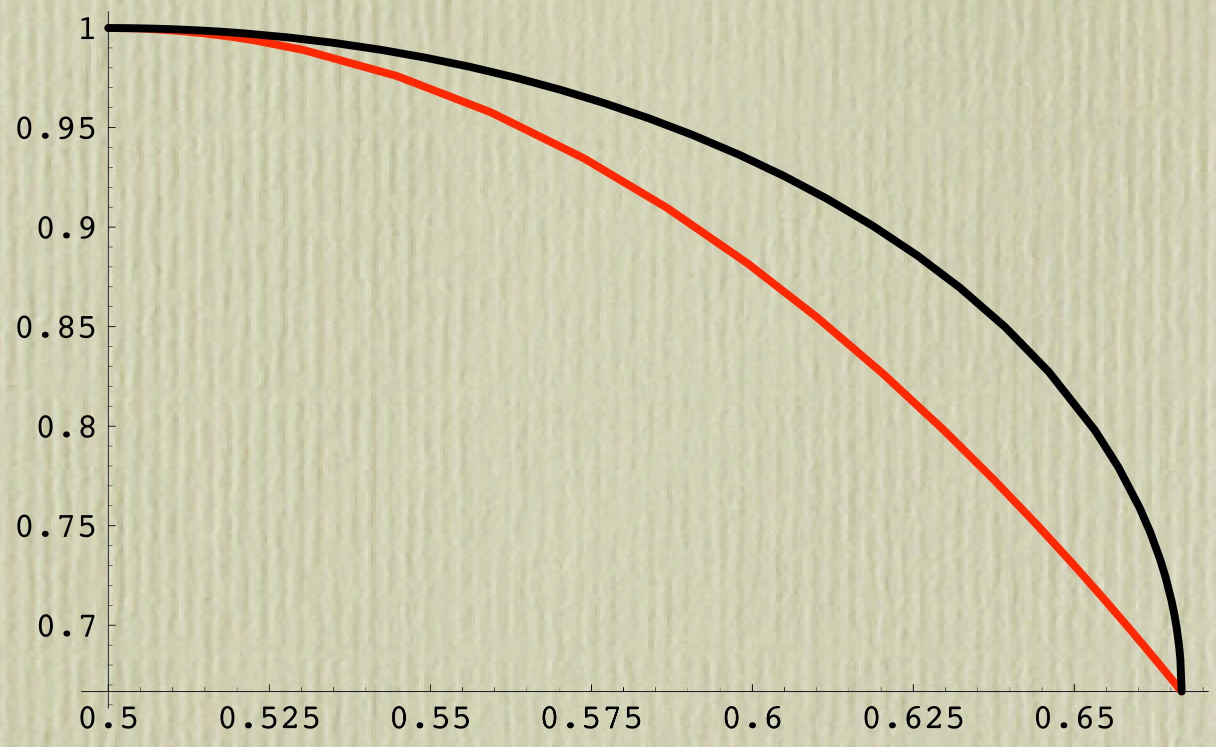
Probes Preparation

$$|\omega\rangle_{Ap} = \cos\theta_A |0\rangle_{Ap} + \frac{\gamma \sin\theta_A}{\sqrt{2}} (|0\rangle_{Ap} + |1\rangle_{Ap})$$

$$|\omega\rangle_{Bp} = \cos\theta_B |0\rangle_{Bp} + \frac{\gamma \sin\theta_B}{\sqrt{2}} (|0\rangle_{Bp} + |1\rangle_{Bp})$$

