Quantum Interferometry

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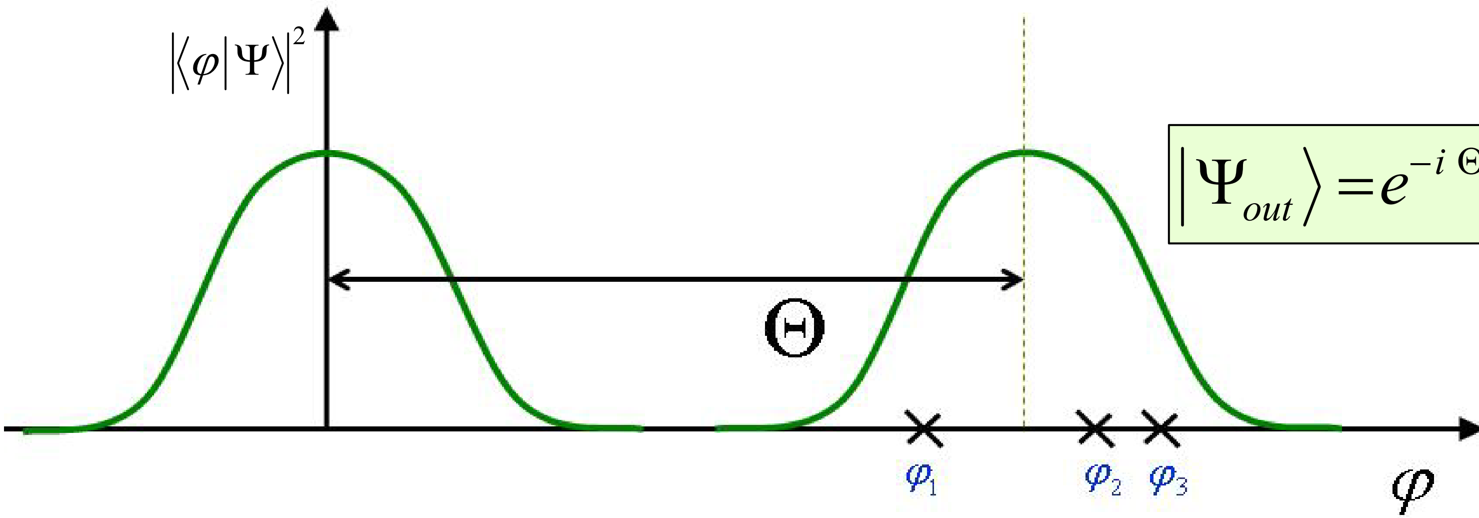


$$\left|\Psi_{out}\right\rangle = e^{-i\,\Theta\,\hat{N}}\left|\Psi_{inp}\right\rangle$$

How precisely the unknown phase shift Θ can be infered from the results of p *independent* measurements $\{\varphi_1, \varphi_2, \ldots \varphi_p\}$?

Standard Quantum vs. Heisenberg limit

Initial state

$$|\Psi_{inp}\rangle \approx |1\rangle|0\rangle + |0\rangle|1\rangle$$

Schroedinger cat (NOON)

$$|\psi_{inp}\rangle \approx |N,0\rangle + |0,N\rangle$$

Phase shift

$$|\psi(\Theta)\rangle = e^{-i\hat{n}\Theta}|\psi_{inp}\rangle = e^{i\Theta}|0\rangle|1\rangle + e^{-i\Theta}|1\rangle|0\rangle$$

$$|\psi_N(\Theta)\rangle = e^{-i\hat{n}\Theta}|\psi_N\rangle = e^{-iN\Theta}|N,0\rangle + e^{iN\Theta}|0,N\rangle$$

Projective measure

$$|\langle\psi_{inp}|\psi(\Theta)\rangle|^2 \approx \cos^2(\Theta)$$

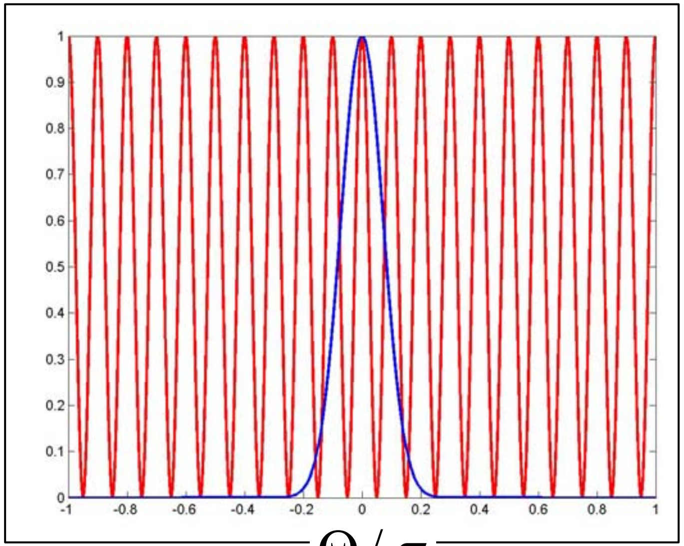
with N uncorreleted particles

$$\cos^{2N}(\Theta) \approx \exp[-\Theta^2/4N]$$

$$\Rightarrow \text{ Orthogonality at } \Theta \approx \frac{1}{\sqrt{N}}$$

$$|\langle\psi_{inp}|\psi(\Theta)\rangle|^2 \approx \cos^2(N\Theta/2)$$

$$\Rightarrow \text{ Orthogonality at } \Theta \approx \frac{1}{N}$$



Θ/π

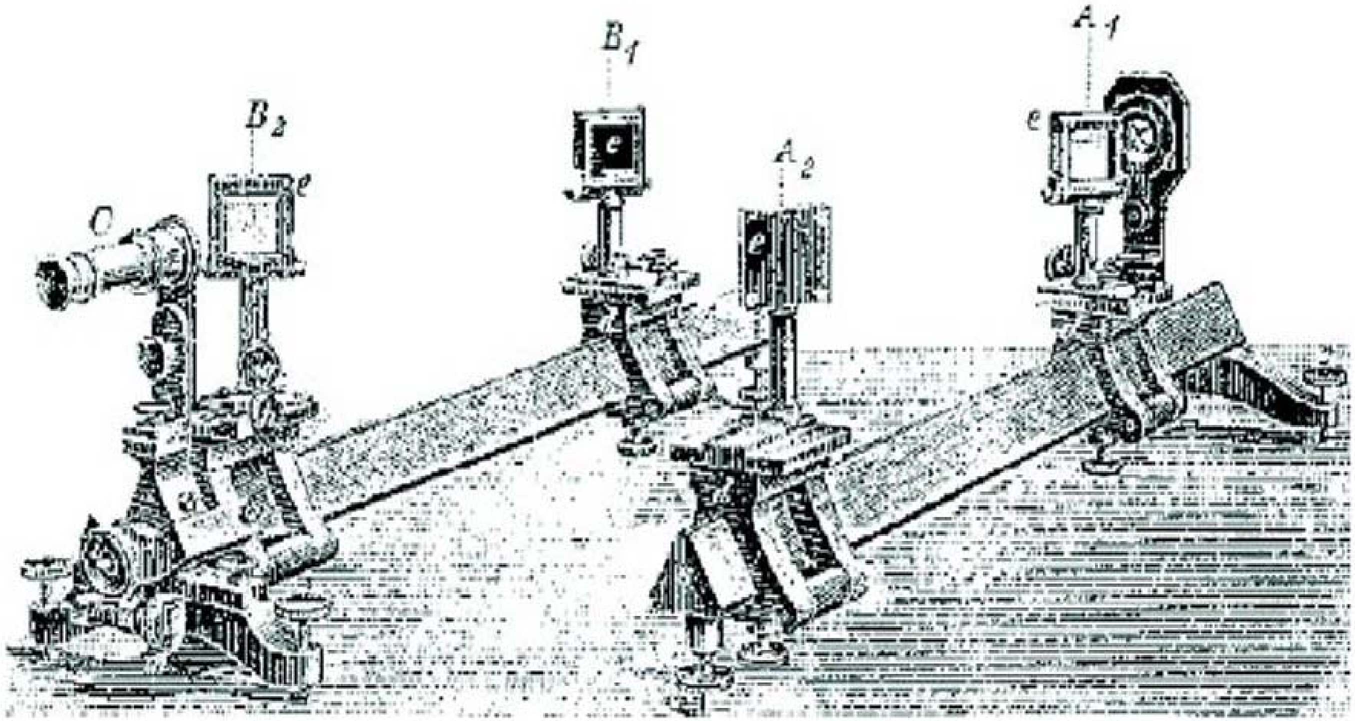
Entanglement (quantum correlations) can provide sensitivity at the Heisenberg limit $\frac{1}{N}$

Content

1. **Standard quantum limit with optical Mach-Zehnder. Exp. @ UCSB**

2. **Sub standard quantum limit sensitivity with trapped ions NOON states. Exp. data @ NIST**

Mach-Zehnder



E. Mach & L. Mach, Wien. Akad. Ber. Klasse 98, 1318 (1889)
L. Zehnder, Zeits. f. Instrumentenk 11, 275 (1891)
L. Mach, Zeits. f. Instrumentenk 12, 89 (1892); ibid. 14, 279 (1894)

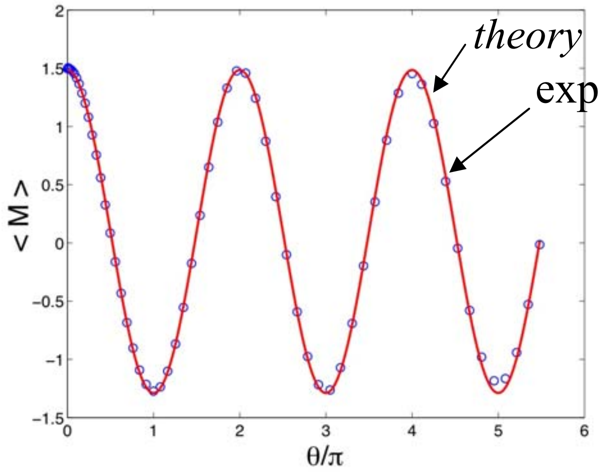
The "classical" Mach-Zehnder

Input state

$$|\Psi_{inp}\rangle = |\alpha\rangle_a |0\rangle_b$$

$$\langle \hat{M} \rangle = \langle \hat{N}_{D1} - \hat{N}_{D2} \rangle = |\alpha|^2 \cos(\Theta)$$

$$\langle \hat{N} \rangle = \langle \hat{N}_{D1} + \hat{N}_{D2} \rangle = |\alpha|^2$$



as estimator, *choose* :

$$\overline{M} \equiv \frac{1}{p} \sum_{i=1}^{p} (N_{D1} - N_{D2}) = |\alpha|^2 \cos(\Theta_{est})$$

The estimated value Θ_{est} of the true phase shift Θ is *defined* as the average of the relative number of particles in p independent measurements.

From error propagation…

$$\Delta\Theta \underset{p>>1}{=} \frac{\Delta \hat{M}}{|\partial \langle \hat{M} \rangle / \partial \Theta|} \frac{1}{\sqrt{p}} = \frac{1}{\sqrt{p|\alpha|^2}} \frac{1}{\sin(\Theta)}$$

Optimal phase sensitivity at $\Theta = \pi / 2$

Is it possible to reach the SQL for <u>any</u> value of the phase shift ?

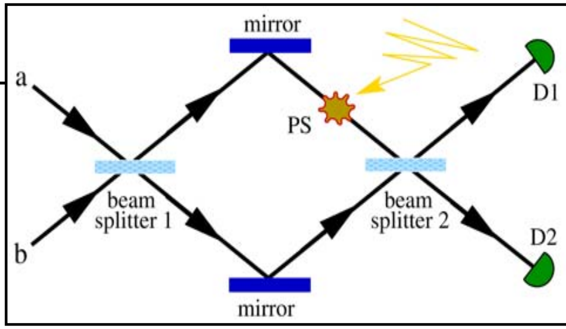
Phase estimation experiment

$\Theta/\pi = 0.69$

$|\alpha|^2 = 1.37$

$|\alpha\rangle_a$

$|0\rangle_b$



Measurements :

$N_{D1}^{(1)}, N_{D2}^{(1)}$

$N_{D1}^{(2)}, N_{D2}^{(2)}$

....

