Microscopic derivation of the Jaynes-Cummings model with cavity losses

M. Scala, B. Militello, A. Messina

Dipartimento di Scienze Fisiche ed Astronomiche,

Università degli Studi di Palermo, Italy

J. Piilo, S. Maniscalco

Department of Physics, University of Turku, Finland

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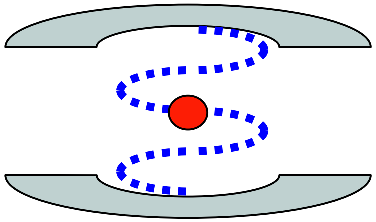
Summary

- The Jaynes-Cummings model
- Open Quantum Systems
- Master equation for the JC
- Comparison with the phenomenological model

The Jaynes-Cummings model



Two-level atom interacting with a cavity mode

$$H_{JC} = \frac{\omega_0}{2}\sigma_z + \omega_0\, a^\dagger a + \Omega\left(a\sigma_+ + a^\dagger\sigma_-\right)$$

Diagonalization:

Singlet

$$|E_0\rangle = |0, g\rangle \qquad E_0 = -\frac{\omega_0}{2}$$

Doublets

$$|E_{N,\pm}\rangle = \frac{1}{\sqrt{2}}\left(|N, g\rangle \pm |N-1, e\rangle\right)$$

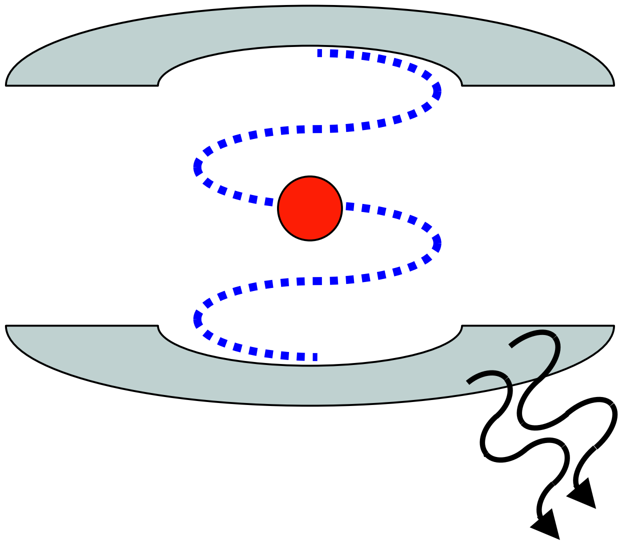
$$E_{N,\pm} = \left(N - \tfrac{1}{2}\right)\omega_0 \pm \Omega\sqrt{N}$$

E.T. Jaynes e F.W. Cummings, Proc. IEEE 51, 89 (1963)

Cavity losses

Phenomenological approach:

$$
\begin{aligned}
\dot{\rho} &= -i\left[H_{JC},\rho\right] \\
&+\gamma\left[n(\omega_0)+1\right]\left[a\rho a^\dagger - \tfrac{1}{2}\left(a^\dagger a\rho + \rho a^\dagger a\right)\right] \\
&+\gamma n(\omega_0)\left[a^\dagger\rho a - \tfrac{1}{2}\left(a a^\dagger\rho + \rho a a^\dagger\right)\right]
\end{aligned}
$$



$n(\omega_0)$: average photon number in the environment at T temperature

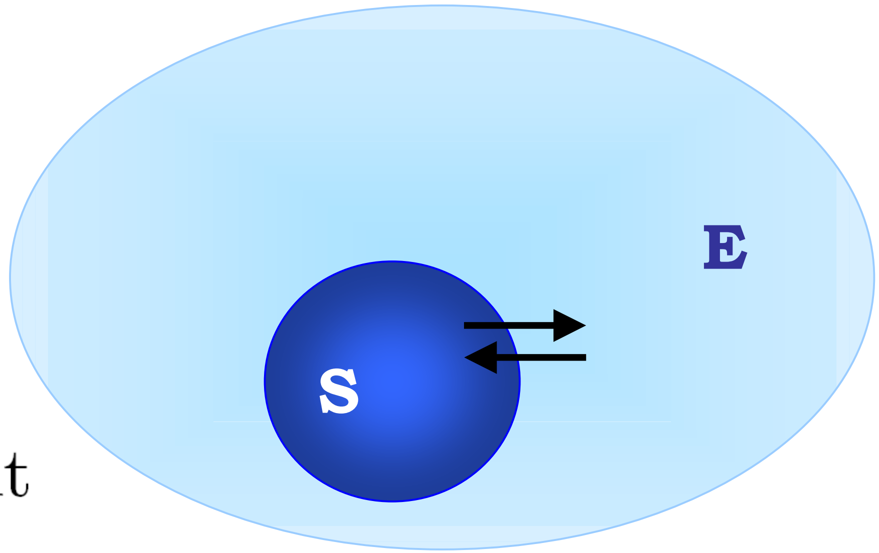
Zero temperature:

$$
\dot{\rho} = -i\left[H_{JC},\rho\right] + \gamma\left(a\rho a^\dagger - \frac{1}{2}a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a\right)
$$

C. Cohen-Tannoudji *et al.*, *Atom-Photon Interactions* (John Wiley, 1998)

Microscopic approach

General formalism:



$$H = H_S + H_E + H_{\text{int}}$$

$$H_{\text{int}} = A \otimes E$$

Expansion: $$A = \sum_{\omega} A(\omega)$$

$$A(\omega) = \sum_{\epsilon' - \epsilon = \omega} \Pi(\epsilon) A \Pi(\epsilon')$$

H.P. Breuer e F. Petruccione, *The theory of open quantum systems* (Oxford, 2002)

Master Equation

$$\dot{\rho}(t) = -i[H_S, \rho(t)]$$

Unitary evolution

$$+ \sum_{\omega>0} \gamma(\omega) \left[A(\omega)\rho(t)A^\dagger(\omega) - \frac{1}{2}\left\{ A^\dagger(\omega)A(\omega), \rho(t) \right\} \right]$$

$$+ \sum_{\omega>0} \gamma(-\omega) \left[A^\dagger(\omega)\rho(t)A(\omega) - \frac{1}{2}\left\{ A(\omega)A^\dagger(\omega), \rho(t) \right\} \right]$$

Nonunitary evolution

Transition rates:

$$\gamma(\omega) = \int_{-\infty}^{+\infty} d\tau \, \mathrm{e}^{i\omega\tau} \left\langle E^\dagger(\tau)E(0) \right\rangle$$

Depending on the environment spectrum

Approximations

1. Weak system-environment coupling

2. Markov (no memory effects)

$$t \gg \tau$$

3. Rotating wave approximation (RWA)

$$\gamma_{\mathrm{max}} \ll \Delta\omega_{\mathrm{min}}$$

Master Equation for the JC

Model

$$\begin{cases} H_S = H_{JC}, \quad H_E = \sum_k \omega_k b_k^\dagger b_k \\ \\ H_{\text{int}} = \left(a + a^\dagger\right) \sum_k g_k \left(b_k + b_k^\dagger\right) \end{cases}$$

Jump operators

$$A \ \left(E_{N',l} - E_{N,m}\right) = \left|E_{N,m}\right\rangle \left\langle E_{N,m}\right| \left(a + a^\dagger\right) \left|E_{N',l}\right\rangle \left\langle E_{N',l}\right|$$

$$= \ \frac{1}{2}\delta_{N,N'-1}\left(\sqrt{N+1} + lm\sqrt{N}\right) \left|E_{N,m}\right\rangle \left\langle E_{N+1,l}\right|$$

$N \geq 1$

$l, m = \pm 1$

$$A\left(E_{1,\pm} - E_0\right) = \frac{1}{\sqrt{2}} \left|E_0\right\rangle \left\langle E_{1,\pm}\right|$$

