# Nonexponential Decay of Unstable Systems at Finite Temperatures

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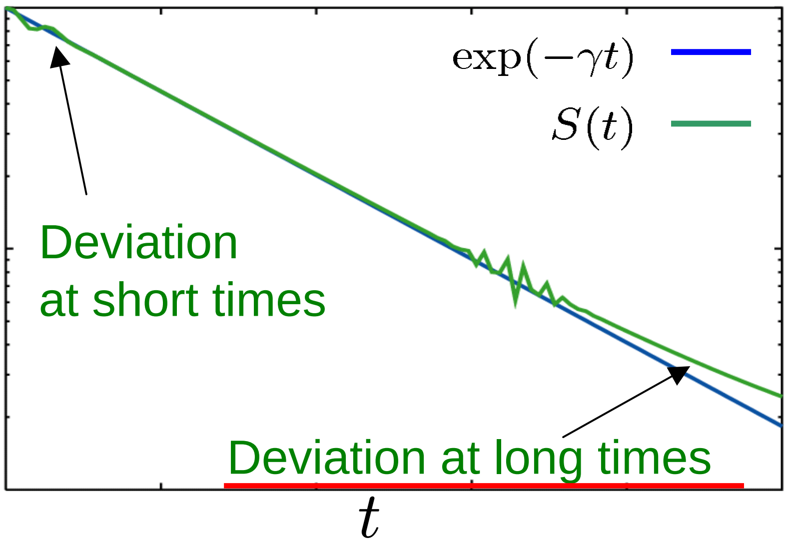
# Introduction



## Decaying Behavior of Unstable Systems

L.Fonda, G.C.Ghirardi, A.Rimini. Rep. Prog. Phys. **41**. 587 (1987)
H.Nakazato, M.Namiki, S.Pascazio, Int. J. Mod. Phys. B **10**, 247 (1996)

$$S(t) = |\langle\psi|e^{-itH}\psi\rangle|^2$$
$$\sim \exp(-\gamma t)$$ **exponential decay**




**Before and after the exponential decay regime**

**Short･･･approved**
S.R.Wilkinson *et al,* Nature (London) **387**, 575 (1997)

**Long･･･approved!**
C.Rothe *et al,* Phys. Rev. Lett. **96**, 163601 (2006)

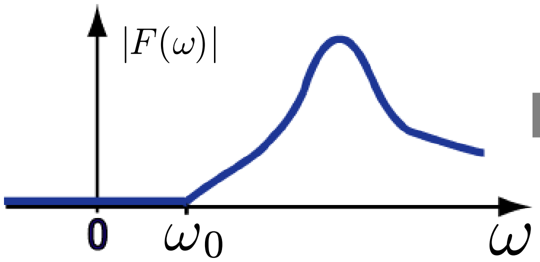
## Long Time Deviation

**L. A. Khalfin, Zh. Eksp. Theor. Fiz. 33 1371 (1957) [ Sov. Phys. JETP 6 , 1053 (1958)] :**

System has the lower-bounded and continuous energy-spectrum

⇒ Long-time decaying behavior becomes Nonexponential !!

Paley-Wiener Theorem




$$\exists\omega_0 > 0 \ \text{s.t.} \ F(\omega) = 0 \ \ (\forall\omega < \omega_0)$$

⇒

$$\int_{-\infty}^{\infty} \frac{\left|\log\left|\int_{-\infty}^{\infty} F(\omega)e^{-it\omega}d\omega\right|\right|}{1+t^2} dt < \infty$$

# Introduction



**Experiment of Rothe *et al***

C.Rothe *et al,* Phys. Rev. Lett. **96**, 163601 (2006)

Nonexponetial decay law at long times was observed

i) *At a finite temperature*

( 77K~room temperature )

ii) Oscillatory behavior at crossing time?

iii) Initial state dependence at long times

(M.M Phys. Rev. A **70**, 032108 2004;

Braz. J. Phys. **35**, 425 2005.)