$N_{D1}^{(p)}, N_{D2}^{(p)}$

Phase probability distribution :

$$P(\phi) = \prod_{i=1}^{p} P(\phi \mid N_{D1}^{(i)}, N_{D2}^{(i)})$$

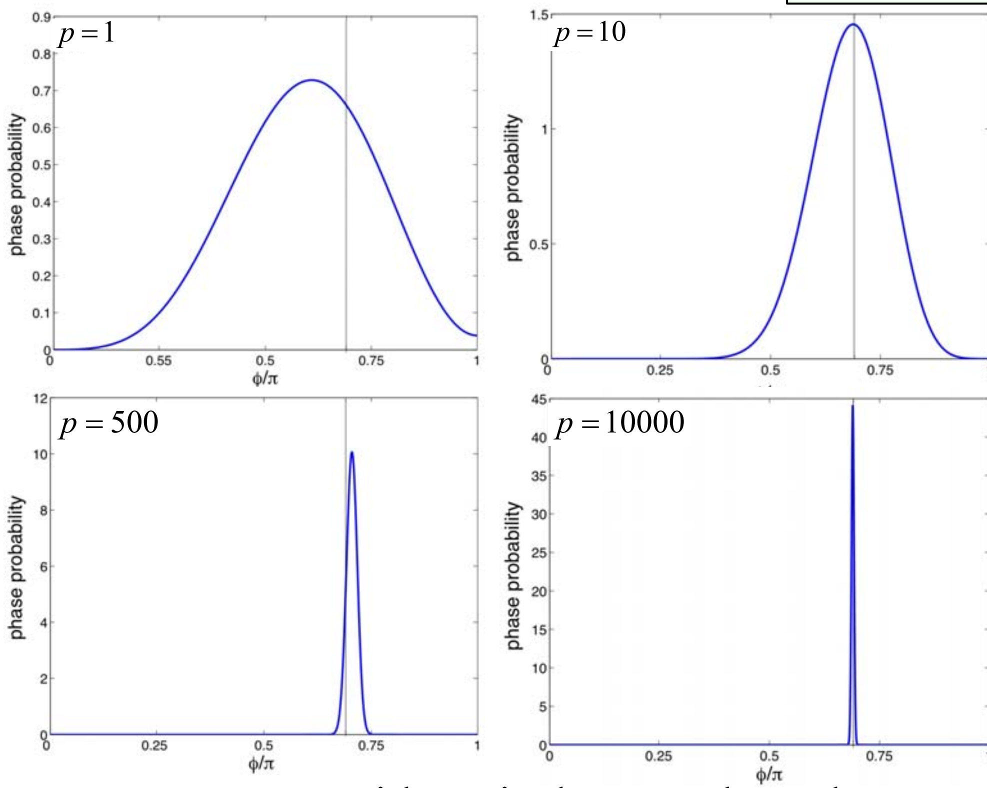
MZ eq. :

$$P(N_1, N_2 \mid \phi) =$$

$$\left| \langle N_1, N_2 | e^{\phi(\hat{a}^+\hat{b} - \hat{a}^+\hat{b})/2} | \alpha, 0 \rangle \right|^2$$

Bayes theorem :

$$P(\phi \mid N_1, N_2) = \frac{P(N_1, N_2 \mid \phi) P(\phi)}{\int_{-\pi}^{+\pi} P(N_1, N_2 | \phi) \, \mathrm{d}\phi}$$



Exp. with optical MZ and number counting photodetectors @ UCSB

L. Pezze', A. Smerzi, G. Khoury, J. Hodelin, D. Bouwmeester, submitted

Advantages of the Bayesian approach:

1) Rigorous analysis without statistical assumptions

2) Possibility to consistently include classical noise and detector efficiency

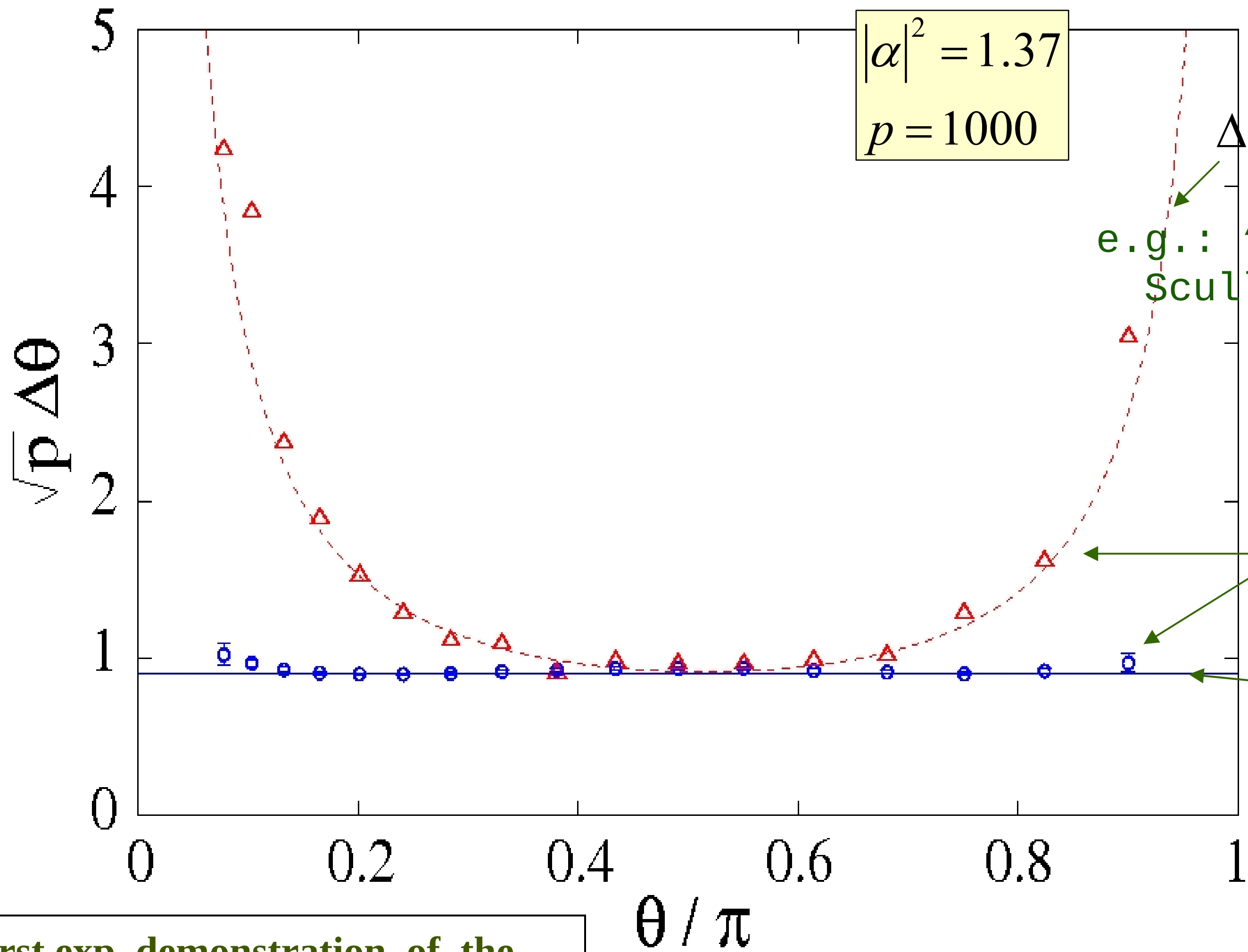
3) Phase estimation with a single measurement. The sensitivity $\Delta\Theta = \frac{1}{\sqrt{N_{D1} + N_{D2}}}$

4) Asymptotically in the number of measurements $\Delta\Theta \underset{p>>1}{=} \frac{1}{\sqrt{p\,|\alpha|^2}}$

cfr. the "classical" theory $\Delta\Theta \underset{p>>1}{=} \frac{1}{\sqrt{p\,|\alpha|^2}} \frac{1}{\sin(\Theta)}$

Experimental test ?

Mach-Zehnder phase sensitivity



$$\Delta\theta = \frac{1}{\sqrt{|\alpha|^2}} \frac{1}{\sin(\theta)}$$

e.g.: “Quantum Optics” Scully and Zubairy

Exp. @ UCSB

$$\Delta\theta = \frac{1}{\sqrt{|\alpha|^2}}$$

First exp. demonstration of the standard quantum limit with photons

Mach-Zehnder phase sensitivity

Cramer - Rao Lower Bound

$$\Delta\theta \geq \frac{1}{\sqrt{F(\theta)}}\frac{1}{\sqrt{p}}$$

$F(\theta)$ is the Fisher information :

$$F(\theta) = \oint d\mu \, \frac{1}{P(\mu \mid \theta)}\left(\frac{\partial P(\mu \mid \theta)}{\partial\theta}\right)^2$$

p : number of independent measurements

"Classical" phase estimation $\Leftrightarrow$ information encoded in $N_1 - N_2$

$$P(N_1 - N_2 \mid \theta) \;\Rightarrow\; F(\theta) = |\alpha|^2 \sin^2(\theta) \;\Rightarrow\; \Delta\theta \geq \frac{1}{\sqrt{p|\alpha|^2}}\frac{1}{\sin(\theta)}$$

Bayesian phase estimation $\Leftrightarrow$ information encoded in N_1 & N_2

$$P(N_1, N_2 \mid \theta) \;\Rightarrow\; F(\theta) = |\alpha|^2 \;\Rightarrow\; \Delta\theta \geq \frac{1}{\sqrt{p|\alpha|^2}}$$

Interferometry with NOON state

Two-mode entangled state (Schroedinger-cat state) recently created experimentally with photons and ions

$$\left|\Psi_N\right\rangle = \frac{\left|N\right\rangle\left|0\right\rangle + \left|0\right\rangle\left|N\right\rangle}{\sqrt{2}}$$

P. Walther, *et al., Nature* **429**, 158 (2004)

M.W. Mitchell, *et al., Nature* **429**, 161 (2004)

Z. Zhao, *et al., Nature* **430**, 54 (2004)

H.S. Einsenberg, *et al., PRL* **94**, 090502 (2005)

Experiments @ NIST

D. Liebfried, *et al., Nature* **422**, 412 (2003)

D. Liebfried, *et al., Science* **304**, 1478 (2004)

D. Liebfried, *et al., Nature* **438**, 639 (2005)

Theoretical analysis

J.J. Bollinger, *et al, PRA* **54**, R4649 (1995)

Gerry & Campos, *PRA* **68**, 025602 (2003)

Applications in lithography and metrology

A.N. Boto, *et al., PRL* **85**, 2733 (2000)

V. Giovannetti, *et al. Science* **306**, 1330 (2004)

NOON states with Beryllium ions

Noon state:

$$\left|N\downarrow\right\rangle \equiv \left|\downarrow\right\rangle_1 \ldots \left|\downarrow\right\rangle_N \quad \text{initial state (gs)}$$

$$\hat{U}_N \equiv e^{-i\frac{\xi\pi}{2}\hat{J}_x} e^{-i\frac{\pi}{2}\hat{J}_x^2} \quad \text{NLBS operator}$$

$$\hat{U}_N\left|N\downarrow\right\rangle = \frac{\left|N\downarrow\right\rangle + i^{\xi+N+1}\left|N\uparrow\right\rangle}{\sqrt{2}}$$

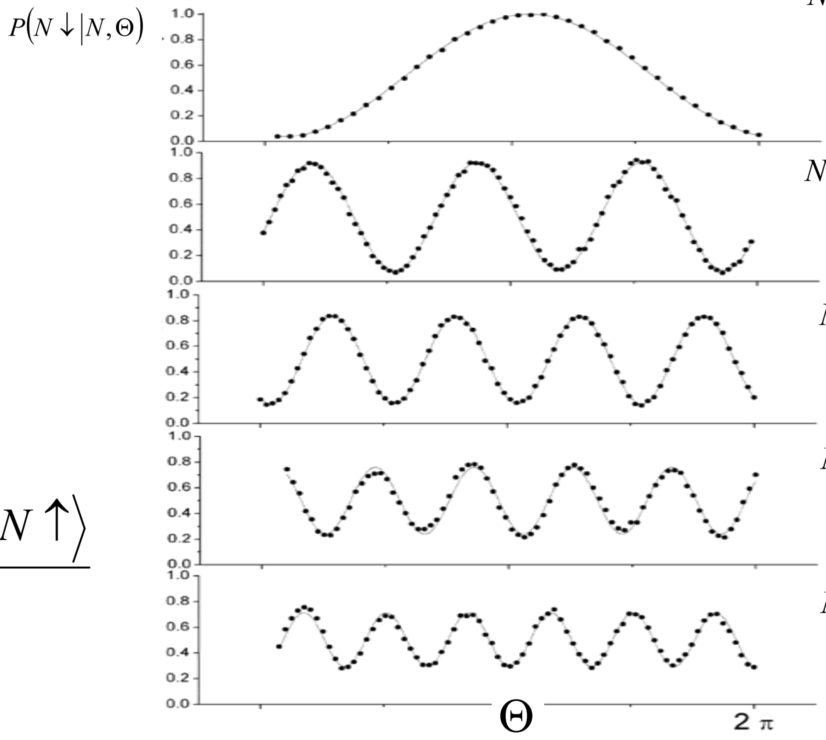
Phase shift:

$$e^{-i\Theta\hat{J}_z}\,\hat{U}_N\left|N\downarrow\right\rangle = \frac{e^{iN\Theta/2}\left|N\downarrow\right\rangle + e^{-iN\Theta/2}\,i^{\xi+N+1}\left|N\uparrow\right\rangle}{\sqrt{2}}$$

Projective measurement:

$$P\left(N\downarrow\middle|N,\Theta\right) = \left|\left\langle N\downarrow\right|\hat{U}_N e^{-i\Theta\hat{J}_z}\hat{U}_N\left|N\downarrow\right\rangle\right|^2 = \cos^2\left(N\Theta/2\right)$$

Experiments @ NIST

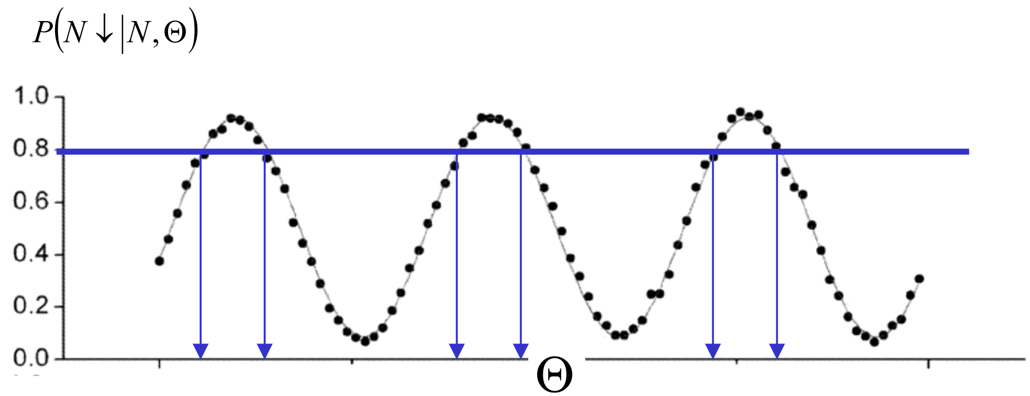


The probability distributions oscillate with period $2\pi/N$

NOON states with Beryllium ions

Example: N=3

The probability distributions oscillate with period $2\pi/N$



There is not a 1 : 1 relation between $P(N\downarrow|N,\Theta)$ and Θ

Estimator : in M measurements calculate the M' times you get the output state $|N\downarrow\rangle$

calculate $P(N\downarrow|N,\Theta)=\frac{M'}{M}$ **= 0.8**

Phase estimation at the Heisenberg limit 1/N with NOON states, requires a priori knowledge at the Heisenberg limit ~1/N

