Dynamics of Qubits in Random Environments

Francesco Petruccione
School of Physics
UNIVERSITY OF KWAZULU-NATAL

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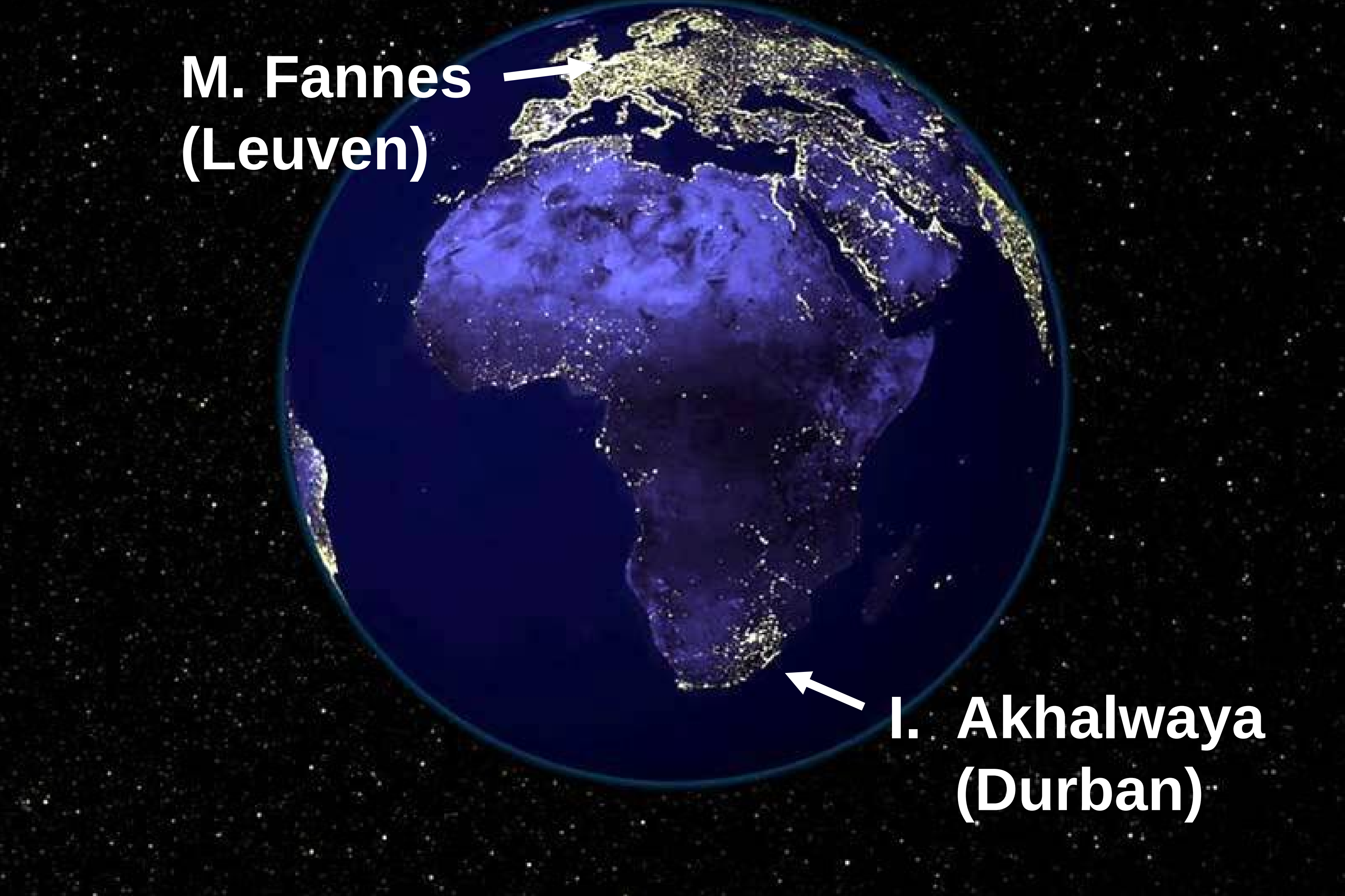
Bertinoro

In collaboration with …



Centre for Quantum Technology

- Quantum Key Distribution
- Single Photon Sources (Defects in Diamonds)
- Open Quantum Systems
  - Non-Markovian phenomena
  - Stochastic simulations
  - High Energy Physics
- Relativistic Quantum Information