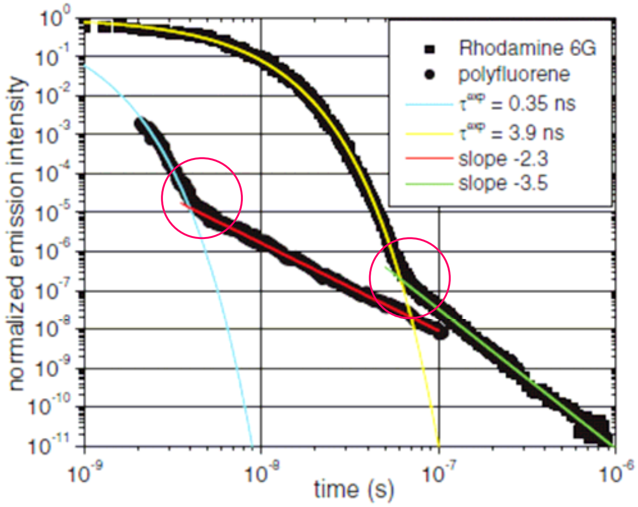
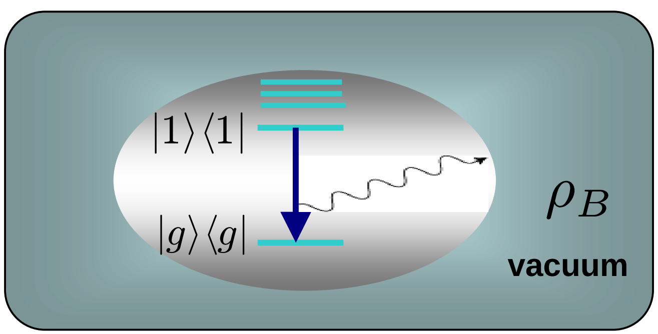



FIG. 2 (color). Corresponding double logarithmic fluorescence decays of the emissions shown in Fig. 1. Exponential and power law regions are indicated by solid lines and the emission intensity at time zero has been normalized.

**Theoretical studies of unstable systems at long times *for a finite temperature* ?**

**system ＋ environmental field**

(discrete) (continuum)




**However we often choose …**

$$\rho_S = |1\rangle\langle 1| \qquad \rho_B = |0\rangle\langle 0|$$

[except, e.g.

A.G.Kofman,G.Kurizki, Phys.Rev.Lett.**93**,130406(2004)]

In this study, we choose $\rho_B$ to be *a thermal state*

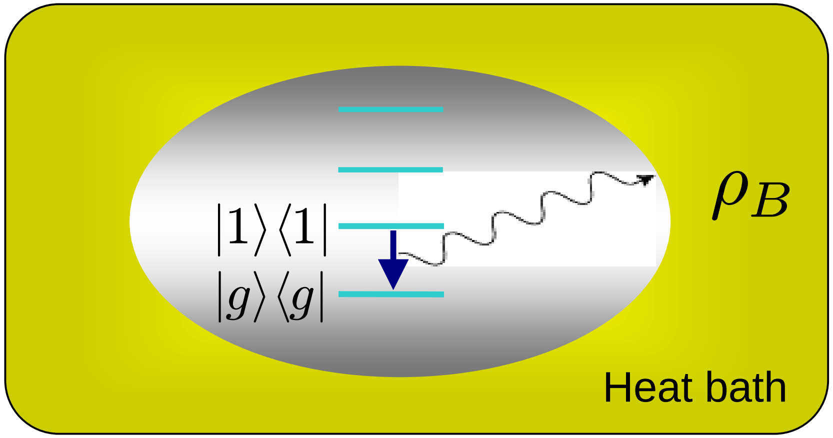
# Harmonic Oscillator coupled to a Bosonic Field