Bayesian analysis

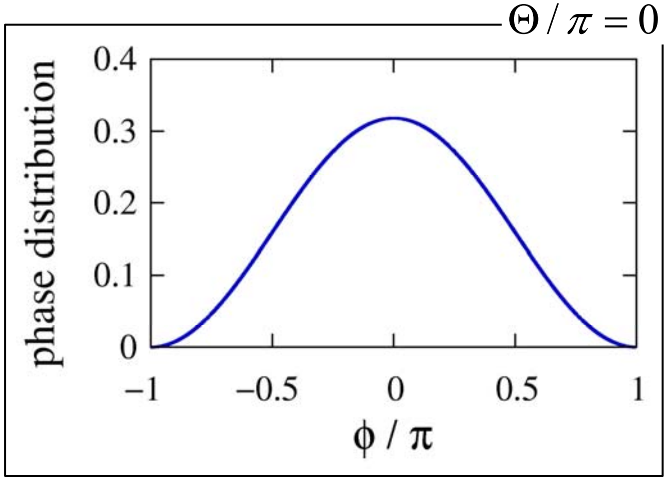
The protocol:

1. Make several independent measurements
2. Carefully choose the number of particles in different measurements

Combining distributions with $N = 1,2,4,\ldots,2^p$

$$P(\phi | N_T, \Theta = 0) \propto \prod_{k=0}^{p} \cos^2\left(\frac{2^k \phi}{2}\right) \approx \exp\left(-\frac{\phi^2}{2}\frac{N_T^2}{6}\right)$$

$$N_T = \sum_{k=0}^{p} 2^k = 2^{p+1} - 1$$



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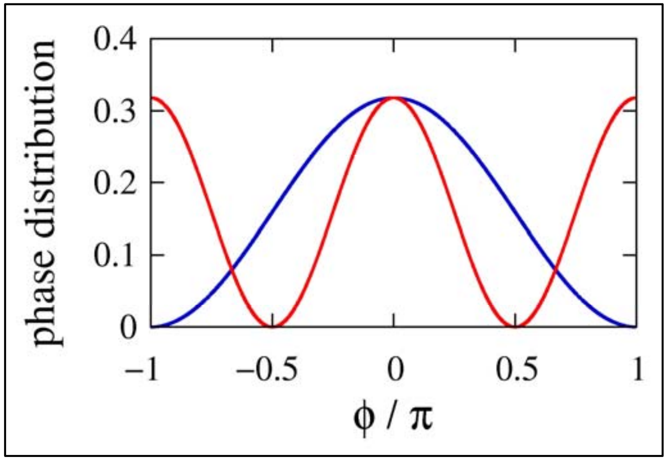

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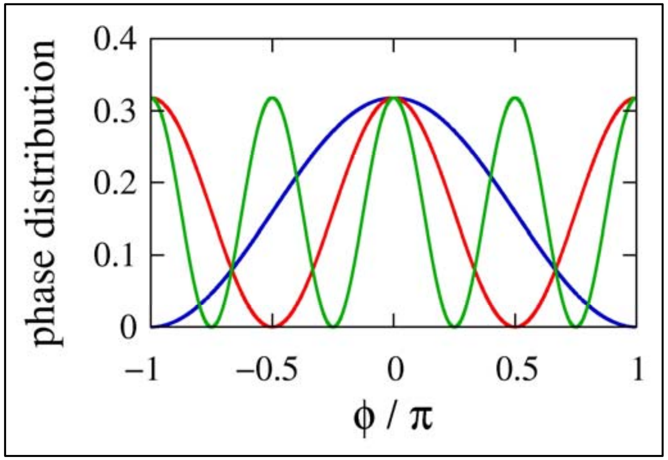

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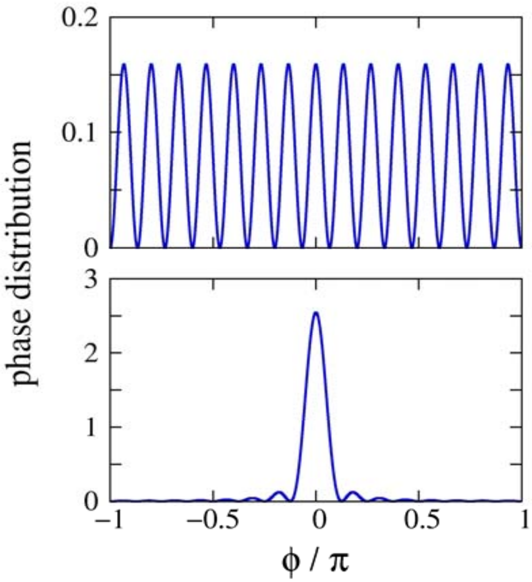
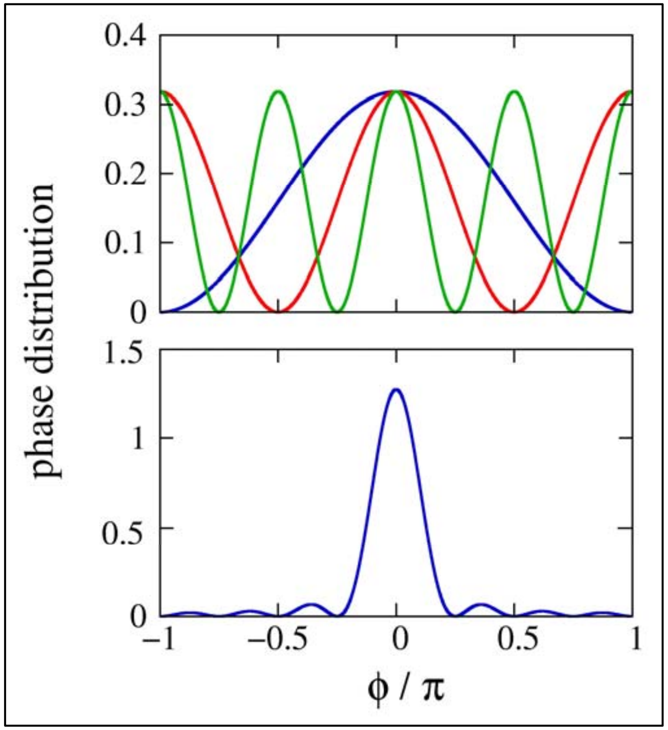


$p = 1$
$N_T = 15$

$p = 4$
$N = 1,2,4,8$
$N_T = 15$



UNBIASED PHASE ESTIMATION with SENSITIVITY AT THE HL

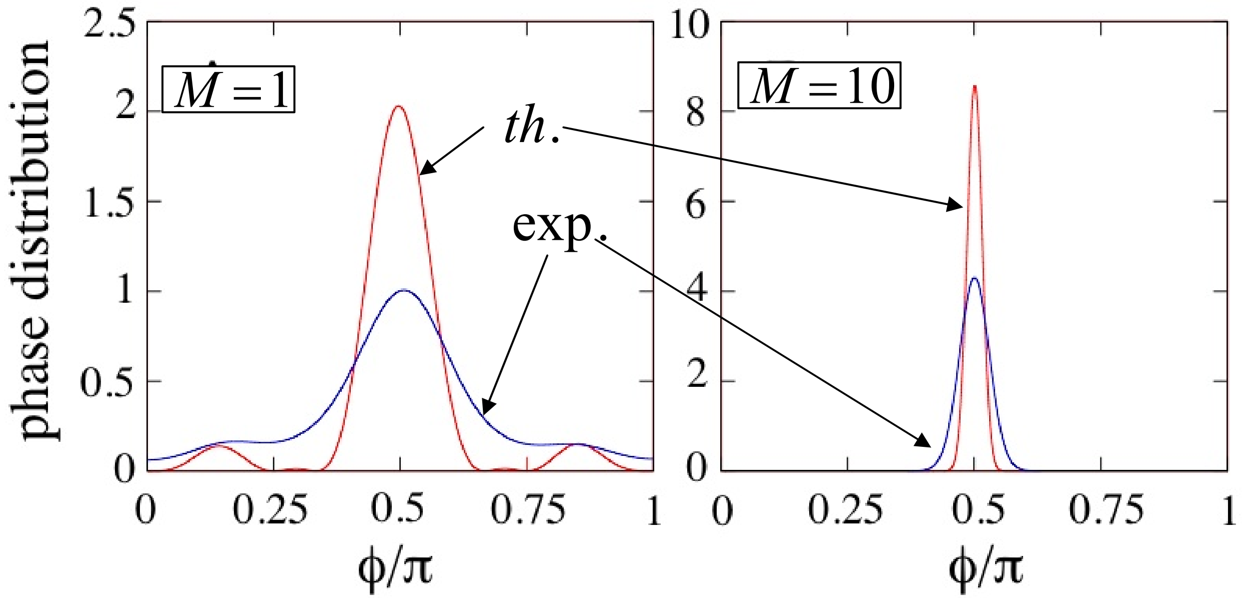
Experimental vs. theoretical (ideal) Phase probability distributions

Theory :

$$\Delta\Phi \approx \frac{1}{N_p}\frac{1}{\sqrt{M}} \approx \frac{1}{12}\frac{1}{\sqrt{N_{tot}}}$$

•) the $\frac{1}{\sqrt{M}}$ term arises from the M *independent* measurements

•) the $\frac{1}{N_p}$ term arises from the Schroedinger cat enhancement in *each* measurement



Data from Exp. @ NIST
Nature **438**, 639 (2005)
Science **304**, 1476 (2004)
Nature **422**, 412 (2003)

First demonstration of sub shot-noise with atoms

L. Pezze' & A. Smerzi, submitted

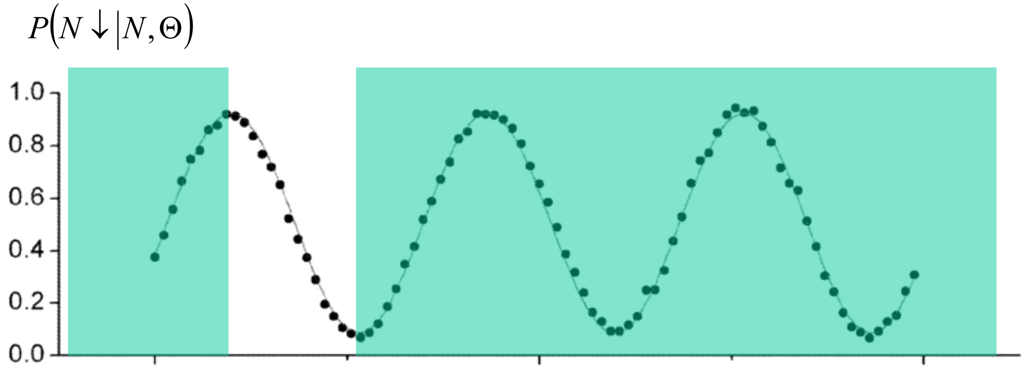
Summary

1. **Standard Quantum limit with Mach-Zehnder**
2. **Heisenberg limit with Scrhroedinger-cat states**
3. **Bayesian phase estimation theory vs. experiments**

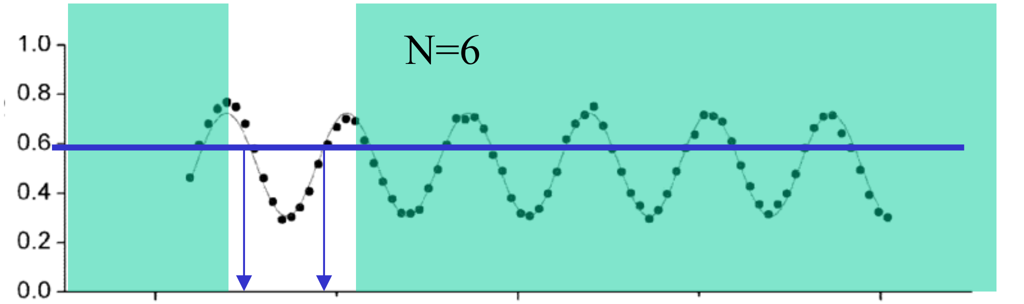
NOON states with Beryllium ions

Example: N=3

Introduce a prior knowledge of the order of $\approx \pi/\mathrm{N}\ (\pi/3)$



To increase the phase sensitivity increase the number of particles in the NOON state



Phase estimation at the Heisenberg limit 1/N with NOON states, requires a priori knowledge at the Heisenberg limit

Experimental gain with respect to the standard quantum limit

<u>Bayesian sensitivity with Be ions</u>

Complete a priori ignorance

i) combine the results with $N = 1,2,3,4,5,6$

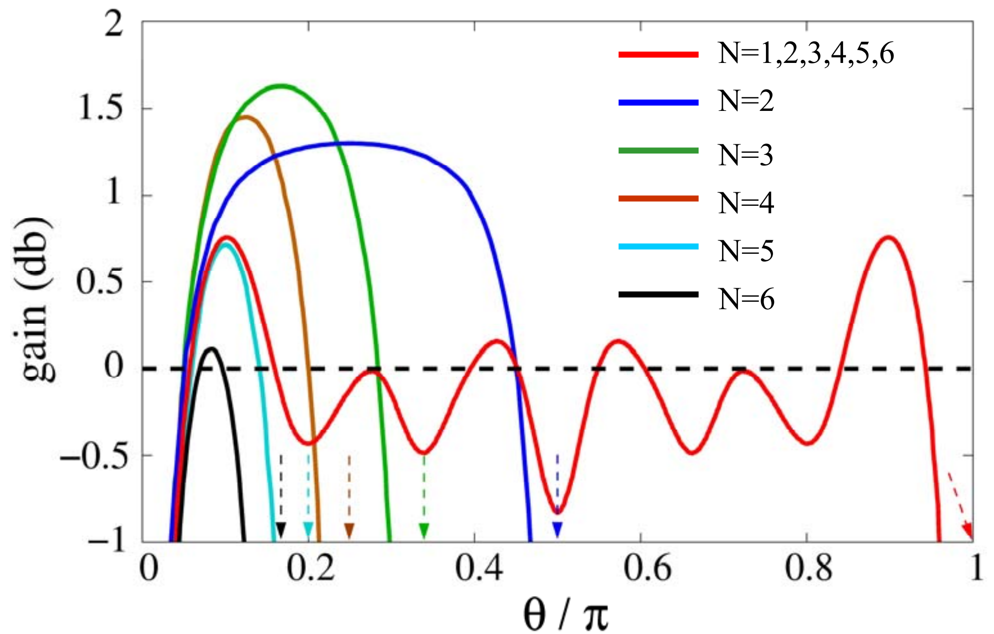
ii) repeat the measurement M times.