Jumps occur between dressed states

Master Equation for the JC

$$\begin{aligned}
\dot{\rho} &= -i\left[H_{JC},\rho\right] + \sum_{l=\pm 1} \frac{\gamma\left(E_{1,l}-E_0\right)}{2}\left(\left|E_0\right\rangle\left\langle E_{1,l}\right|\rho\left|E_{1,l}\right\rangle\left\langle E_0\right| - \frac{1}{2}\left\{\left|E_{1,l}\right\rangle\left\langle E_{1,l}\right|,\rho\right\}\right) \\
&+ \sum_{l,m=\pm;N=1,+\infty} \frac{\gamma\left(E_{N+1,l}-E_{N,m}\right)}{4}\left(\sqrt{N+1}+lm\sqrt{N}\right)^2\left(\left|E_{N,m}\right\rangle\left\langle E_{N+1,l}\right|\rho\left|E_{N+1,l}\right\rangle\left\langle E_{N,m}\right|\right. \\
&- \left.\frac{1}{2}\left\{\left|E_{N+1,l}\right\rangle\left\langle E_{N+1,l}\right|,\rho\right\}\right) \\
&+ \sum_{l=\pm 1} \frac{\gamma\left(E_0-E_{1,l}\right)}{2}\left(\left|E_{1,l}\right\rangle\left\langle E_0\right|\rho\left|E_0\right\rangle\left\langle E_{1,l}\right| - \frac{1}{2}\left\{\left|E_0\right\rangle\left\langle E_0\right|,\rho\right\}\right) \\
&+ \sum_{l,m=\pm;N=1,+\infty} \frac{\gamma\left(E_{N,m}-E_{N+1,l}\right)}{4}\left(\sqrt{N+1}+lm\sqrt{N}\right)^2\left(\left|E_{N+1,l}\right\rangle\left\langle E_{N,m}\right|\rho\left|E_{N,m}\right\rangle\left\langle E_{N+1,l}\right|\right. \\
&- \left.\frac{1}{2}\left\{\left|E_{N,m}\right\rangle\left\langle E_{N,m}\right|,\rho\right\}\right)
\end{aligned}$$

Kubo-Martin-Schwinger (KMS) condition:

$$\gamma(-\omega) = \exp\left(-\frac{\omega}{k_B T}\right)\gamma(\omega)$$

Validity of RWA

- Free system evolution: minimum Rabi frequency;
- To compare with the maximum decay rate.

$$2\Omega \gg \gamma_{\max}$$

Phenomenological Master Equation:

$$\omega_0 \gg \gamma_{\max}$$

Comparison between the two Master Equations

1. The stationary state at T temperature

2. The phenomenological master equation for weak damping

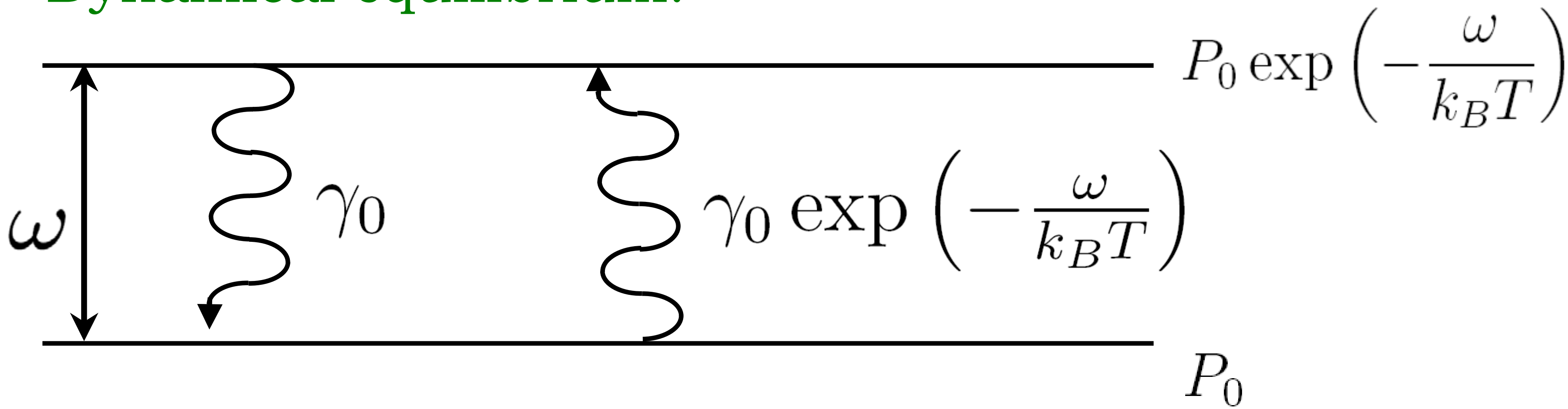
3. Dynamics at T=0 with one initial excitation
   - Rabi oscillations
   - Decay of a Bell state