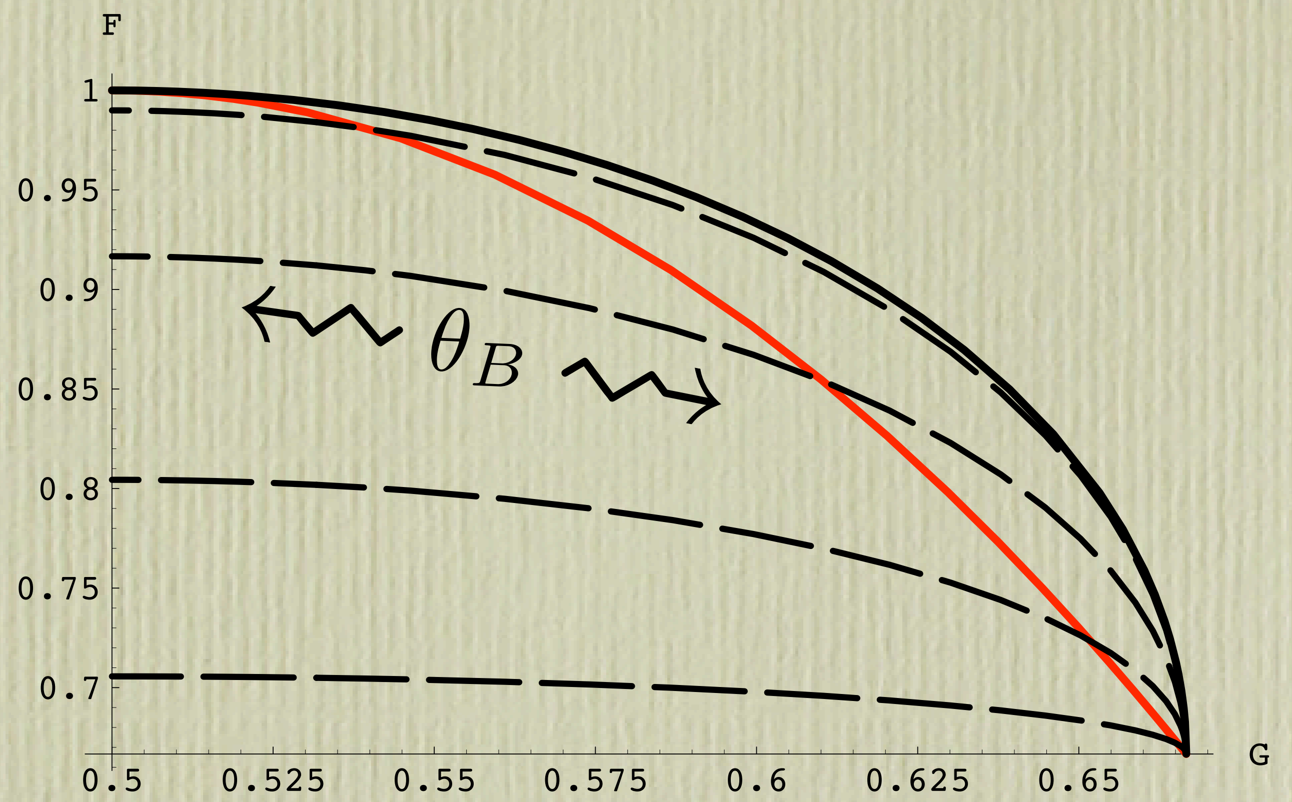
Two parameters : θ_A and θ_B

10. 2-user measurement scheme for qubit (2)



- Optimal trade-off: black
- $\theta_B = \theta_A$ red

10. 2-user measurement scheme for qubit (2)



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- fixed θ_A varying θ_B dashed

from above to bottom $\theta_A = \frac{\pi}{9} j$

$j = \{1, 2, 3, 4\}$

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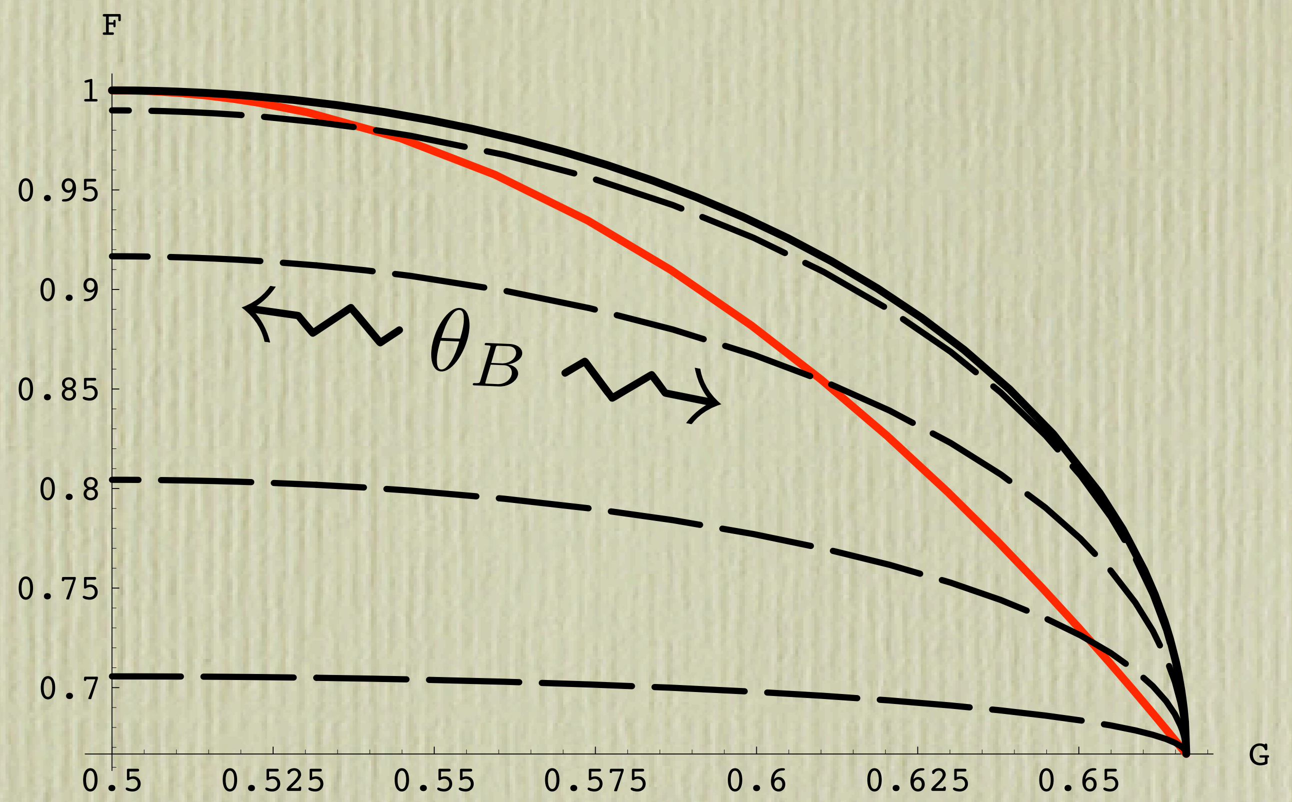