iii) multiply Bayesian distributions

For $M >> N$ we have

$$\Delta\Theta \approx \frac{\alpha}{\sum_i N_i \sqrt{M}} \approx \frac{\alpha'}{\sqrt{N_T}}$$

gain $\Rightarrow -10\log_{10}(\alpha)$



The maximum gain is 0.8 db (3.18 db with ideal distributions)

Prior constraint of the phase shift

i) constrain the phase shift in $\left[0, \frac{2\pi}{N}\right]$

ii) combine the results with fixed N

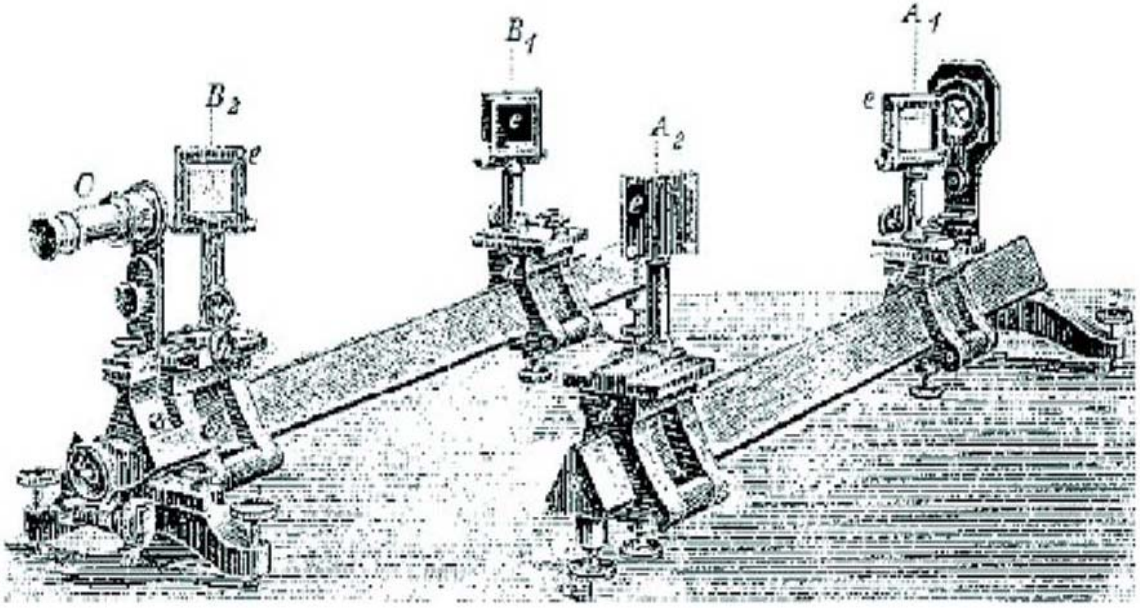
iii) repeat the measurement M times

iv) multiply Bayesian distributions

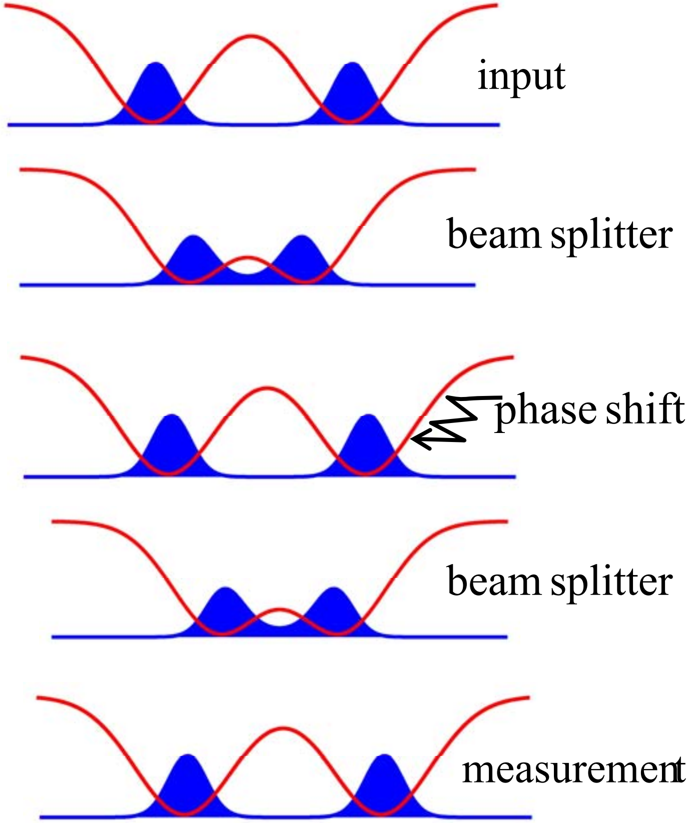
First demonstration of sub shot-noise with atoms

L. Pezze', A. Smerzi, submitted

Mach-Zehnder



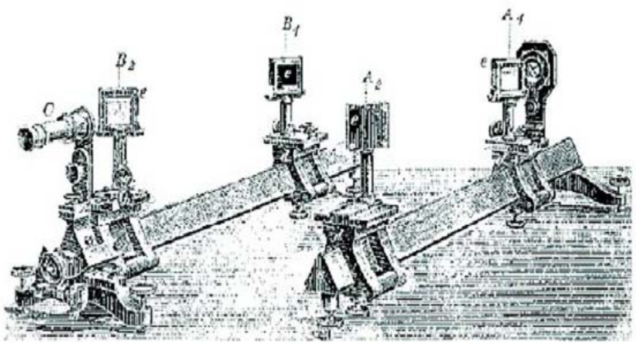
E. Mach & L. Mach, Wien. Akad. Ber. Klasse 98, 1318 (1889)
L. Zehnder, Zeits. f. Instrumentenk 11, 275 (1891)
L. Mach, Zeits. f. Instrumentenk 12, 89 (1892); ibid. 14, 279 (1894)



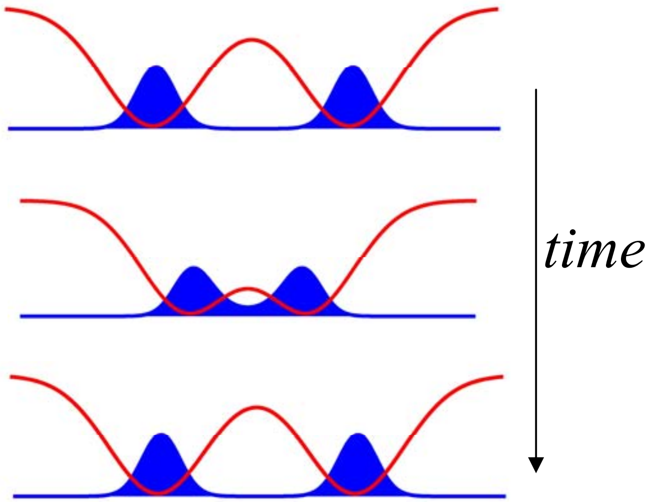
Double-well experiments:
Ketterle & coll. PRL 92, 5 (2004),
Oberthaler & coll. PRL95, 010402 (2005), J.
Schmiedmayer & coll. NaturePhys 1, 57 (2005)

Nonlinear Beam Splitter with BEC

Goal: creation of a Schroedinger cat (NOON) with a number-squeezed state + beam splitter



$$\left|\psi_{bs}\right\rangle = e^{-i\,(\pi/2)\,(\hat{a}^{+}\hat{b}+\hat{b}^{+}\hat{a})/2}\left|\psi_{inp}\right\rangle$$



$$\left|\psi_{bs}\right\rangle = e^{-i\left(E_c\,(\hat{a}^{+}\hat{a}\,\hat{b}^{+}\hat{b})/2+K(t)(\hat{a}^{+}\hat{b}+\hat{b}^{+}\hat{a})\right)t}\left|\psi_{inp}\right\rangle$$

There are different ways to create a BEC beam splitter.
The common problem is to evaluate the role of nonlinearity

L. Pezze', A. Smerzi, G.P. Berman, A.R. Bishop, L.A. Collins, PRA74, 033610 (2006)

Non-interacting limit $\boxed{E_c = 0}$

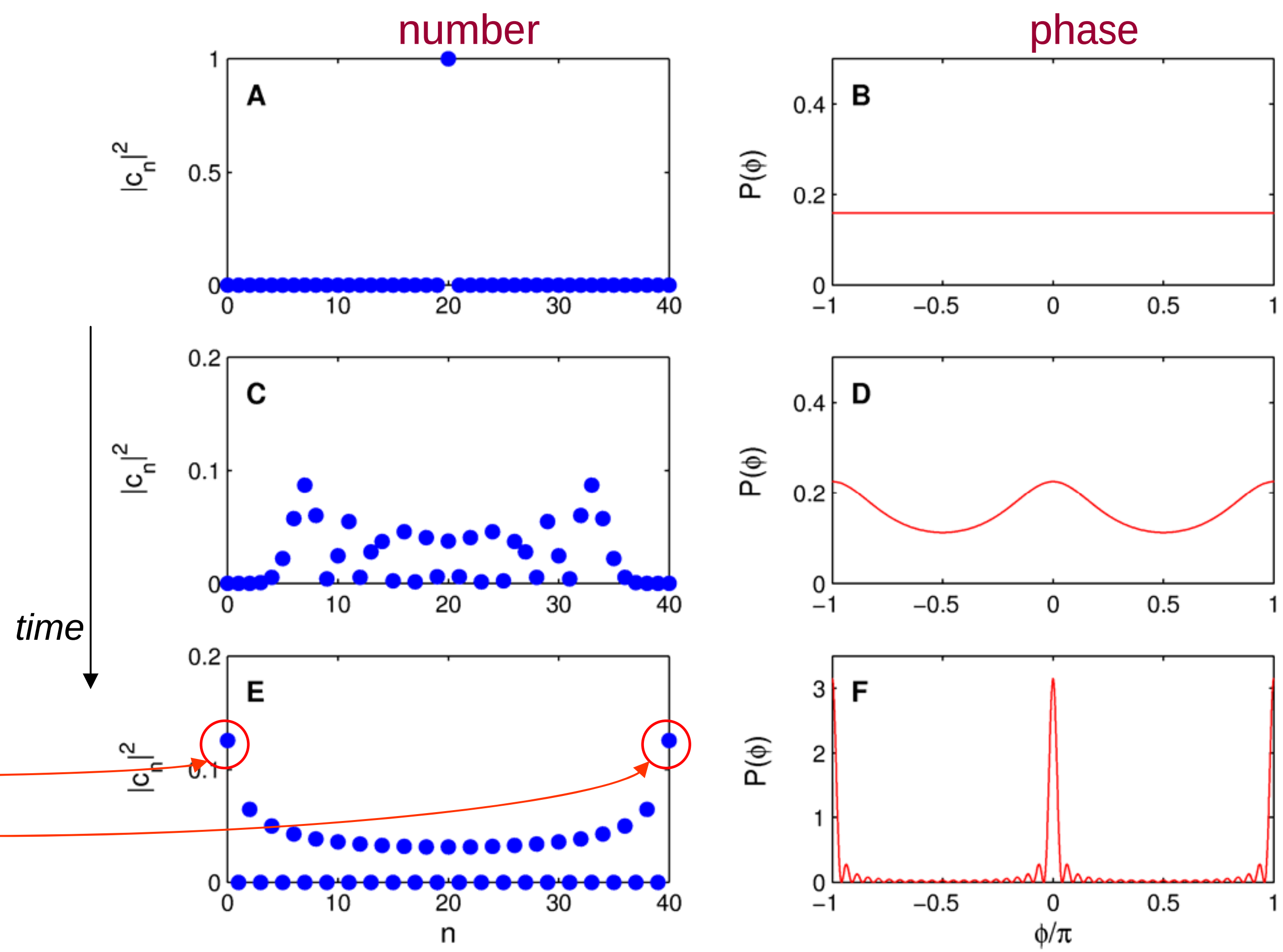
$$\left|\psi_{bs}\right\rangle = e^{-i\left(K(t)\,(\hat{a}^{+}\hat{b}+\hat{b}^{+}\hat{a})\right)t}\left|N/2, N/2\right\rangle = \sum_{n=0}^{N} c_n \left|n\right\rangle\left|N-n\right\rangle$$

M.J. Holland & K. Burnett
PRL71, 1355 (1993)

50/50 beam-splitter

$$\int_0^{t_{bs}} 2K(t)\,dt = \frac{\pi}{2}$$

The linear beam-splitter transforms number-squeezing in phase-squeezing and *viceversa*



$$\left|N,0\right\rangle + \left|0,N\right\rangle$$

Role of interaction $E_c \neq 0$

$$|\psi_{bs}\rangle = e^{-i\left(E_c\,(\hat{a}^+\hat{a}\,\hat{b}^+\hat{b})/2 + K(t)(\hat{a}^+\hat{b} + \hat{b}^+\hat{a})\right)t}\,|\psi_{inp}\rangle$$