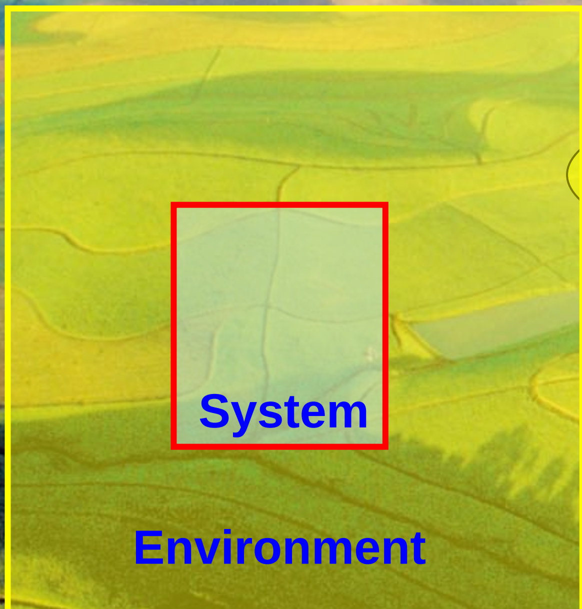
18. Chris Engelbrecht Summer School in Theoretical Physics

Theoretical Foundations of
Quantum Information Processing
and Communication

14- 24 January 2007

… where the Dolphins come to play!

Microscopic Theory



Total system:

$$H = H_S \otimes I_B + I_S \otimes H_B + \alpha H_I$$

$$\frac{d}{dt}\rho(t) = -i\left[H, \rho(t)\right]$$

Observables: $A \otimes I_B$

$$\langle A \rangle = \mathrm{tr}_S\left\{A\rho_S\right\}$$

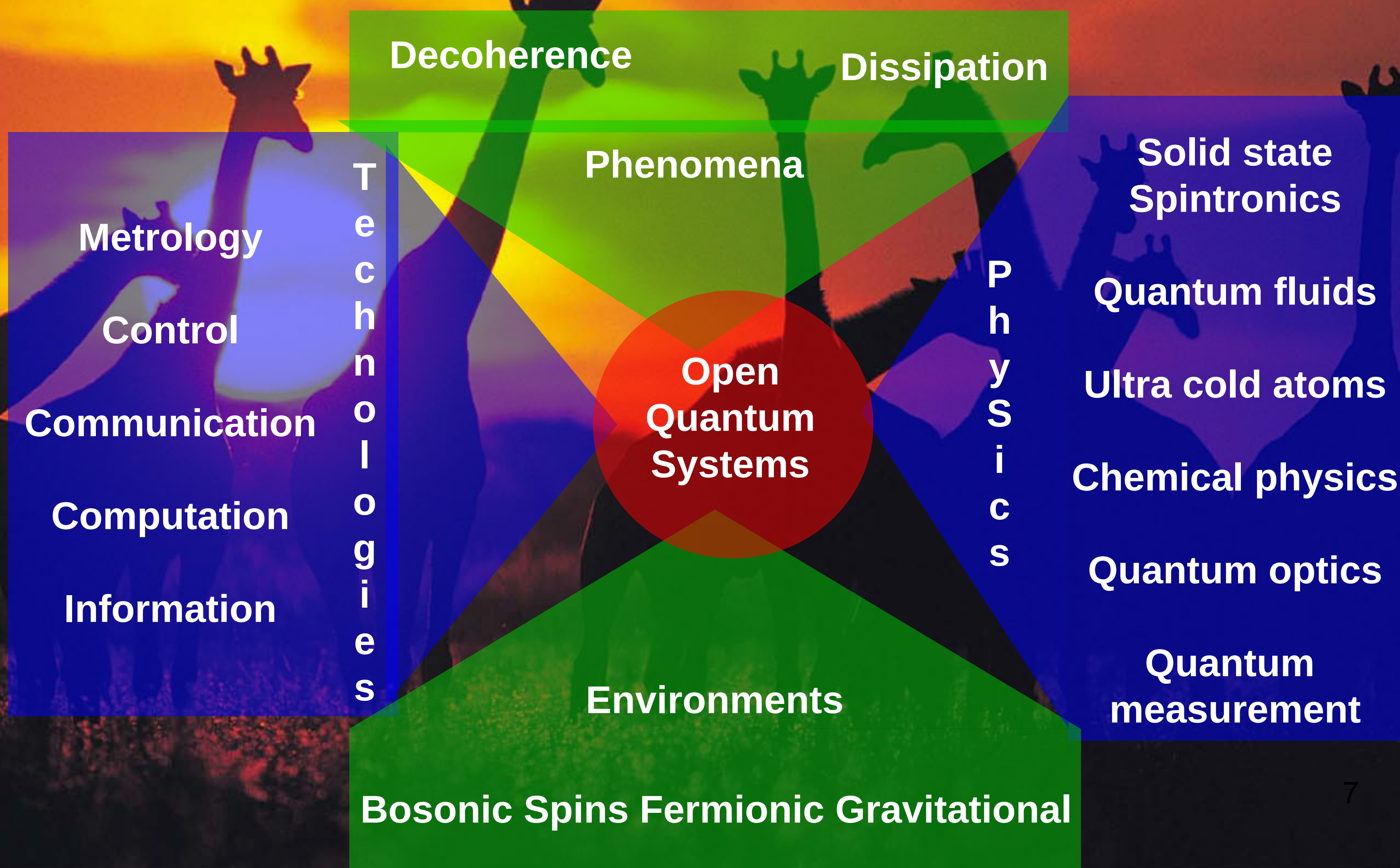
$$\rho_S = \mathrm{tr}_B\,\rho$$

$$\frac{d}{dt}\rho_S(t) = -i\,\mathrm{tr}_B\left[H, \rho(t)\right]$$

Dissipation and Decoherence

- **Useful: magnetic resonance and laser spectroscopy (medical diagnostics)**
- **Essential: Laser cooling**
- **Crucial: Quantum measurement and quantum cosmology**
- **Noxious: Quantum computing**

Open Quantum Systems



The Big Five





Hermitean Random Matrices

Off-diagonal elements:

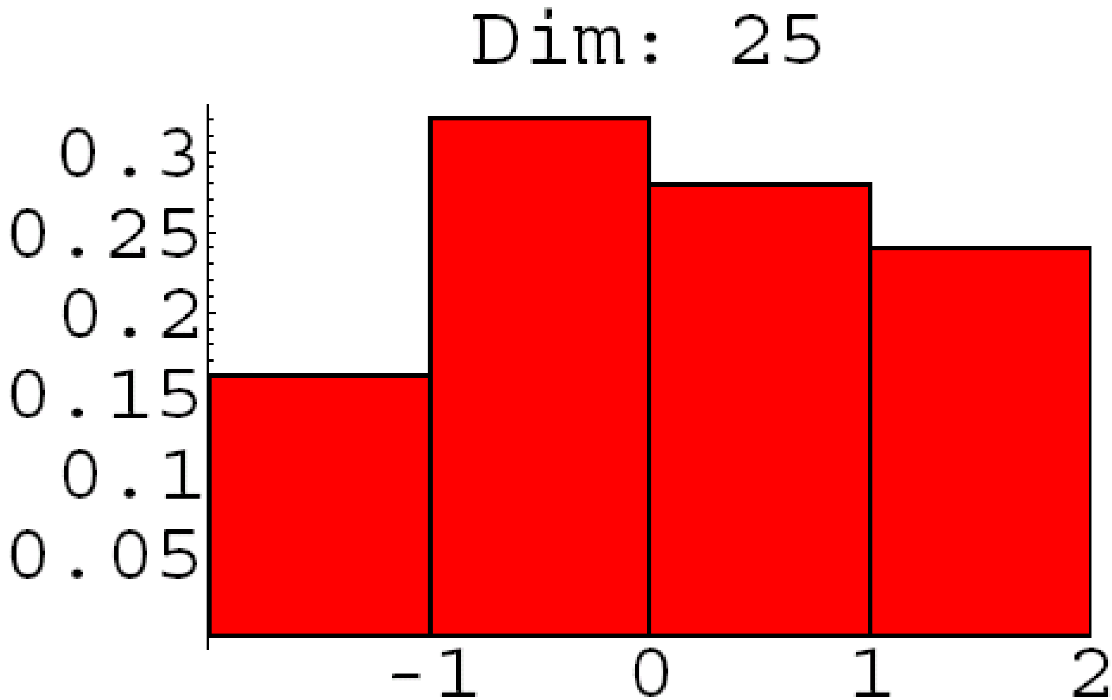
$$A_{jk} := \mathrm{Random}(0, \frac{1}{\sqrt{2N}}) + i\ \mathrm{Random}(0, \frac{1}{\sqrt{2N}})$$