The stationary state

KMS

$$\rho_{\mathrm{th}} = \frac{\exp\left(-\frac{H_{JC}}{k_B T}\right)}{Tr\left\{\exp\left(-\frac{H_{JC}}{k_B T}\right)\right\}}$$

Dynamical equilibrium:



H.P. Breuer e F. Petruccione, *The theory of open quantum systems* (Oxford, 2002)

The stationary state

Microscopic Master Equation

$$\rho_{\text{th}} = \frac{\exp\left(-\frac{H_{JC}}{k_B T}\right)}{Tr\left\{\exp\left(-\frac{H_{JC}}{k_B T}\right)\right\}}$$

Phenomenological Master Equation

$$\rho_{\text{th}}^{\text{ph}} = \frac{\exp\left(-\frac{\frac{\omega_0}{2}\sigma_z + \omega_0\, a^\dagger a}{k_B T}\right)}{Tr\left\{\exp\left(-\frac{\frac{\omega_0}{2}\sigma_z + \omega_0\, a^\dagger a}{k_B T}\right)\right\}}$$

In the phenomenological master equation the interaction energy is neglected

R.R. Puri, *Mathematical methods of quantum optics* (Spriger, 2001)

The dressed state approximation

Phenomenological master equation for weak enough damping:

$$\gamma \ll \Omega/(2N^{3/2})$$

- Interaction picture with respect to H_{JC}
- Use of the basis $\{|E_{N,\pm}\rangle\}$
- Dropping of the terms oscillating at frequencies $n\Omega$

R.R. Puri e G.S. Agarwal, PRA **33**, 3610 (1986); PRA **35**, 3433 (1987);

J. Gea-Banacloche, PRA **47**, 2221 (1993).

The dressed state approximation

Same set equations for the matrix elements

$$\langle E_{N,+}|\,\dot{\rho}\,|E_{N,+}\rangle \;,\; \langle E_{N,-}|\,\dot{\rho}\,|E_{N,-}\rangle \text{ e } \langle E_{N,\pm}|\,\dot{\rho}\,|E_{N,\mp}\rangle$$

also starting from the microscopic master equation

⇓

Explanation of the success of the phenomenological master equation for the description of experiments

Assumption: flat spectrum for the environment

Dynamics at T=0 with 1 initial excitation

T=0, the microscopic master equation reduces to:

$$\dot{\rho} = -i\left[H_{JC}, \rho\right]$$
$$+\gamma\left(\omega_0+\Omega\right)\left(\tfrac{1}{2}\left|E_0\right\rangle\left\langle E_{1,+}\right|\rho\left|E_{1,+}\right\rangle\left\langle E_0\right| - \tfrac{1}{4}\left\{\left|E_{1,+}\right\rangle\left\langle E_{1,+}\right|, \rho\right\}\right)$$
$$+\gamma\left(\omega_0-\Omega\right)\left(\tfrac{1}{2}\left|E_0\right\rangle\left\langle E_{1,-}\right|\rho\left|E_{1,-}\right\rangle\left\langle E_0\right| - \tfrac{1}{4}\left\{\left|E_{1,-}\right\rangle\left\langle E_{1,-}\right|, \rho\right\}\right)$$

to compare with:

$$\dot{\rho} = -i\left[H_{JC}, \rho\right] + \gamma\left(a\rho a^\dagger - \frac{1}{2}a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a\right)$$

H.J. Briegel e B.G. Englert, PRA **47**, 3311 (1993)

Rabi oscillations

Initial state $|0, e\rangle$

The system oscillates between $|0, e\rangle$ and $|1, g\rangle$

Consequence of losses: damping of the oscillations

Monitoring of the population of the ground state $|0, g\rangle$

Decay rate: $\gamma\left(\omega_0 - \Omega\right) = \gamma\left(\omega_0 + \Omega\right) = \gamma = 2\Omega/10$

RWA on the microscopic model is valid, while the dressed state approximation on the phenomenological model is not