**Hamiltonian**

$$H = H_0 + \lambda V$$

$$H_0 = \omega_0 a^\dagger a + \int_0^\infty d\omega \omega b_\omega^\dagger b_\omega$$

$$V = \int_0^\infty d\omega [g^*(\omega) a^\dagger b_\omega + g(\omega) a b_\omega^\dagger]$$




**Initial State**

$$\rho(0) = \rho_S(0) \otimes \rho_B(0)$$

$$\rho_S(0) = a^\dagger |g\rangle\langle g| a = |1\rangle\langle 1|$$ First Excited state

$$\rho_B(0) = e^{-\beta H_B} / \mathrm{tr_B} \left\{ e^{-\beta H_B} \right\}$$ **Thermal State**

**Reduced Density Operator**

$$\rho_S(t) = \mathrm{tr_B} \{\rho(t)\}$$

**Survival probability**

$$\langle 1 | \rho_S(t) | 1 \rangle$$

**Form factor is assumed to be meromorphic**

(e.g. a system of hydrogen atom interacting with an EM field)

$$|g(\omega)|^2 = \varphi(\omega) = \Lambda \tilde{\varphi}(\omega/\Lambda) = \Lambda \frac{\pi(\omega/\Lambda)}{\rho(\omega/\Lambda)}$$

$$\underset{\omega \to 0}{=} \Lambda \left[ \frac{\tilde{\varphi}^{(n)}(0)}{n!} (\omega/\Lambda)^n + O((\omega/\Lambda)^{n+1}) \right]$$

P.Facchi, S.Pascazio, Phys. Lett. A, **241**, 139 (1998);I. Antoniou *et al.*, Phys. Rev. A, **63**, 062110 (2001)

# Survival Probability at Finite Temperature

**Generating Function in $Q$-Representation**

$$\mathcal{G}^{(a)}(J,J^*) := \mathrm{tr}_S\left\{e^{aJ^*}e^{a^\dagger J}\rho_S(t)\right\} \quad \Longrightarrow \quad \rho_S(t) = \int \frac{d^2J}{\pi}\mathcal{G}^{(a)}(J,-J^*)e^{-a^\dagger J}e^{aJ^*}$$

**Survival Probability at a Finite Temperature**

$$\langle 1|\rho_S(t)|1\rangle = |y(t)|^2 \frac{1-\int d\omega n_\beta(\omega)|G(\omega,t)|^2}{\left(1+\int d\omega n_\beta(\omega)|G(\omega,t)|^2\right)^3} + \frac{\int d\omega n_\beta(\omega)|G(\omega,t)|^2}{\left(1+\int d\omega n_\beta(\omega)|G(\omega,t)|^2\right)^2}$$

**New term**

Survival amplitude when the initial state of the field is vacuum

$$y(t) = \frac{1}{2\pi i}\int_B \frac{e^{st}}{s+i\omega_0+\lambda^2\int_0^\infty d\omega' \frac{\varphi(\omega')}{s+i\omega'}}ds = \frac{1}{2\pi i}\int_B \frac{e^{st}}{h(s)}ds$$

Survival amplitude of an emitted photon

$$G(\omega,t) = \frac{-\lambda}{2\pi}\left[\int_B \frac{e^{st}}{h(s)}\frac{1}{s+i\omega}ds\right]g^*(\omega)$$

Bose distribution : $n_\beta(\omega) = (e^{\beta\omega}-1)^{-1}$

Probability conservation law

$$|y(t)|^2 + \int d\omega\, |G(\omega,t)|^2 = 1$$

Mean excited-energy level

$$\langle a^\dagger(t)a(t)\rangle = \langle a^\dagger(t)a(t)\rangle_{\mathrm{vac}} + \int d\omega n_\beta(\omega)|G(\omega,t)|^2 \qquad \langle a^\dagger(t)a(t)\rangle_{\mathrm{vac}} = |y(t)|^2$$