$$\forall\ 1 < j < k \leq N$$

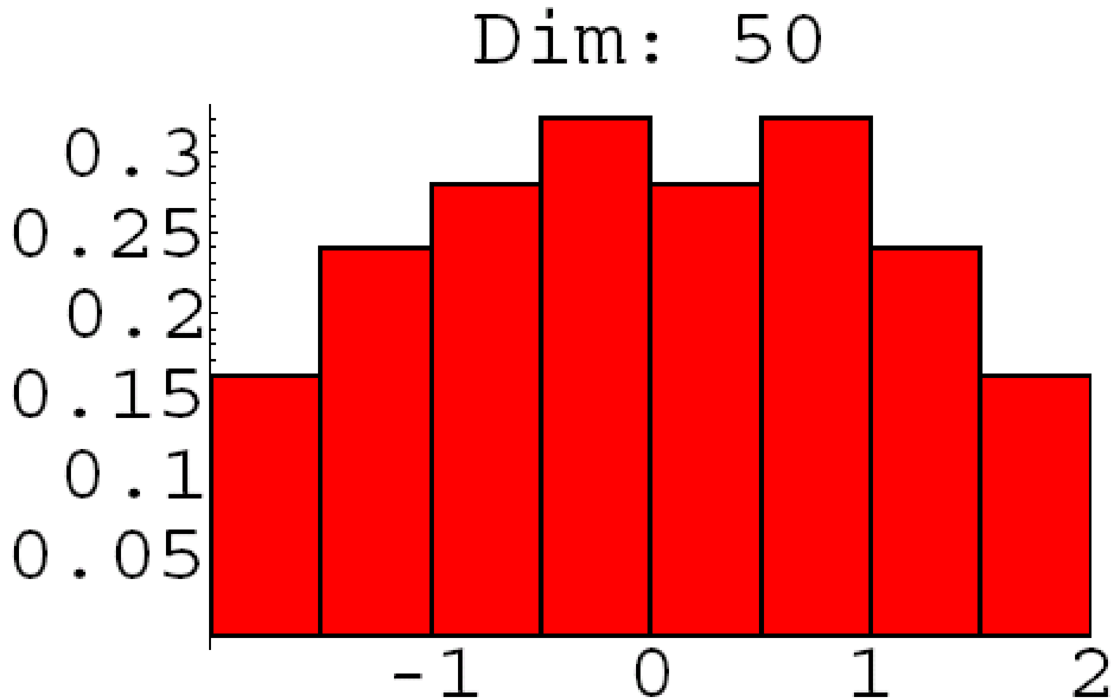
Diagonal elements:

$$\underbrace{A_{jj} := \mathrm{Random}(0, \frac{1}{\sqrt{N}})}_{\text{Gaussian distributed}}$$

Eigenvalue spectrum of random matrices



Eigenvalue Spectrum of random matrices



Wigner's Semi-Circle Law

Limiting distribution of eigenvalues of random matrices

$$f(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$

If X is standard semicircular distributed:

$$\langle X^{2n+1} \rangle = 0$$

$$\langle X^{2n} \rangle = \frac{1}{n+1}\binom{2n}{n} = C_n$$

Lit: E P Wigner, Ann. Math. 62 (1955)

Freeness

In the limit of large random matrices:

$$\boxed{\left\langle \left(x^{m_1} - \langle x^{m_1}\rangle\right)\left(y^{n_1} - \langle y^{n_1}\rangle\right)\cdots\left(y^{n_k} - \langle y^{n_k}\rangle\right)\right\rangle = 0}$$

$\Rightarrow$ x, y are in free relation!

E. g.: $\langle xyxy\rangle = \langle x^2\rangle\langle y\rangle^2 + \langle x\rangle^2\langle y^2\rangle - \langle x\rangle^2\langle y\rangle^2$

(For free probability theory see, e.g., Voiculescu, Speicher)

Qubit in random environment

Hamiltonian of the coupled system:

$$h = \begin{bmatrix} \varepsilon + x & \lambda y \\ \lambda y & x \end{bmatrix}$$

x, y: Hermitean random
environment operators

λ: coupling constant

ε: gap between ground and excited state

The initial condition

Qubit decoupled from environment:

$$\rho_0 = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$$

$$\rho_{11} \geq 0,\ \rho_{11} + \rho_{22} = 1,\ \rho_{21} = \overline{\rho_{12}},\ \text{and}\ |\rho_{12}|^2 \leq \rho_{11}\rho_{22}$$