Rabi oscillations

Monitoring of the population of the ground state $|0,g\rangle$

Decay rate: $\gamma\left(\omega_0-\Omega\right)=\gamma\left(\omega_0+\Omega\right)=\gamma=2\Omega/10$

1. Microscopic master equation

$$\langle 0,g|\,\rho_{\mathrm{r}}(t)\,|0,g\rangle=1-\mathrm{e}^{-\frac{\gamma}{2}t}$$

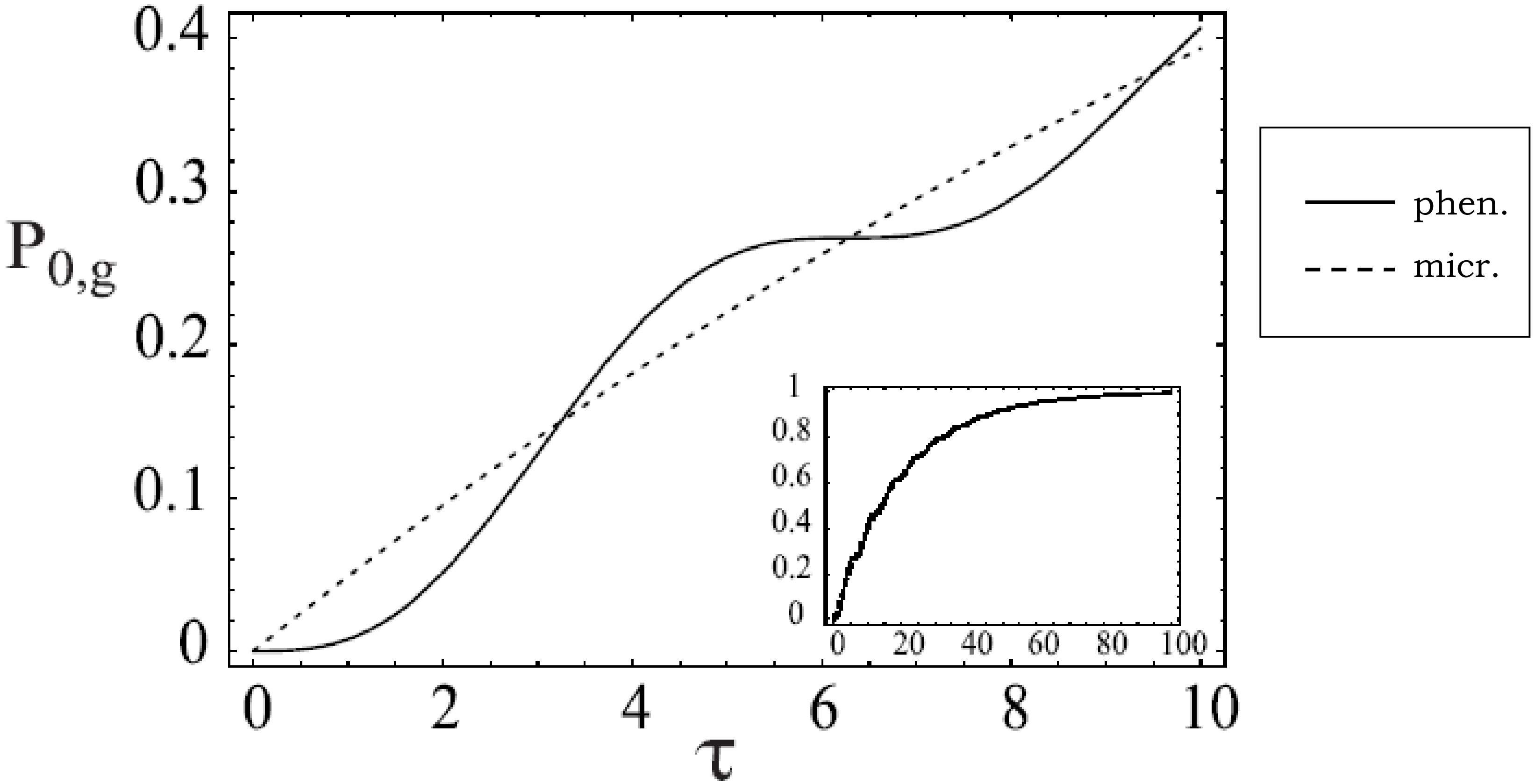
2. Phenomenological master equation

First order in γ/Ω

$$\langle 0,g|\,\rho_{\mathrm{r}}^{\mathrm{ph}}(t)\,|0,g\rangle=1-\frac{16\Omega^2}{16\Omega^2-\gamma^2}\mathrm{e}^{-\frac{\gamma}{2}t}$$