# Survival Probability at Finite Temperature

**Zero temperature limit**

$$\langle 1|\rho_S(t)|1\rangle$$

$$= |y(t)|^2 \frac{1 - \int d\omega n_\beta(\omega)|G(\omega,t)|^2}{\left(1 + \int d\omega n_\beta(\omega)|G(\omega,t)|^2\right)^3} + \frac{\int d\omega n_\beta(\omega)|G(\omega,t)|^2}{\left(1 + \int d\omega n_\beta(\omega)|G(\omega,t)|^2\right)^2}$$

$$\lim_{\beta\to\infty} \int d\omega n_\beta(\omega)|G(\omega,t)|^2 = 0$$

$\to |y(t)|^2 \quad (\beta\to\infty)$ … agree with the time evolution at zero temperature!

Long time behavior when an initial state of the field is vacuum（c.f. Friedrichs Model）

$$|y(t)|^2 \simeq \lambda^4 \frac{1}{\Lambda^{2(n-1)}} \frac{(\tilde{\varphi}^{(n)}(0))^2}{\left(\omega_0 - \lambda^2 \int d\omega \frac{\varphi(\omega)}{\omega}\right)^4} \frac{1}{t^{2(n+1)}} \qquad t \to \infty$$

P.Facchi, S.Pascazio, Phys. Lett. A, **241**, 139 (1998)

I. Antoniou *et al*., Phys. Rev. A, **63**, 062110 (2001)

**The inverse power of $t$ is determined by $n$ in the form factor**

Correction to the long time behavior → **What we need to evaluate is …**

$$\int d\omega n_\beta(\omega)|G(\omega,t)|^2$$

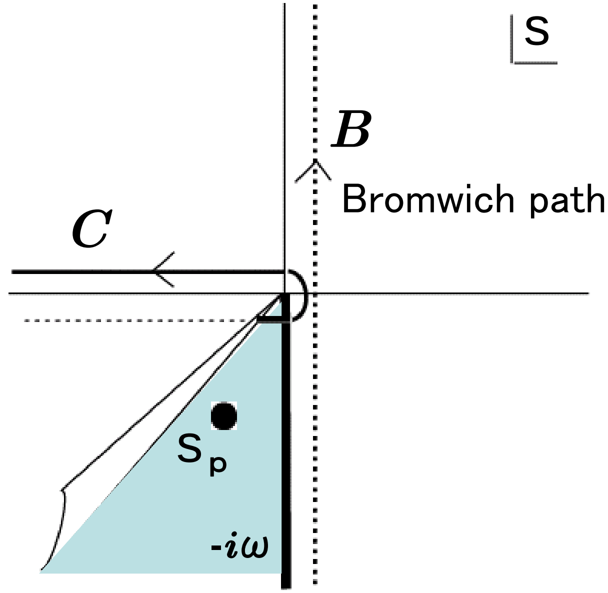
# Evaluation of $\int d\omega n_\beta(\omega)|G(\omega,t)|^2$



$$G(\omega,t)=\frac{-\lambda}{2\pi}\left[\int_B \frac{e^{st}}{h(s)}\frac{1}{s+i\omega}ds\right]g^*(\omega)$$

$$=\ -i\lambda g^*(\omega)\left[\frac{e^{-i\omega t}}{h(-i\omega+0)}+\frac{e^{s_p t}}{h_2'(s_p)}\frac{1}{s_p+i\omega}+\Xi(\omega,t)\right]$$

**pole ($s$=-$i\omega$)** **pole of $h_2(s)$** **cut**




Corresponds to the equilibrium state

$$|G(\omega,t)|^2$$

$$=\lambda^2\varphi(\omega)\left[\frac{1}{|h(-i\omega+0)|^2}+\frac{e^{2Re(s_p)t}}{|h_2'(s_p)|^2}\frac{1}{|s_p+i\omega|^2}+|\Xi(\omega,t)|^2\right.$$

$$\left.+\left(\frac{e^{-i\omega t+s_p^* t}}{h(-i\omega+0)(h_2'(s_p))^*(s_p^*-i\omega)}+\frac{e^{-i\omega t}}{h(-i\omega+0)}\Xi^*(\omega,t)+\frac{e^{s_p t}}{h_2'(s_p)}\frac{1}{s_p+i\omega}\Xi^*(\omega,t)+\text{h.c.}\right)\right]$$