Rabi Regime :

$$\frac{K}{E_c} >> N \Rightarrow \Delta\Theta \approx 1/N$$

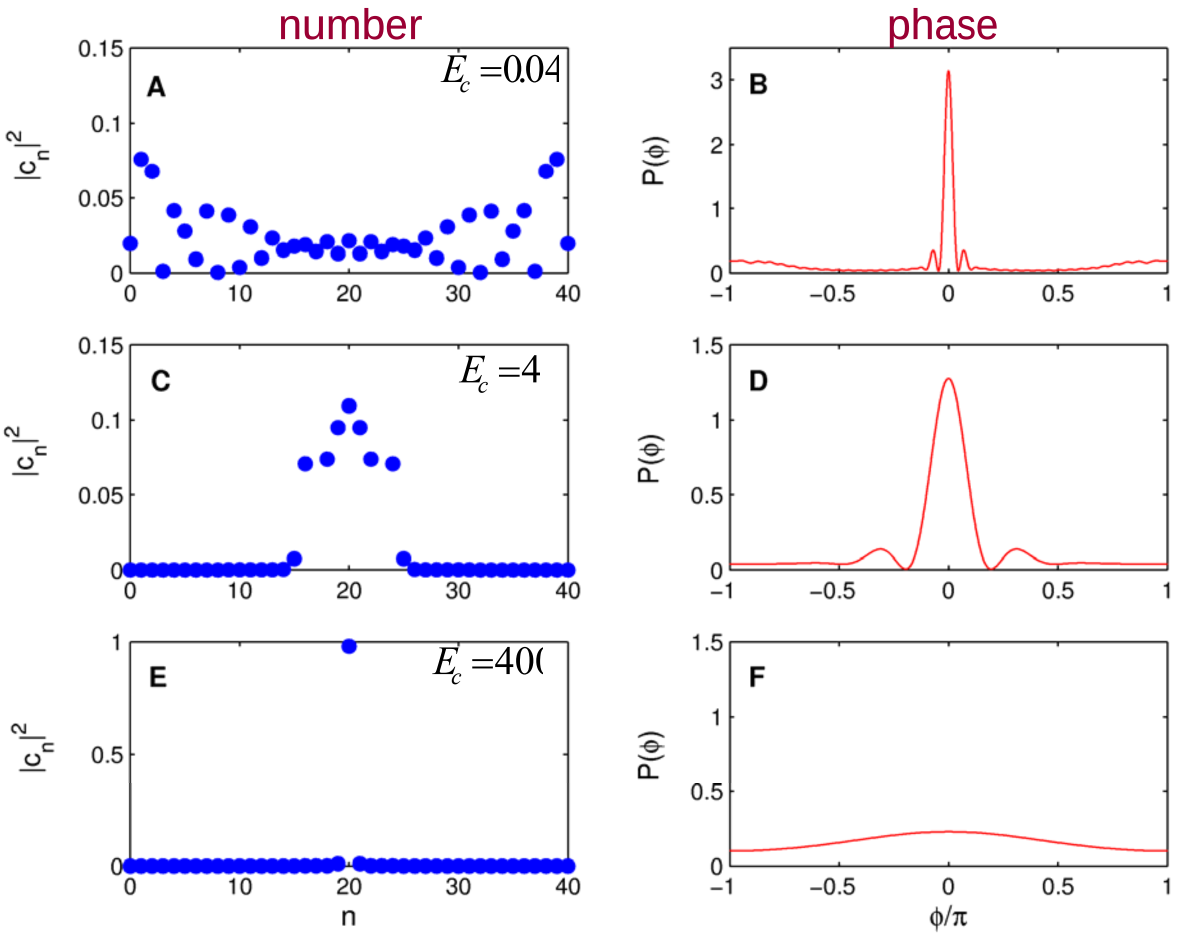
Josephson Regime

$$\frac{1}{N} << \frac{K}{E_c} << N \Rightarrow \Delta\Theta = 1/\sqrt{N}$$

Fock Regime

$$\frac{K}{E_c} << \frac{1}{N} \Rightarrow \Delta\Theta = 2\pi$$

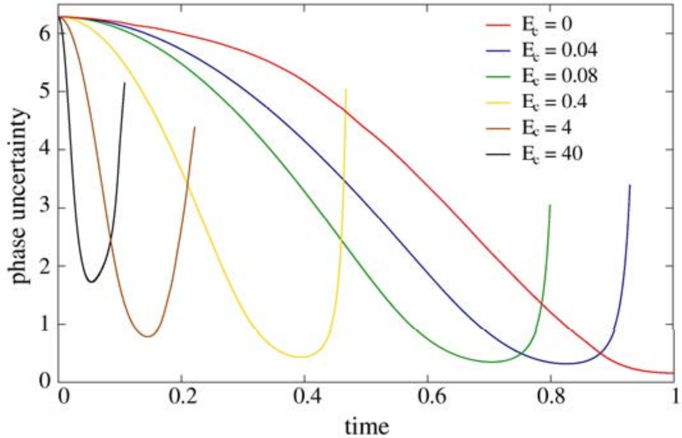
Distributions after the beam-splitter

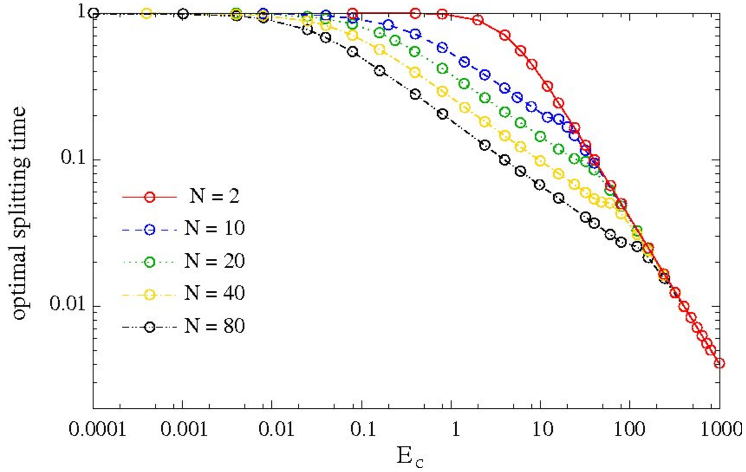


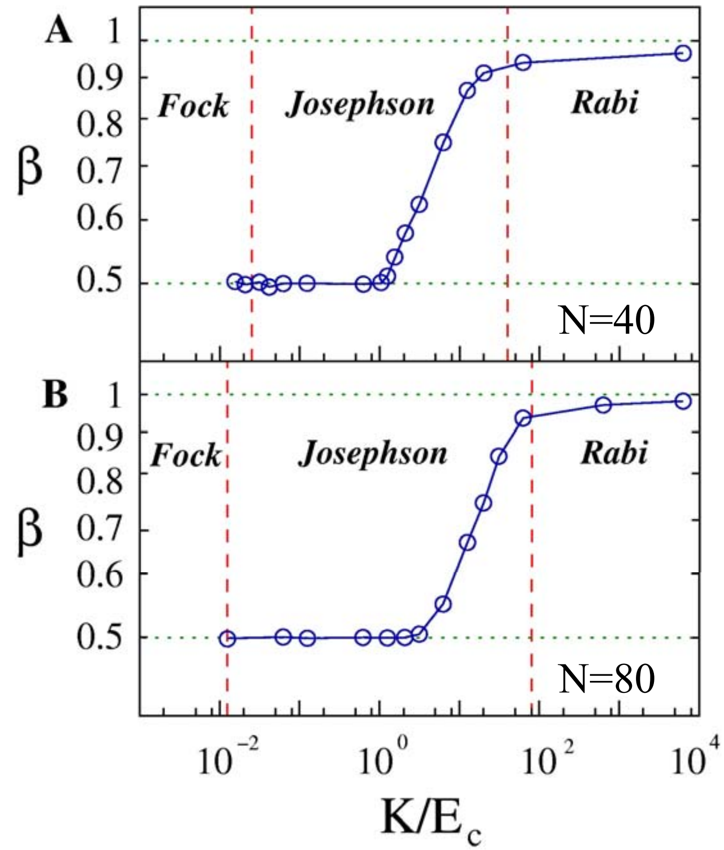
Role of interaction

$$\left|\psi_{bs}\right\rangle = e^{-i\left(E_c\,(\hat{a}^{+}\hat{a}\,\hat{b}^{+}\hat{b})/2+K(t)(\hat{a}^{+}\hat{b}+\hat{b}^{+}\hat{a})\right)t}\left|\psi_{inp}\right\rangle$$

$$\Delta\Theta \approx \frac{1}{N^{\beta}}$$







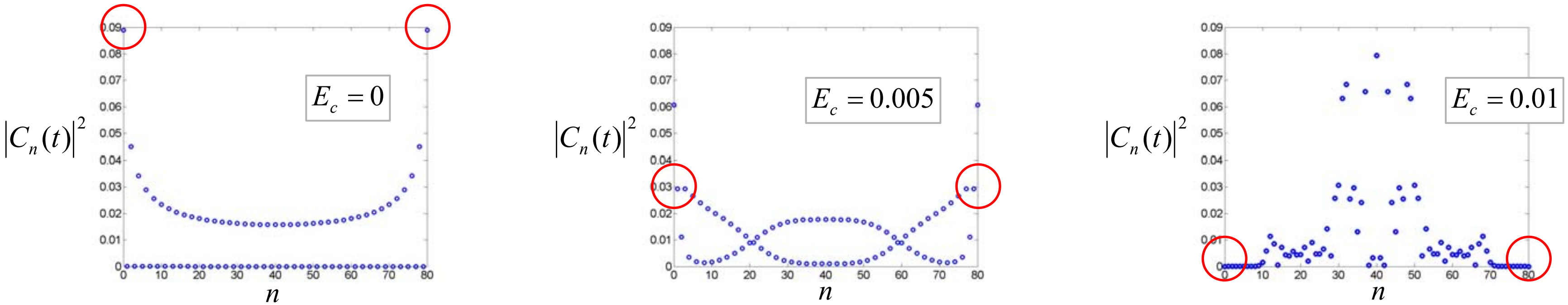
Summary

1. **The Heisenberg limit of phase sensitivity requires the creation of “noon” (maximally entangled) states and an efficient phase estimation protocol.**

2. **Optimal phase sensitivity of the “classical” Mach-Zehnder with Bayesian analysis.**

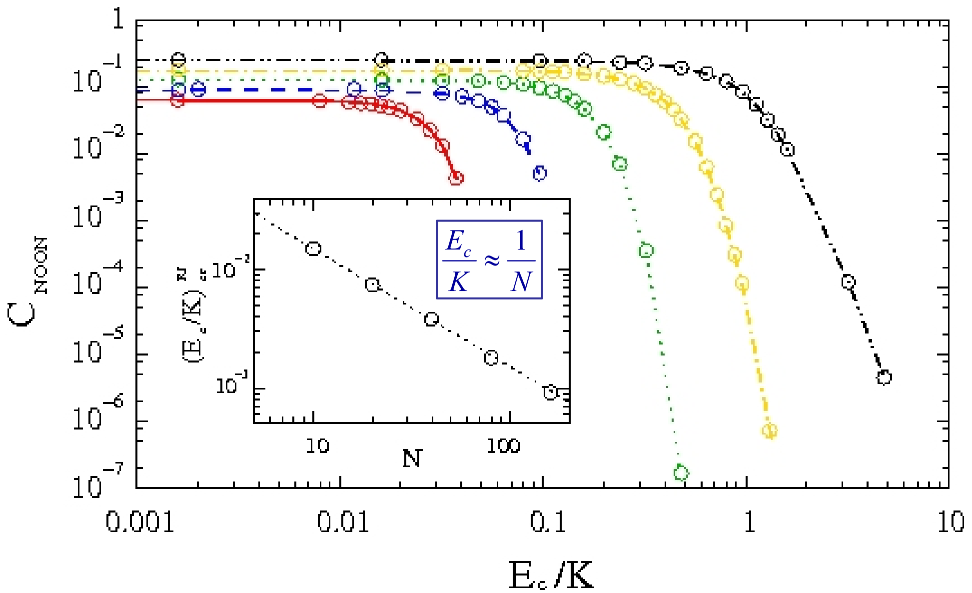
3. **Conditions for the creations of BEC beam-splitters.**