$$+\frac{\gamma^2+\gamma\sqrt{\gamma^2-16\Omega^2}}{2(16\Omega^2-\gamma^2)}\mathrm{e}^{\frac{-\gamma+\sqrt{\gamma^2-16\Omega^2}}{2}t}+\frac{\gamma^2-\gamma\sqrt{\gamma^2-16\Omega^2}}{2(16\Omega^2-\gamma^2)}\mathrm{e}^{\frac{-\gamma-\sqrt{\gamma^2-16\Omega^2}}{2}t}$$

Rabi oscillations



$$P_{0,g} = \langle 0, g | \rho(t) | 0, g \rangle \text{ vs. } \tau = 2\Omega t$$

Rabi oscillations

Different decay mechanisms

1. Phenomenological master equation

   only the mode directly decays and then one has the decay only if the population of $|1,g\rangle$ is nonzero: signature of the Rabi oscillations

2. Microscopic master equation

   both $|E_{1,+}\rangle$ and $|E_{1,-}\rangle$ directly decay with the same rate and so do $|1,g\rangle$ and $|0,e\rangle$: no oscillations

Atomic ground state

Population: $P_g(\rho) = \langle 0, g|\,\rho\,|0, g\rangle + \langle 1, g|\,\rho\,|1, g\rangle$

1. Microscopic master equation

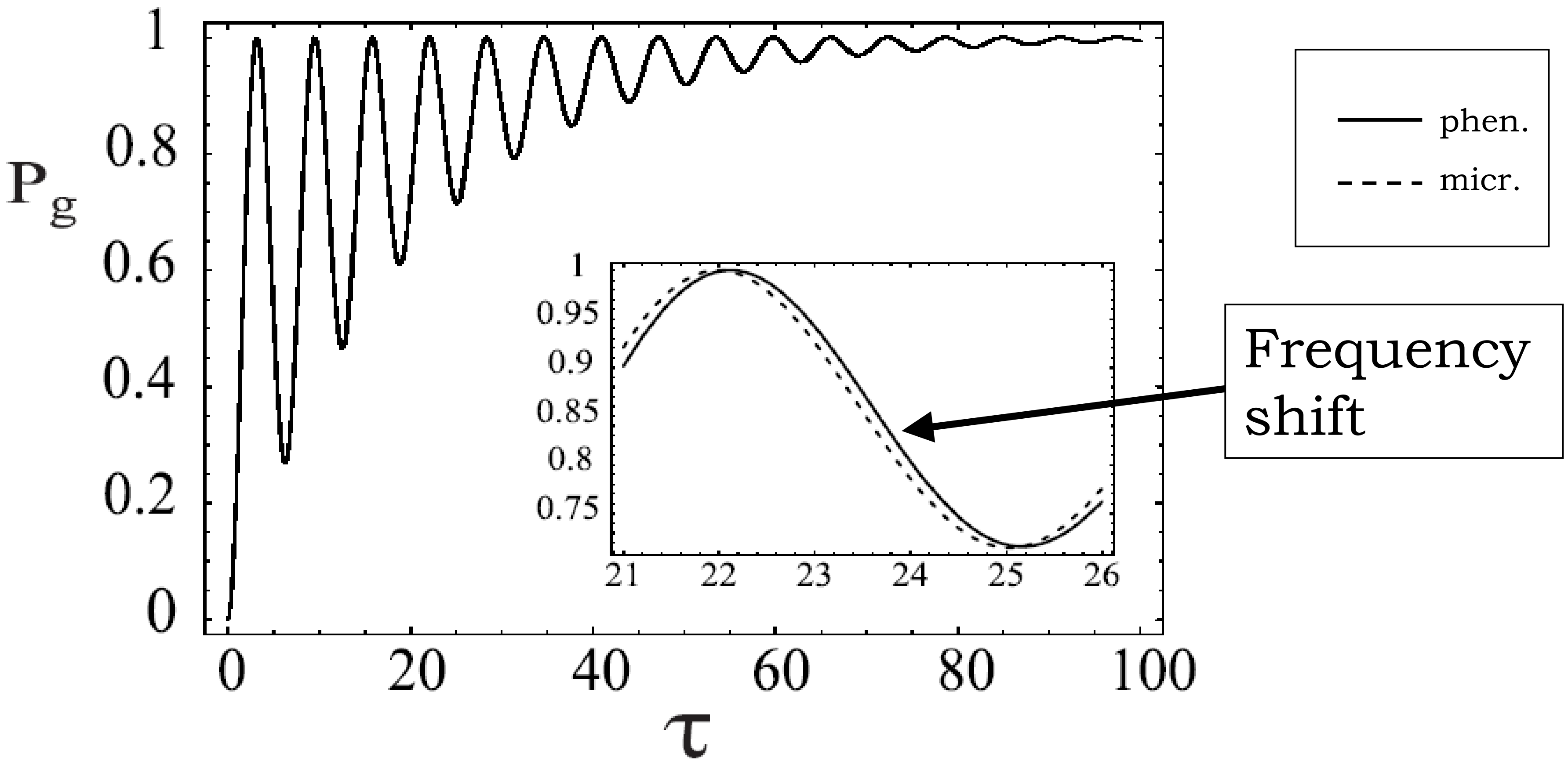
$$P_g\left(\rho_{\mathrm{r}}(t)\right) = 1 - \tfrac{1}{2}\mathrm{e}^{-\frac{\gamma}{2}t} - \tfrac{1}{4}\left(\mathrm{e}^{\left(2i\Omega-\frac{\gamma}{2}\right)t} - \mathrm{e}^{\left(-2i\Omega-\frac{\gamma}{2}\right)t}\right)$$