**Non exponential decay**

**(cut contributions)**

**Negligible! ( exponentially-decaying terms )**

# Asymptotes of nonexponentially decaying terms

As $t \to \infty$

Equilibrium Value $\langle a^\dagger(\infty)a(\infty)\rangle$

$$\int d\omega n_\beta(\omega)|G(\omega,t)|^2 \simeq \lambda^2 \int d\omega n_\beta(\omega)\frac{\varphi(\omega)}{|h(-i\omega+0)|^2} + \lambda^2 \int d\omega n_\beta(\omega)\varphi(\omega)|\Xi(\omega,t)|^2$$

$$+\lambda^2\left(\int d\omega n_\beta(\omega)\varphi(\omega)\frac{e^{-i\omega t}}{h(-i\omega+0)}\Xi^*(\omega,t) + \mathrm{h.c.}\right)$$

$$\lambda^2 \int d\omega n_\beta(\omega)\varphi(\omega)|\Xi^*(\omega,t)|^2 \le \lambda^6 \frac{[C_1\Gamma(n+1-\alpha)]^2}{t^{2(n+1)-2\alpha}} \int d\omega \frac{n_\beta(\omega)\varphi(\omega)}{\omega^{2-2\alpha}} = O(t^{-2n-2(1-\alpha)})$$

$$0 \le \alpha < 1$$

$$\lambda^2 \int_0^\infty d\omega n_\beta(\omega)\varphi(\omega)\frac{e^{-i\omega t}}{h(-i\omega+0)}\Xi^*(\omega,t) = -\lambda^4 \frac{1}{t^{2n}}\frac{1}{\beta\Lambda^{2(n-1)}}\frac{(\tilde{\varphi}^{(n)}(0))^2}{n^2(\omega_0-\lambda^2\int d\omega \frac{\varphi(\omega)}{\omega})^3} + O(t^{-2n})$$

**The latter term may dominate over the former one for all $n$ !**

$$\langle 1|\rho_S(t)|1\rangle = |y(t)|^2 \frac{1-\int d\omega n_\beta(\omega)|G(\omega,t)|^2}{\left(1+\int d\omega n_\beta(\omega)|G(\omega,t)|^2\right)^3} + \frac{\int d\omega n_\beta(\omega)|G(\omega,t)|^2}{\left(1+\int d\omega n_\beta(\omega)|G(\omega,t)|^2\right)^2}$$

# Long Time Behavior of Survival Probability

$$\langle 1|\rho_{\mathrm{S}}(t)|1\rangle$$

Equilibrium Value

$$\simeq \lambda^2 \frac{\int d\omega n_\beta(\omega)\frac{\varphi(\omega)}{|h(-i\omega+0)|^2}}{\left(1+\lambda^2\int d\omega n_\beta(\omega)\frac{\varphi(\omega)}{|h(-i\omega+0)|^2}\right)^2}$$

**New additional term**

**comparable to $|y(t)|^2$!**

$$+\left\{\lambda^4\frac{1}{\Lambda^{2(n-1)}}\frac{(\tilde{\varphi}^{(n)}(0))^2}{\left(\omega_0-\lambda^2\int d\omega\frac{\varphi(\omega)}{\omega}\right)^4}\underline{\frac{1}{t^{2(n+1)}}}-\lambda^4\frac{1}{\beta\Lambda^{2(n-1)}}\frac{2(\tilde{\varphi}^{(n)}(0))^2}{n^2\left(\omega_0-\lambda^2\int d\omega\frac{\varphi(\omega)}{\omega}\right)^3}\underline{\frac{1}{t^{2n}}}+O(t^{-2n})\right\}$$

$$\times\frac{1-\lambda^2\int d\omega n_\beta(\omega)\frac{\varphi(\omega)}{|h(-i\omega+0)|^2}}{\left(1+\lambda^2\int d\omega n_\beta(\omega)\frac{\varphi(\omega)}{|h(-i\omega+0)|^2}\right)^3}$$