Equivalently: $\rho_0 = \frac{1}{2}\left(1 + \overline{r}_0 \cdot \overline{\sigma}\right)$

Bloch vector

Time evolution

$U(t) = e^{-iHt}$

- Numerically carry out for sample H.
- Apply to $\rho(0)$
- Environment at infinite temperature: $1_{N\times N}$
- Trace out the environment.

$$\rho(0) = \begin{pmatrix} \rho_{11}(0) & \rho_{12}(0) \\ \rho_{21}(0) & \rho_{22}(0) \end{pmatrix} \otimes 1_{N\times N}$$

Reduced dynamics of the qubit

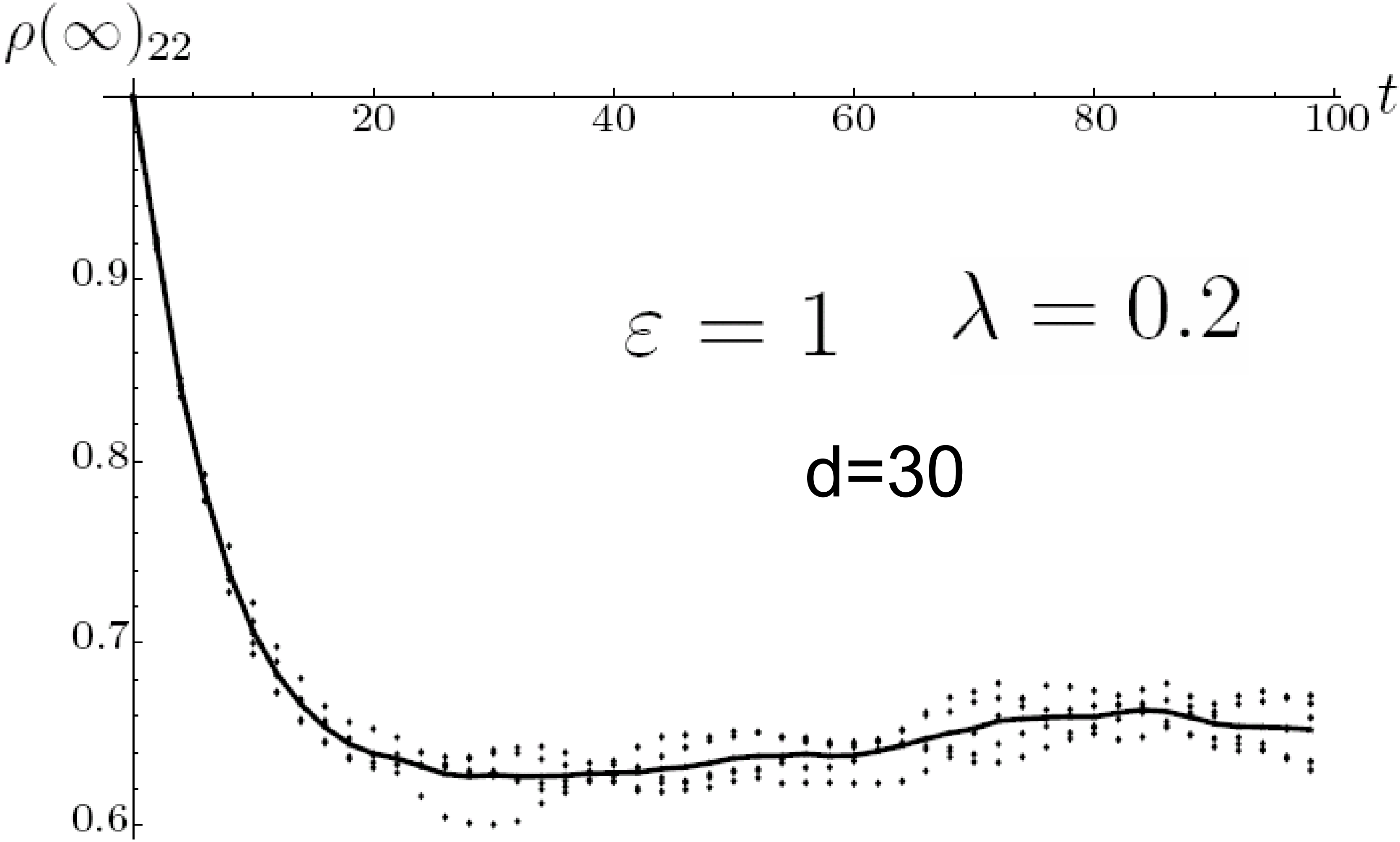
$$\boxed{t \mapsto \left\langle \mathrm{Tr}\big(\rho_0\, u(t)\, \overline{\sigma}\, u(t)^\dagger\big) \right\rangle}$$