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Estimation fidelity

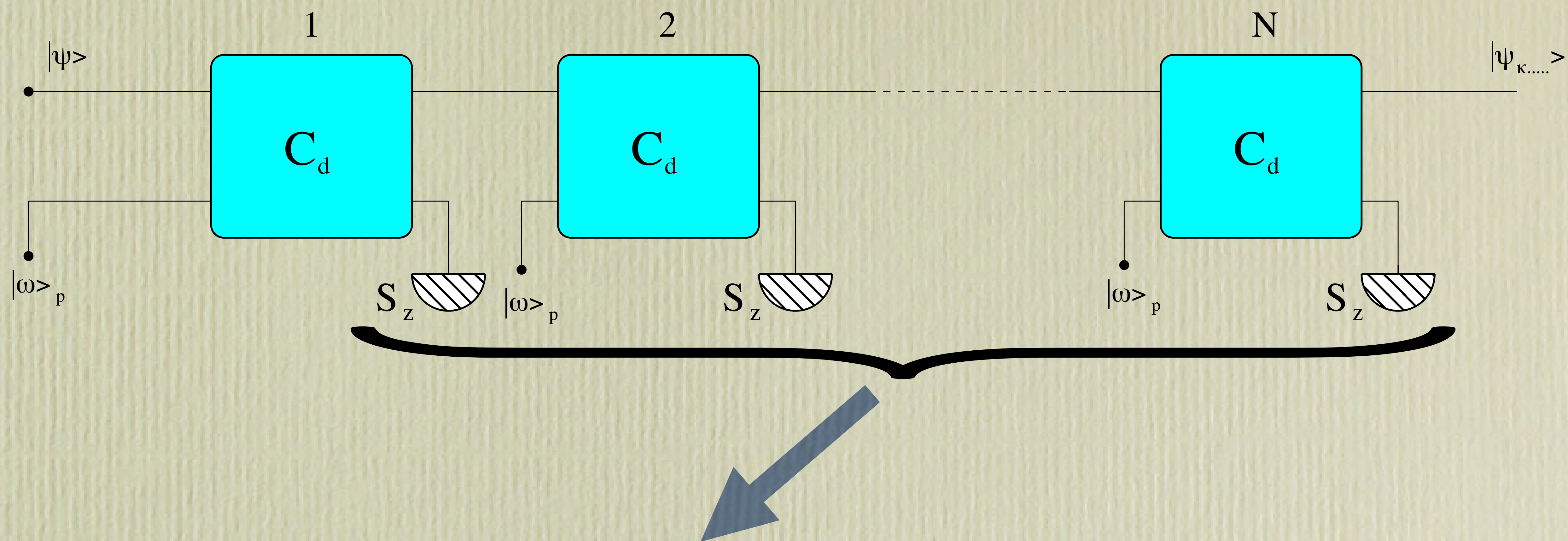
$$\frac{1}{2} < G < \frac{2}{3}$$

Transmission fidelity

$$\frac{2}{3} < F < \frac{5 - \cos 2\theta_A}{6}$$

- *"Near-optimal"* region achievable

11. Single user and N-step measurement scheme



$$\{k_i\} = \{k_1, k_2, \ldots, k_N\}$$

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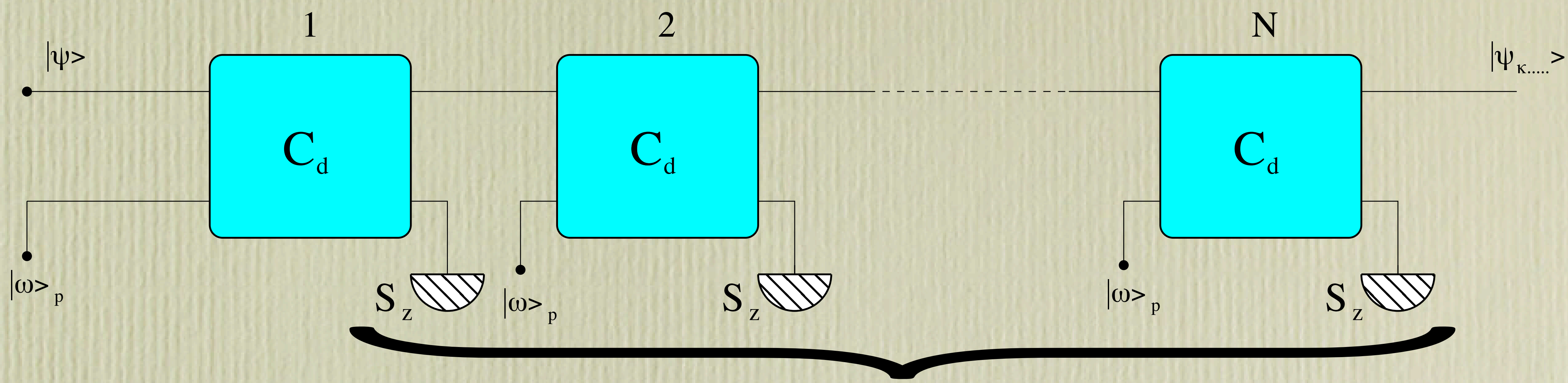

$$\{k_i\} = \{k_1, k_2, \ldots, k_N\}$$

Collective inference rule

$$\varrho_{\{k_i\}} = \frac{n_0}{N}|0\rangle\langle 0| + \frac{n_1}{N}|1\rangle\langle 1| + \cdots + \frac{N-(n_0+n_1+\cdots+n_{d-2})}{N}|d-1\rangle\langle d-1|$$

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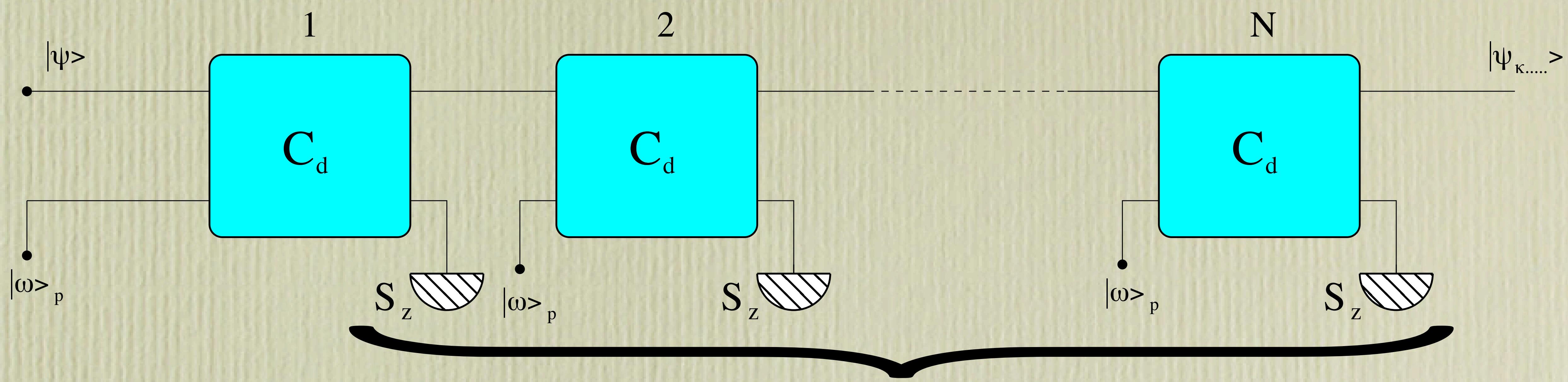

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Same results as those obtained with the *single-measure inference rule*

13. Outro

- Qudit: measurement schemes optimizing the information/disturbance tradeoff may be achieved using a single d-dim probe

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13. Outro

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- N-user measurement schemes: trade-off gets worse increasing N

- 2-user measurement scheme: *“near-optimal”* region achievable

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12. Problems, outlooks and applications

- Set of symmetric states $|\psi_g\rangle = U_g|\psi_e\rangle$

  (reducible and irreducible)

- Parametrization of sectors of the Hilbert space

- Application in eavesdropping strategies