Rabi-Josephson transition



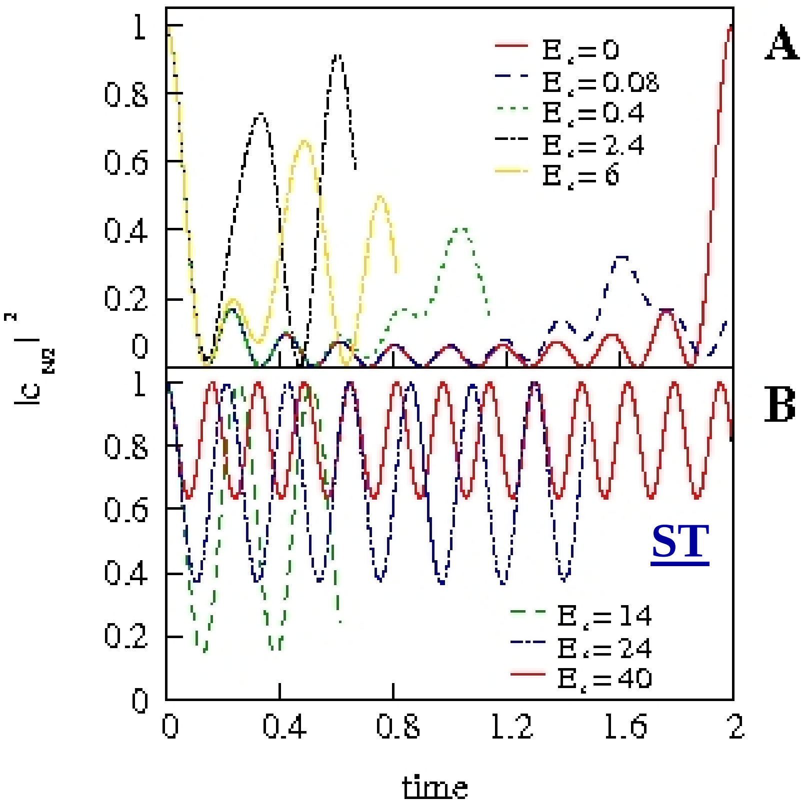
Height of the NOON

$$C_{NOON}(t) \equiv |C_0(t)|^2 + |C_N(t)|^2$$



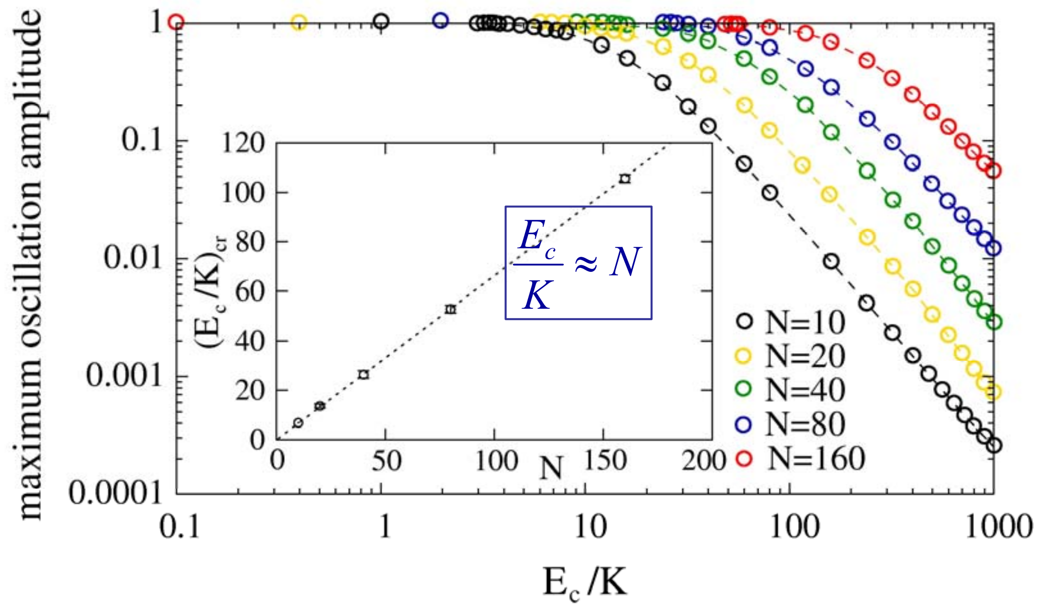
Josephson-Fock Transition

Self-Trapping of particle number fluctuations



initially: $C_n(0) = 1$ for $n = N/2$

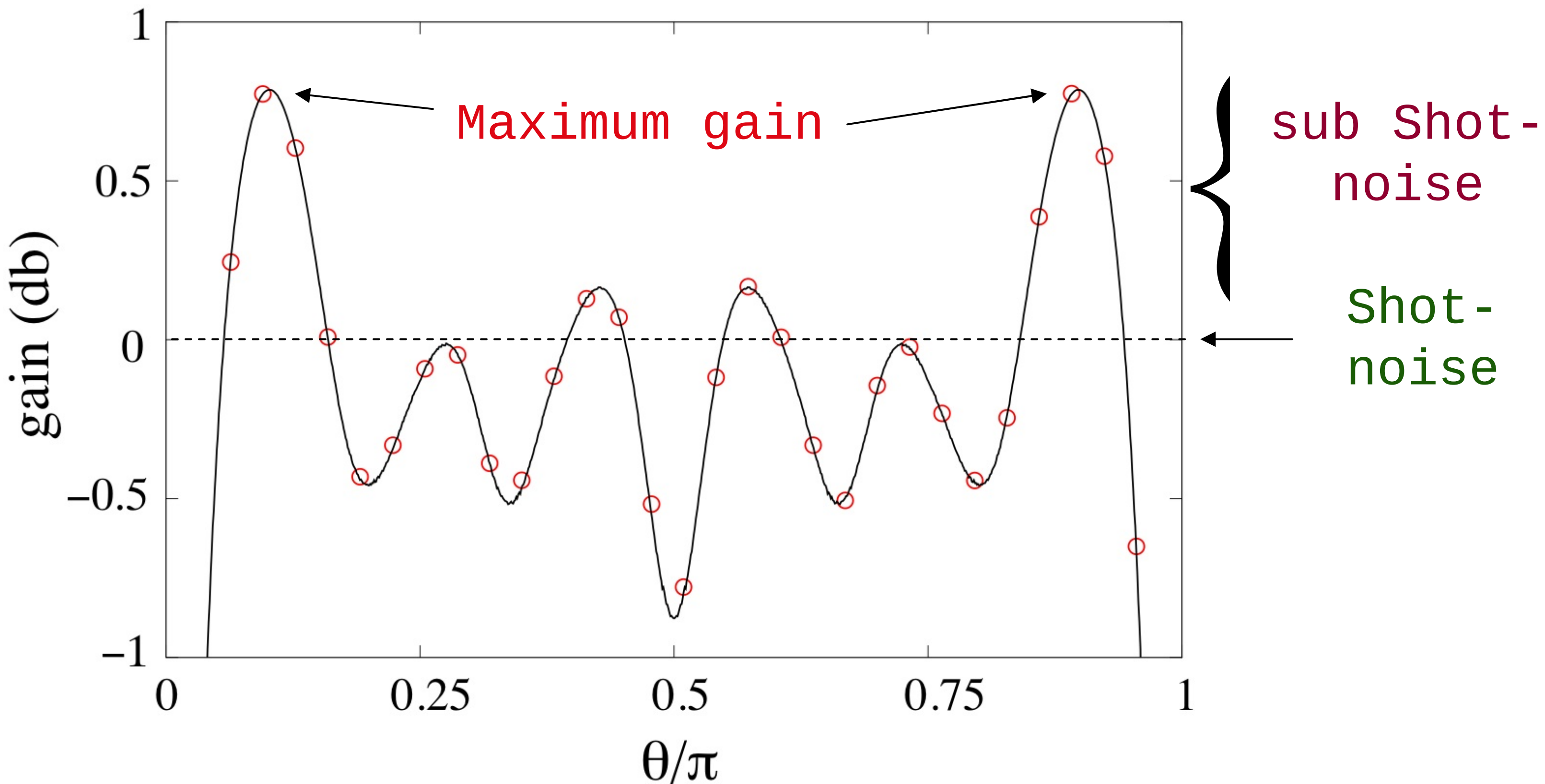
$C_n(0) = 0$ otherwise



Self-Trapping: the initial energy does not redistribute over different modes–

ineffective beam splitter

Experimental gain with respect to the "standard quantum limit" (shot-noise)



$$\text{gain (db)} \equiv 10\,\text{Log}_{10}\left(\frac{\Delta\Theta_{exp}}{\Delta\Theta_{shot-noise}}\right)$$

Fluctuations are due to experimental "classical" noise

Summary

1. I discussed the problem of phase estimation in interferometry giving the recipe for a rigorous analysis (based on the Bayes theorem).

2. I have shown rigorous calculations of MZ sensitivities

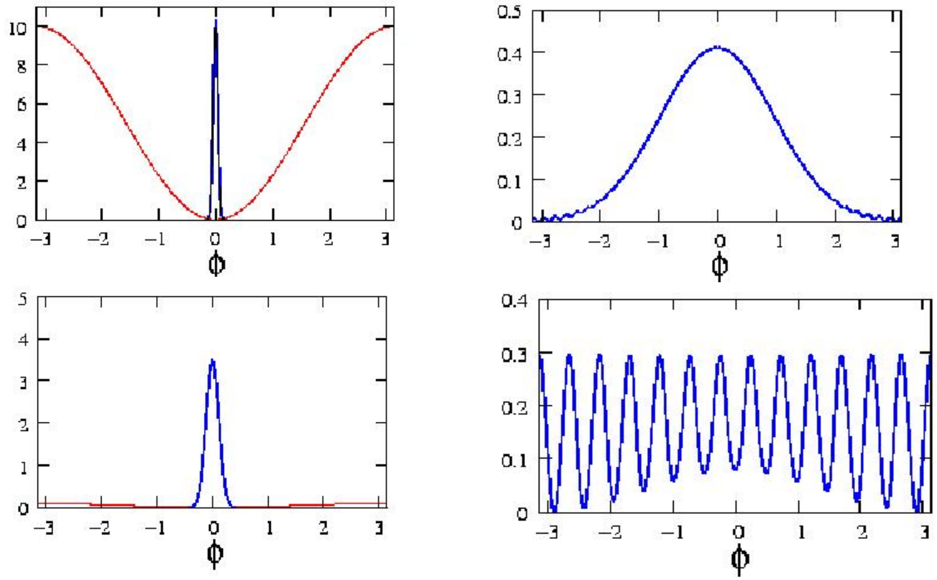
3. I discussed how to reach the ultimate limit on phase sensitivity imposed by Quantum Mechanics

4. I addressed plus and minus of using BEC
   - + non classical states
   - - degrade sensitivity

Creation of number squeezing

$$i\frac{\partial\Psi(\phi,t)}{\partial t}=\left[-\frac{E_c}{2}\frac{\partial^2}{\partial\phi^2}-K\,N\cos(\phi)\right]\Psi(\phi,t)$$

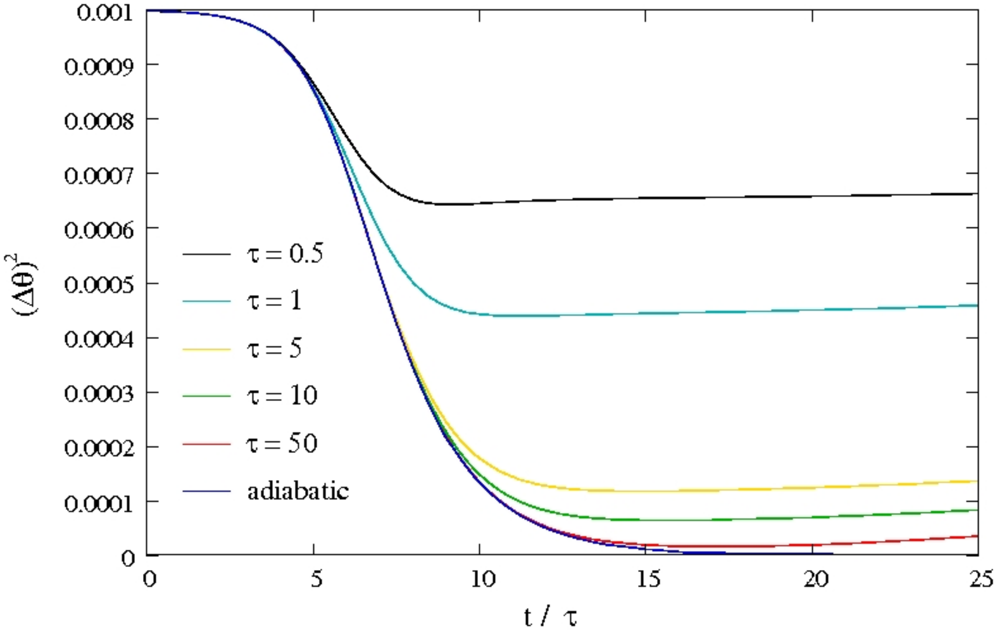
A) Initially the phase amplitude is narrow and centered in the minimum of the effective potential.

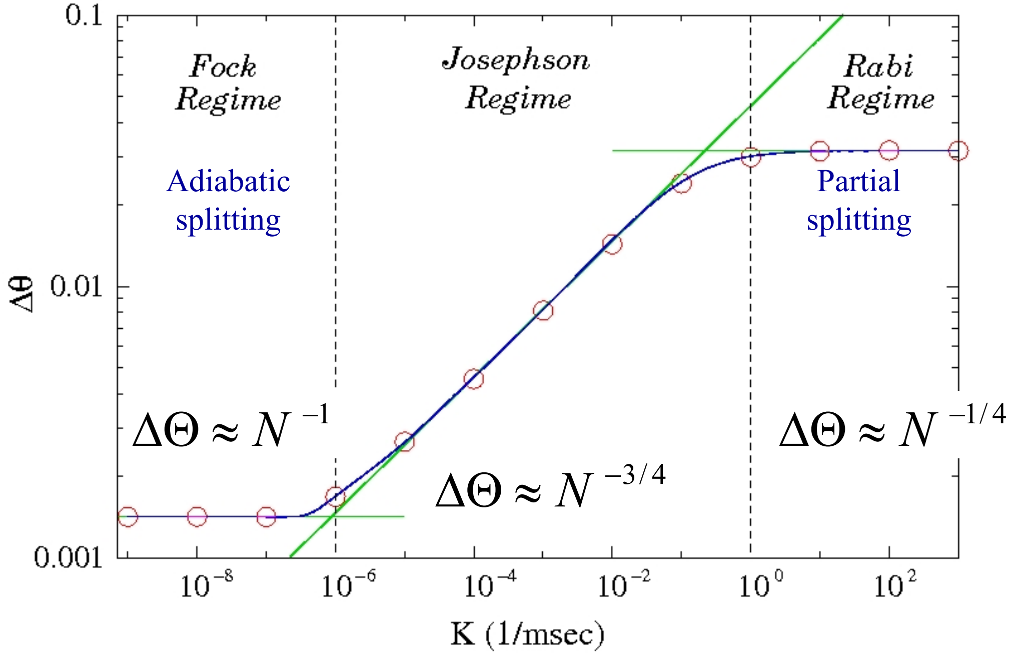


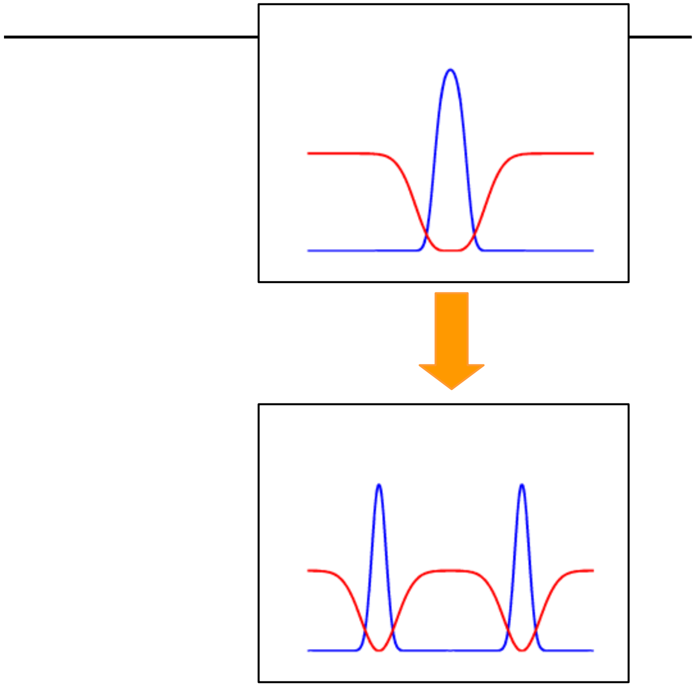
B) The splitting of the potential wells corresponds to the decrease of the effective potential. Initially the process is adiabatic.