2. Phenomenological master equation

$$P_g\left(\rho_{\mathrm{r}}^{\mathrm{ph}}(t)\right) = 1 - \tfrac{8\Omega^2}{16\Omega^2-\gamma^2}\mathrm{e}^{-\frac{\gamma}{2}t}$$

$$+\tfrac{2\gamma^2+2\gamma\sqrt{\gamma^2-16\Omega^2}-16\Omega^2}{4(16\Omega^2-\gamma^2)}\mathrm{e}^{\frac{-\gamma+\sqrt{\gamma^2-16\Omega^2}}{2}t} + \tfrac{2\gamma^2-2\gamma\sqrt{\gamma^2-16\Omega^2}-16\Omega^2}{4(16\Omega^2-\gamma^2)}\mathrm{e}^{\frac{-\gamma-\sqrt{\gamma^2-16\Omega^2}}{2}t}$$

Atomic ground state



$$P_g \text{ vs. } \tau = 2\Omega t$$

Decay of a Bell state

Initial state: $|E_{1,+}\rangle = \frac{1}{\sqrt{2}}\left(|0,e\rangle + |1,g\rangle\right)$

Population of the atomic ground state:

1. Microscopic master equation

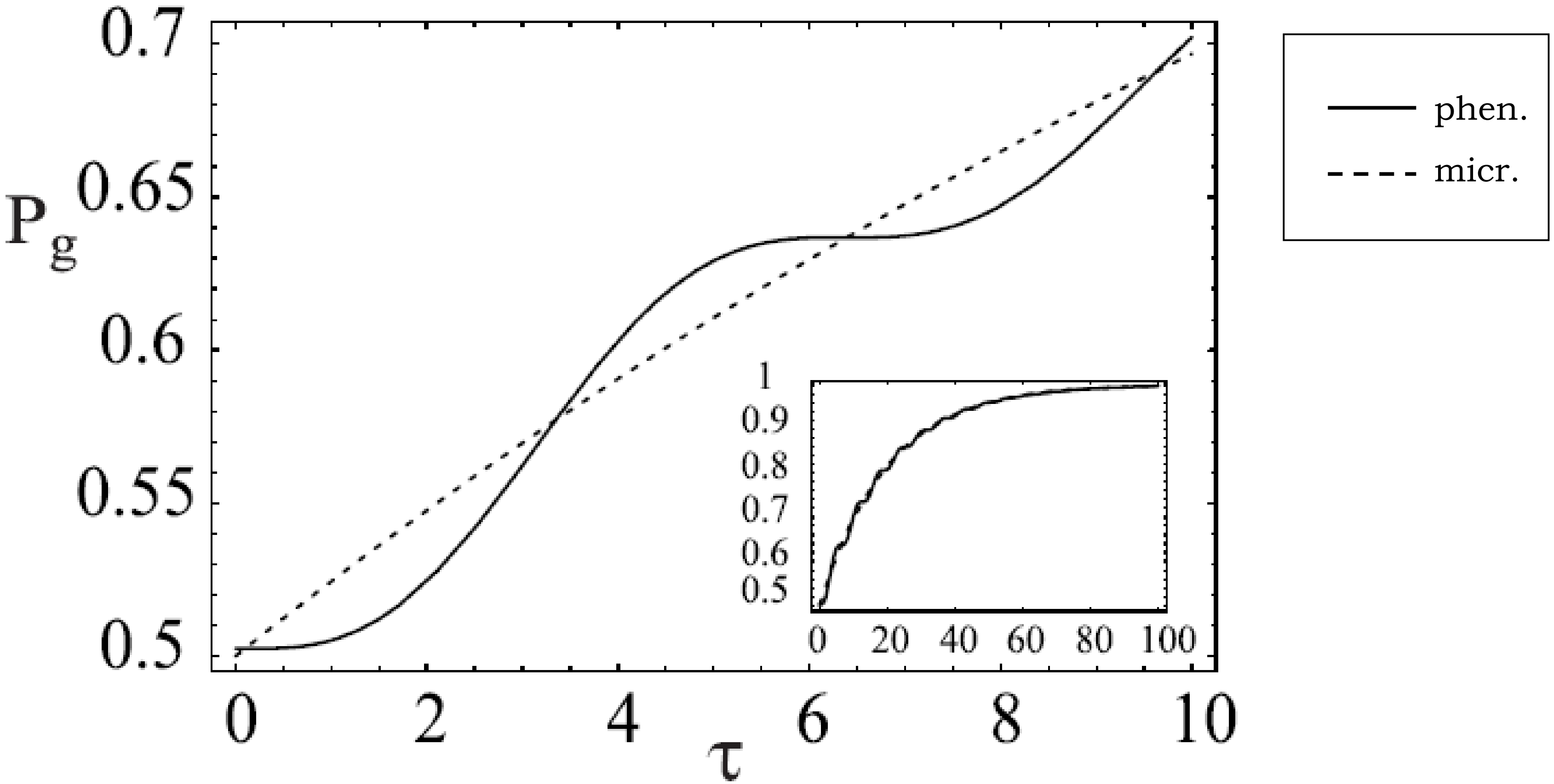
$$P_g\left(\rho_{\mathrm{b}}(t)\right) = 1 - \frac{1}{2}\mathrm{e}^{-\frac{\gamma}{2}t}$$

2. Phenomenological master equation

$$P_g\left(\rho_{\mathrm{b}}^{\mathrm{ph}}(t)\right) = 1 - \frac{8\Omega^2}{16\Omega^2-\gamma^2}\mathrm{e}^{-\frac{\gamma}{2}t}$$

$$+\frac{\gamma^2+\gamma\sqrt{\gamma^2-16\Omega^2}}{4(16\Omega^2-\gamma^2)}\mathrm{e}^{\frac{-\gamma+\sqrt{\gamma^2-16\Omega^2}}{2}t} + \frac{\gamma^2-\gamma\sqrt{\gamma^2-16\Omega^2}}{4(16\Omega^2-\gamma^2)}\mathrm{e}^{\frac{-\gamma-\sqrt{\gamma^2-16\Omega^2}}{2}t}$$

Decay of a Bell state



$$P_g \text{ vs. } \tau = 2\Omega t$$

Conclusions

- Microscopic derivation of a master equation for the Jaynes-Cummings model
- Comparison with the predictions of the phenomenological master equation: stationary state, weak damping, dynamics at T=0
- Different physical mechanisms in the dissipative dynamics

Perspectives

- Dynamics for non flat spectrum
- Non markovian theory
- Beyond the RWA