In the interaction picture:

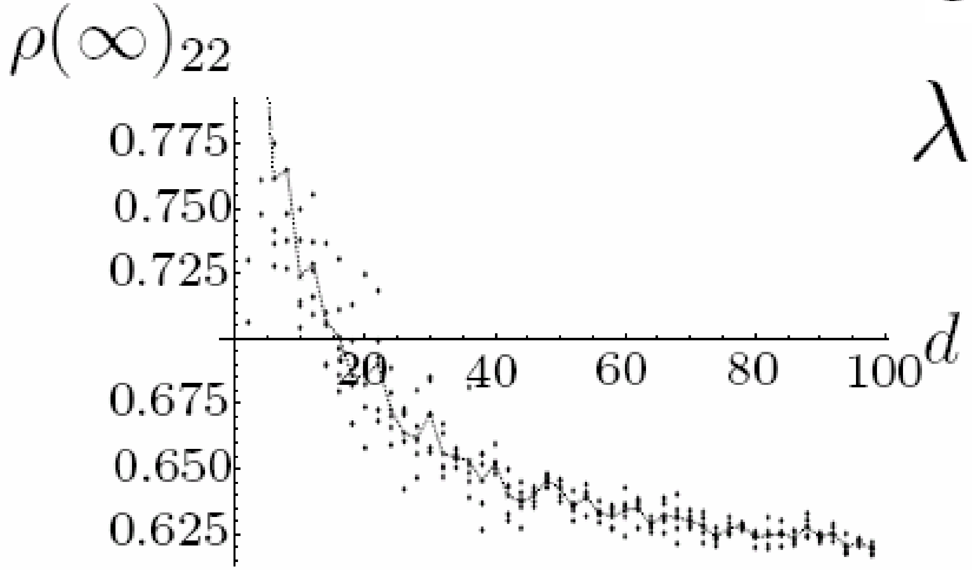
$$t \mapsto \left\langle \mathrm{Tr}\big(\rho_0\, U_0^\dagger(t)\, U(t)\, \overline{\sigma}\, U(t)^\dagger\, U_0(t)\big) \right\rangle =: \mathrm{Tr}\, \rho(t)\, \overline{\sigma}$$

$$H_0 := \begin{bmatrix} \varepsilon + X & 0 \\ 0 & X \end{bmatrix}$$

Realizations of reduced density matrix



Long time limit



$$\varepsilon = 1$$

$$\lambda = 0.2$$

Dyson expansion

$$h = h_0 + \lambda p$$

For an observable *a* of the total system:

$$\begin{aligned} &u_0^\dagger(t)\, u(t)\, a\, u^\dagger(t)\, u_0(t) \\ &= a + (i\lambda) \int_0^t ds_1\, [p_{t-s_1}, a] + \cdots \\ &\quad + (i\lambda)^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n\, [p_{t-s_1}, [p_{s_1-s_2}, \ldots [p_{s_{n-1}-s_n}, a] \ldots]] + \cdots \end{aligned}$$

Freeness:

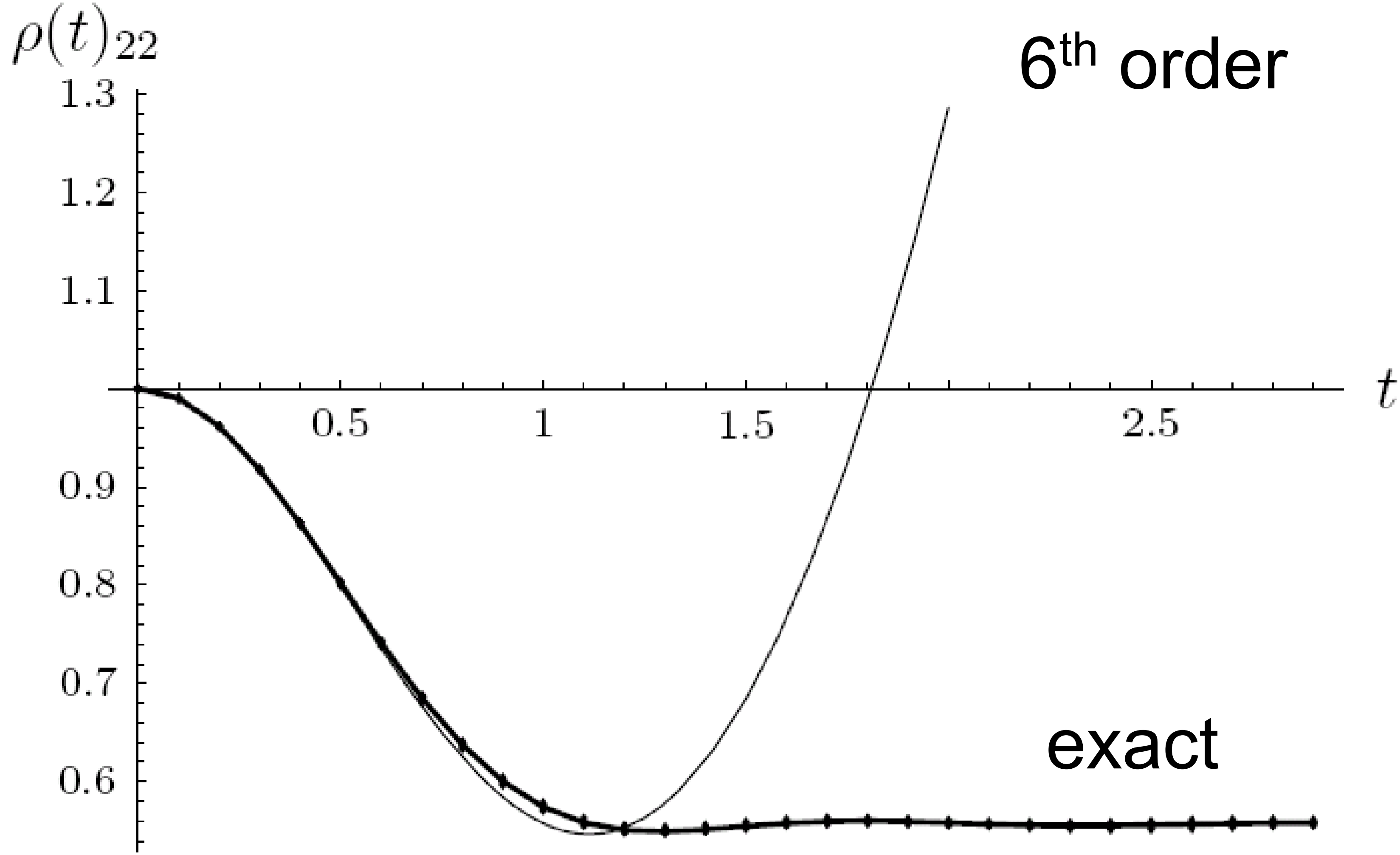
$\Longrightarrow$ $$\langle \exp(isx) \rangle = \frac{1}{2\pi} \int_{-2}^{2} d\zeta\, \sqrt{4 - \zeta^2}\, \mathrm{e}^{is\zeta} = \frac{\mathrm{J}_1(2s)}{s}.$$