**Usual survival-probability term, i.e.,** $|y(t)|^2$

- **Finite temperature correction seems significant at long time**

- **The mathematical origin of nonexponential decay is as same as usual one**
  [Paley-Wiener theorem L.A.Khalfin, Zh. Eksp. Theor. Fiz. **33**, 1371 (1957) [Sov. Phys. JETP **6**, 1053(1958)] ]

  ⇒ Examination of nonexponetial decay at finite temperature is still valid !

# Summary

As a first trial, we examine the long time behavior of unstable systems at a finite temperature based on a specific model

- We have found how the finite temperature effect appears in the asymptote of the survival probability at long time
- A finite temperature correction may be significant comparable to the usual survival-probability term

# Future Problems

- Validity of the asymptotic expansion?
- The first time at which the asymptotic expansion is valid?
- Oscillatory behavior at cross over time?
- Finite temperature effect on the short time behavior?

# 水素原子系への応用



## ベキ崩壊則の遷移時刻の見積もり

$$\langle 1|\rho_S(t)|1\rangle \simeq \lambda^2 \frac{\int d\omega n_\beta(\omega)\frac{\varphi(\omega)}{|h(-i\omega+0)|^2}}{\left(1+\lambda^2\int d\omega n_\beta(\omega)\frac{\varphi(\omega)}{|h(-i\omega+0)|^2}\right)^2}$$

$$+\left\{\lambda^4\frac{1}{\Lambda^{2(n-1)}}\frac{(\tilde{\varphi}^{(n)}(0))^2}{\left(\omega_0-\lambda^2\int d\omega\frac{\varphi(\omega)}{\omega}\right)^4}\frac{1}{t^{2(n+1)}}+\lambda^4\frac{(-1)^n}{\beta\Lambda^{2(n-1)}}\frac{2(\tilde{\varphi}^{(n)}(0))^2}{n^2\left(\omega_0-\lambda^2\int d\omega\frac{\varphi(\omega)}{\omega}\right)^3}\frac{1}{t^{2n}}\right\}\frac{1-\lambda^2\int d\omega n_\beta(\omega)\frac{\varphi(\omega)}{|h(-i\omega+0)|^2}}{\left(1+\lambda^2\int d\omega n_\beta(\omega)\frac{\varphi(\omega)}{|h(-i\omega+0)|^2}\right)^3}$$

大小の交代

電磁場と相互作用する水素原子系　・・・　形状因子が解析的に求まっている ($n$=1)

J.Seke,S. Physica A **203**, 269 (1994)

$$t_{\rm th} \simeq \sqrt{\frac{\beta}{\omega_0+\frac{-15\pi\Lambda\lambda^2}{96}}} = \begin{cases} 0.8\times 10^{-6}\tau & (300\ \mathrm{K}) \\ 0.8\times 10^{-1}\tau & (30\ \mathrm{nK}) \end{cases}$$

$\tau$ ： 寿命

$|y(t)|^2$ のcutの漸近展開は $10^{-6}\tau$ より有効

P.Facchi,S.Pascazio Phys. Lett. A 241,139 (1998)

温度が少しでも有限ならば，温度補正によるベキ崩壊しか見えない！？

- 主要項の漸近形の評価が不十分！！
- 場の初期状態が真空の場合，指数崩壊からベキ崩壊への遷移時刻は 90 $\tau$（生存確率は非常に小！！）

➡ ベキ崩壊を検出する系には不適

- 高準位の影響