C) The relative phase is spread over the whole interval.
We have complete defasing.

Input state





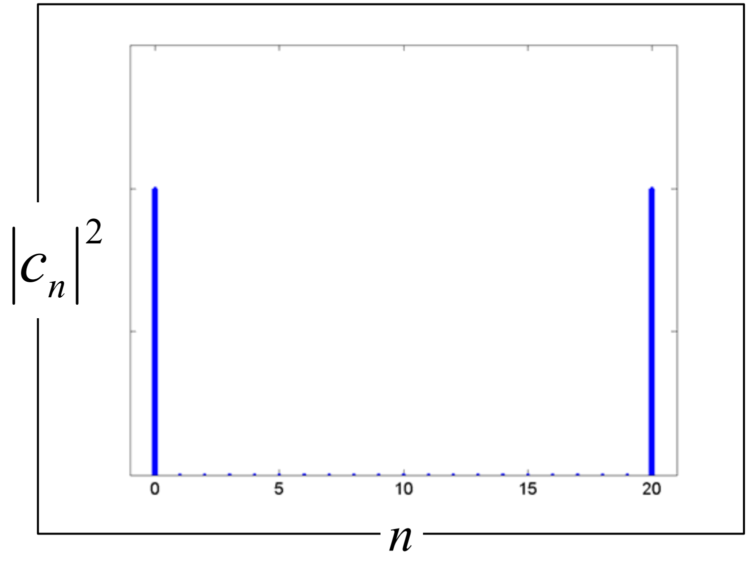


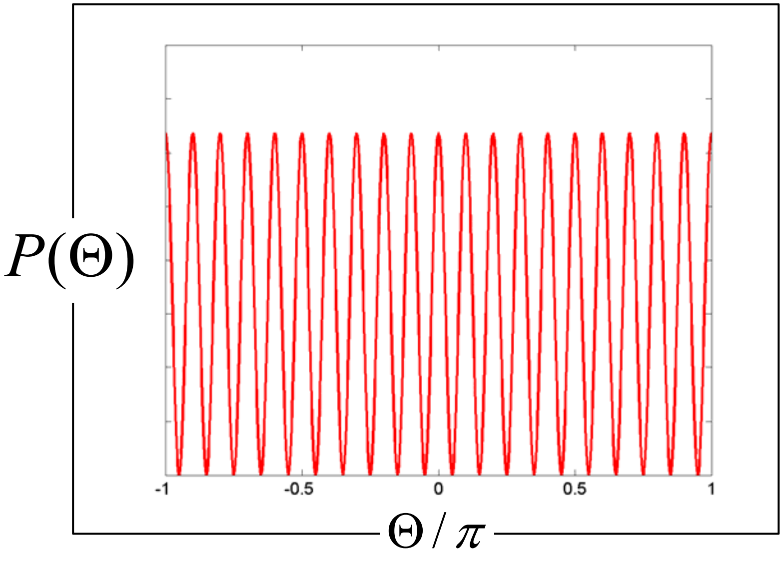
+

Linear MZ interferometry

… more states giving the HL

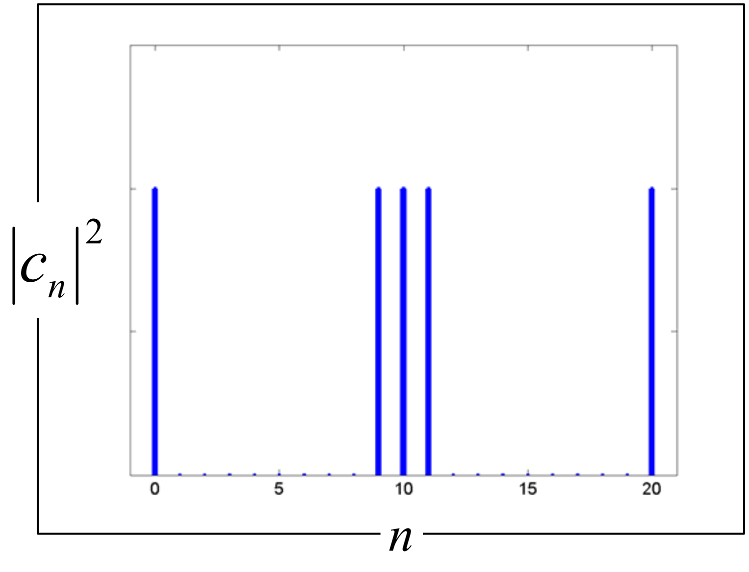
$$|\Psi_N\rangle = \frac{|N\rangle|0\rangle + |0\rangle|N\rangle}{\sqrt{2}} = \sum_0^N c_n\, |n\rangle|N-n\rangle$$

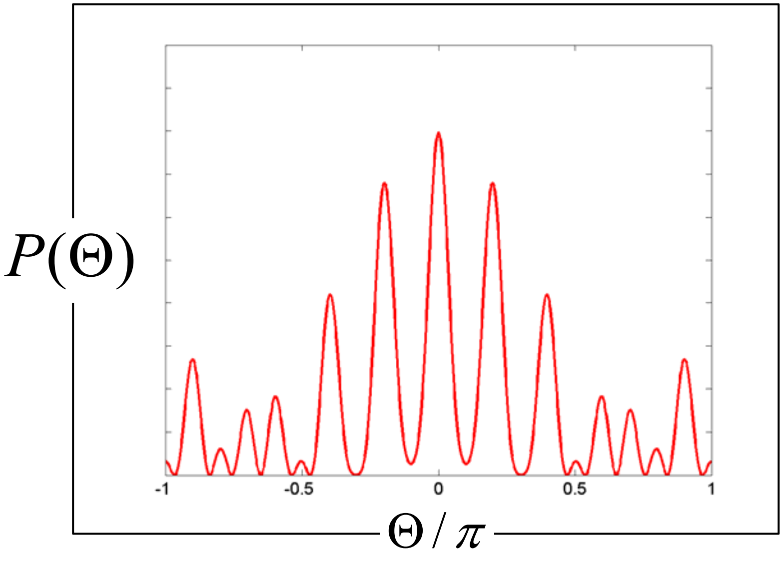




… more states giving the HL

$$|\Psi\rangle = \frac{|N\rangle|0\rangle + |0\rangle|N\rangle}{\sqrt{2}} + \ldots = \sum_{0}^{N} c_n |n\rangle|N-n\rangle$$





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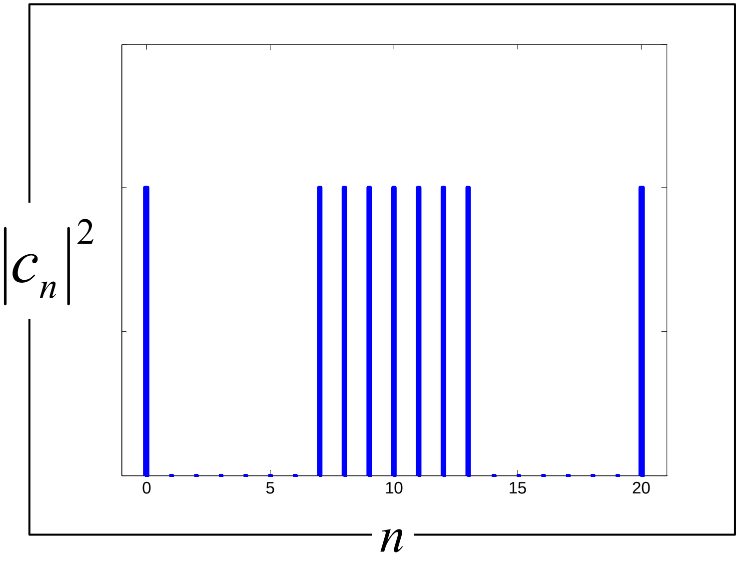

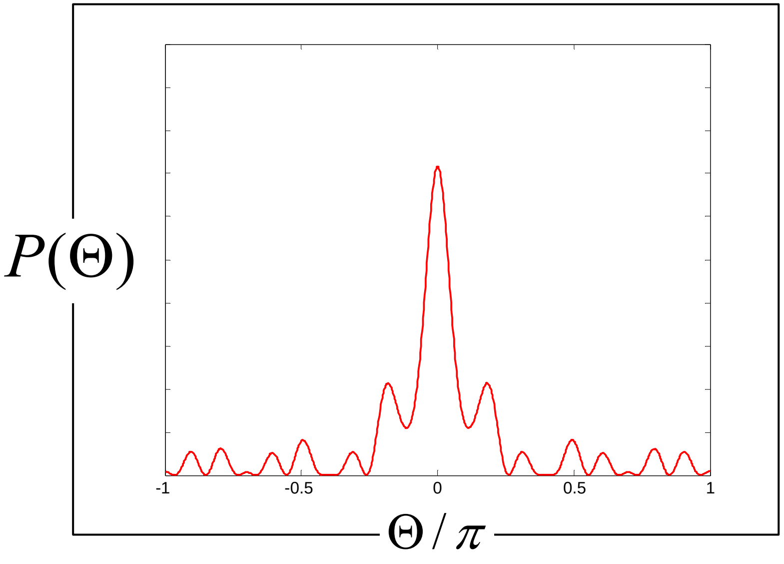

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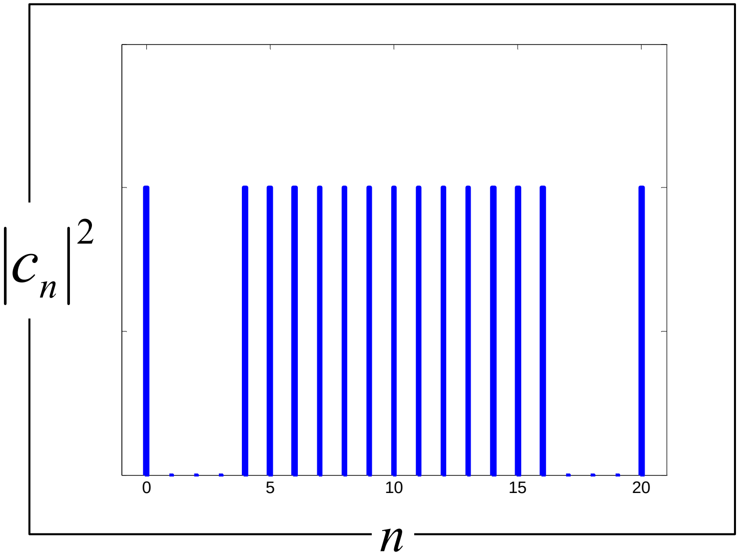