Reduced density matrix

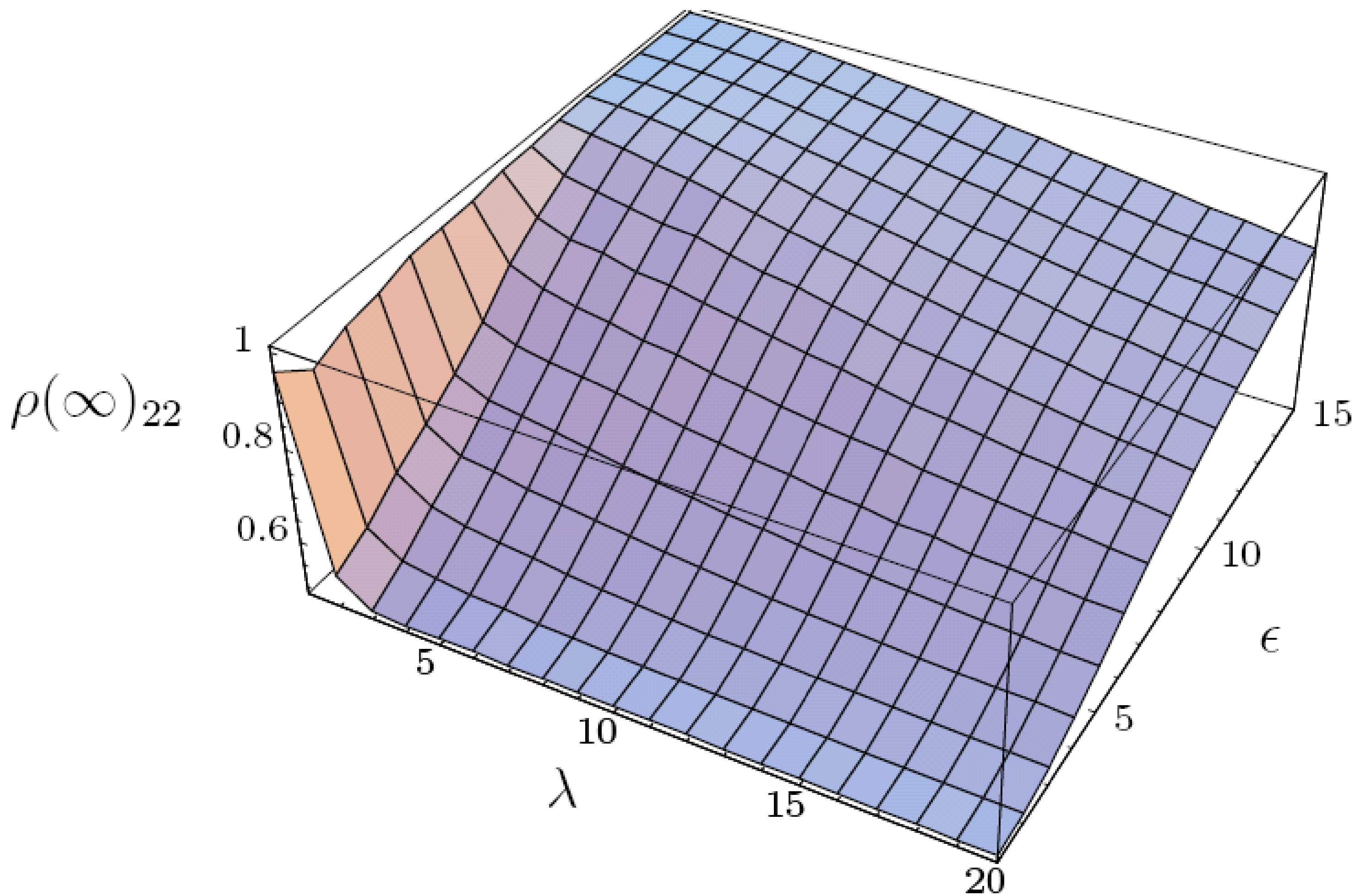
$$\langle (1-\sigma^z)/2 \rangle = \rho_{22} =$$

$$1 - 2\lambda^2 \int_0^t dx\,(t-x)\,\frac{\cos(\varepsilon x)\mathrm{J}_1^2(2x)}{x^2}$$
$$+\frac{4}{3}\lambda^4 \int_0^t dx_1 \int_0^t dx_2\,(t-x_1)\,(t-x_2)\,\Big\{\cos\big(\varepsilon(x_1-x_2)\big)$$
$$\frac{\mathrm{J}_1^2(2x_1)\,\mathrm{J}_1^2(2x_2)\,\mathrm{J}_1^2\big(2(x_1-x_2)\big)}{x_1^2\,x_2^2\,(x_1-x_2)^2} + \cos\big(\varepsilon(x_1+x_2)\big)\,\frac{\mathrm{J}_1^2(2x_1)\,\mathrm{J}_1^2(2x_2)\,\mathrm{J}_1^2\big(2(x_1+x_2)\big)}{x_1^2\,x_2^2\,(x_1+x_2)^2}\Big\}$$
$$+\,\mathrm{O}(\lambda^6)$$

Dyson versus exact



Asymptotic values



Ongoing studies

- Analytical solution
- Extended models
- Physical applications

- Lit: I. Akhalwaya, M. Fannes, F. P. (almost submitted)

Thank you!

- petruccione@ukzn.ac.za
- http://quantum.ukzn.ac.za

The Big Five

- Open Quantum Systems
- General strategies:
  - Markovian
  - Non-Markovian
- Simple spin systems:
  - One qubit in a spin bath
  - Two qubits in a spin bath
- Qubit in Random Environment
- **Outlook**