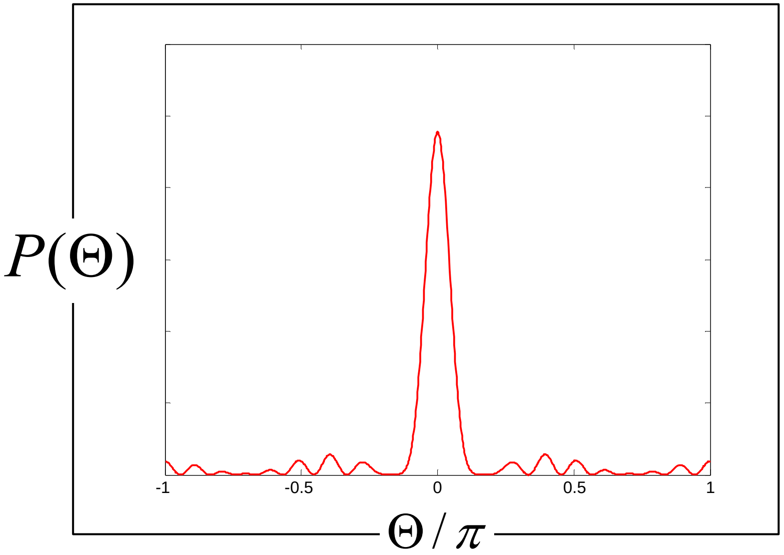

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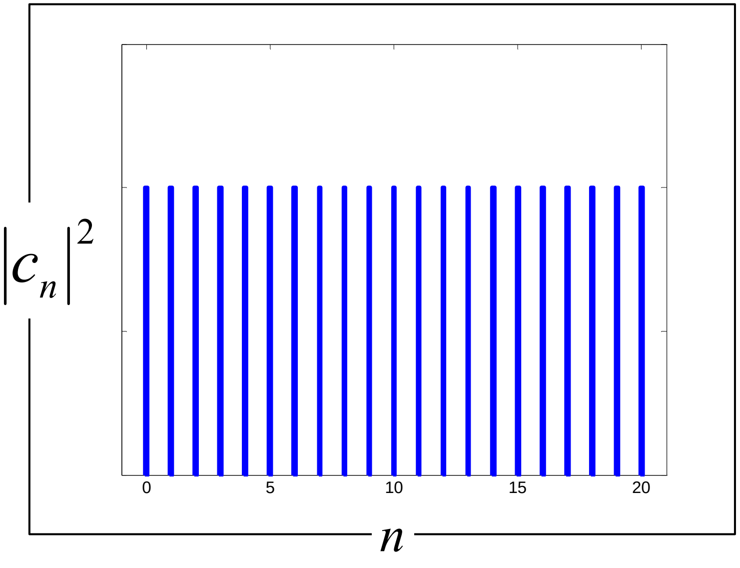
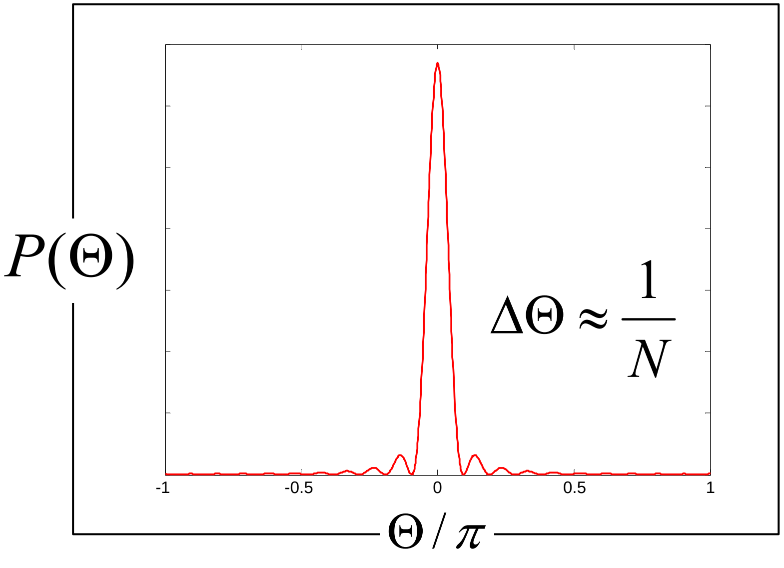
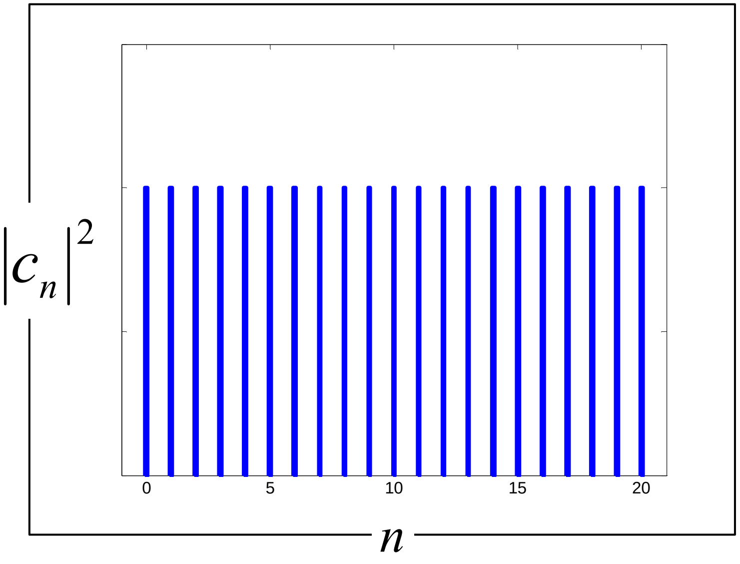


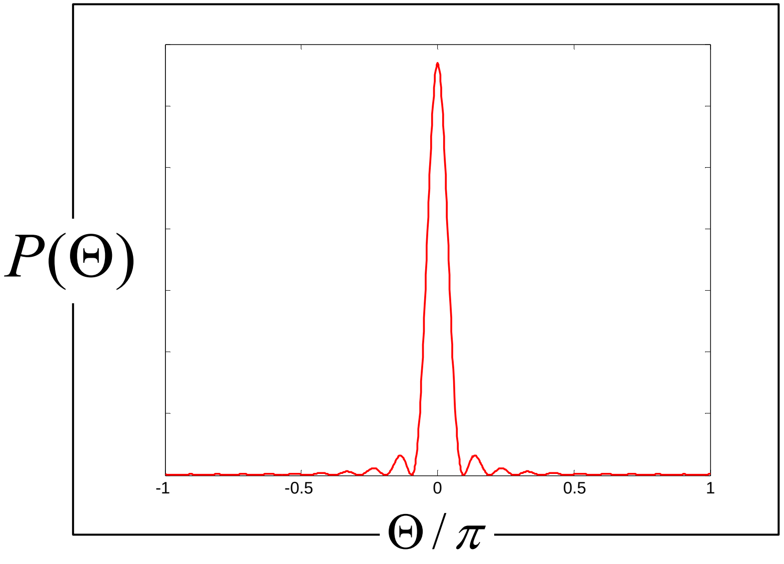


… more states giving the HL

$$|\Psi\rangle = \varepsilon \frac{|N\rangle|0\rangle + |0\rangle|N\rangle}{\sqrt{2}} + \ldots = \sum_{0}^{N} c_n |n\rangle|N-n\rangle$$

$$\varepsilon \to 0$$





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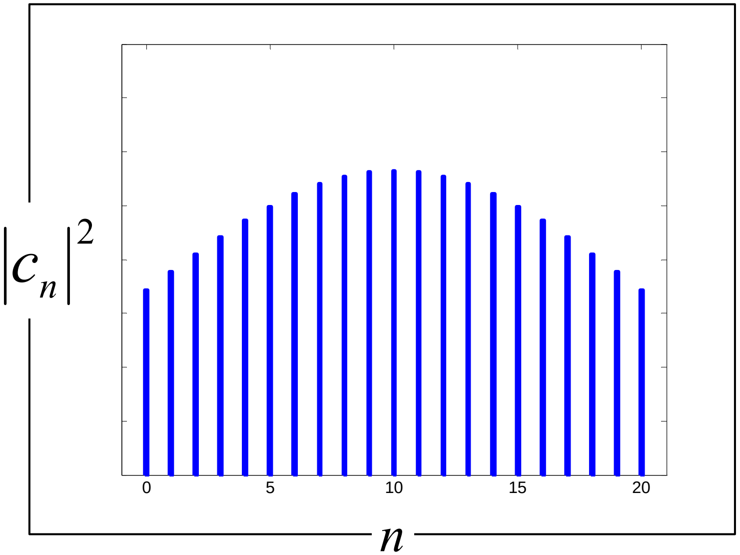

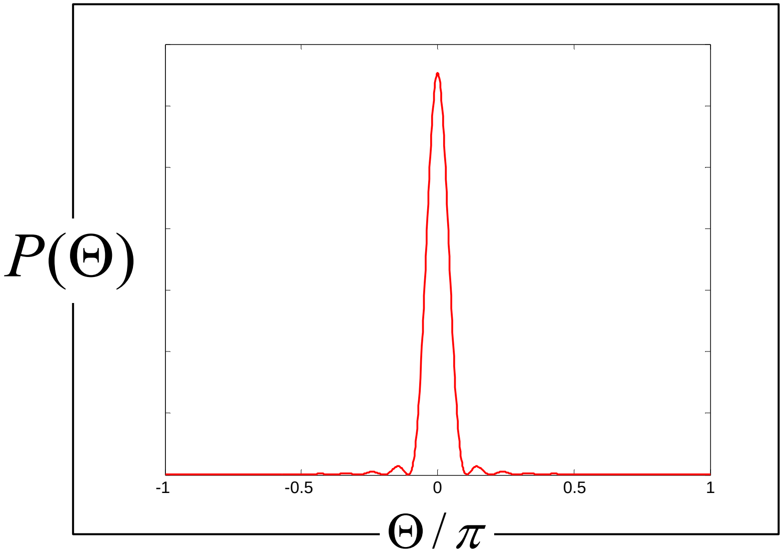

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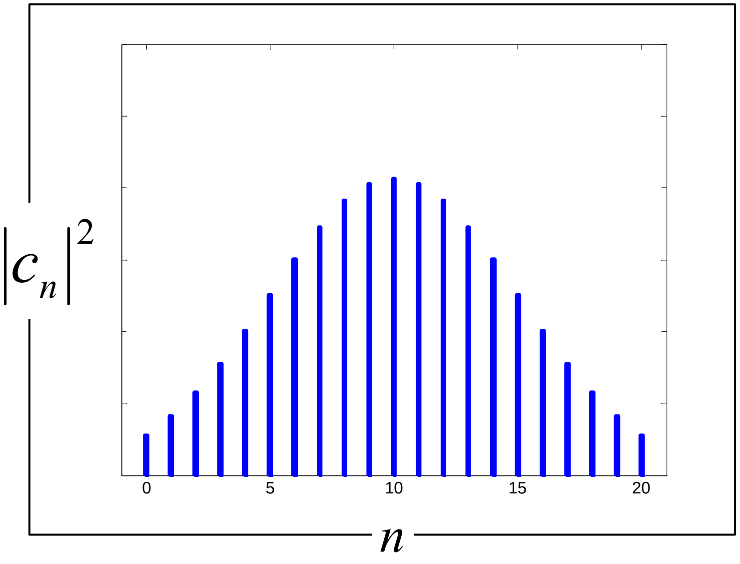
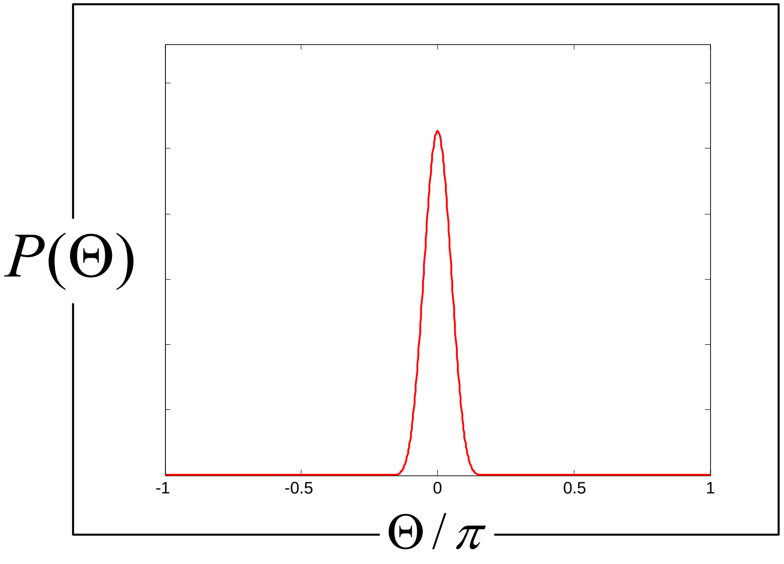
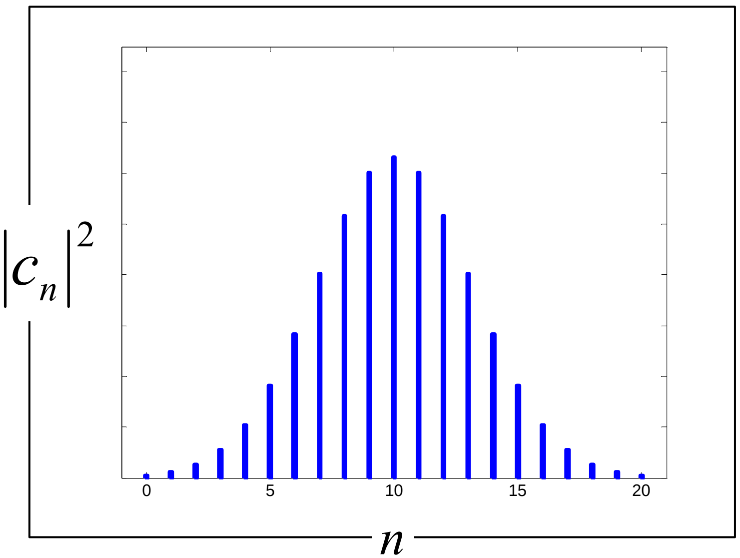


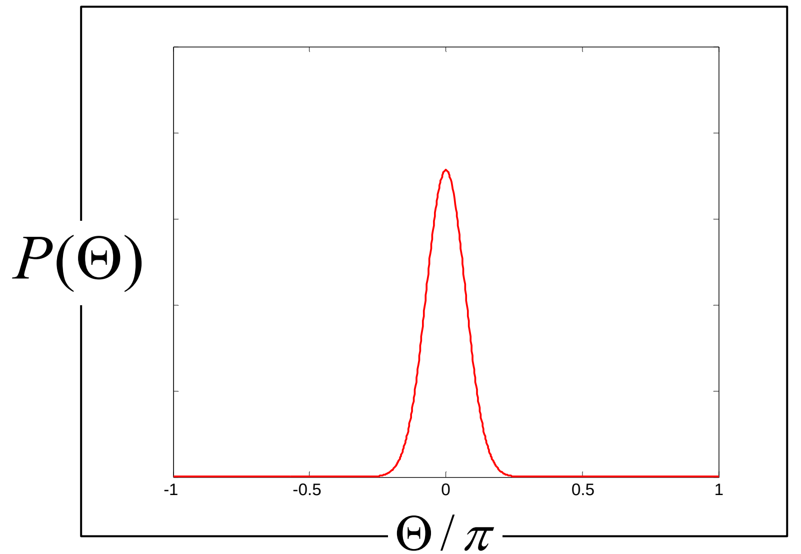


... more states giving the HL

$$\left|\Psi\right\rangle = \varepsilon \frac{\left|N\right\rangle\left|0\right\rangle + \left|0\right\rangle\left|N\right\rangle}{\sqrt{2}} + \ldots = \sum_{0}^{N} c_n \left|n\right\rangle\left|N-n\right\rangle$$

$$\varepsilon \to 0$$





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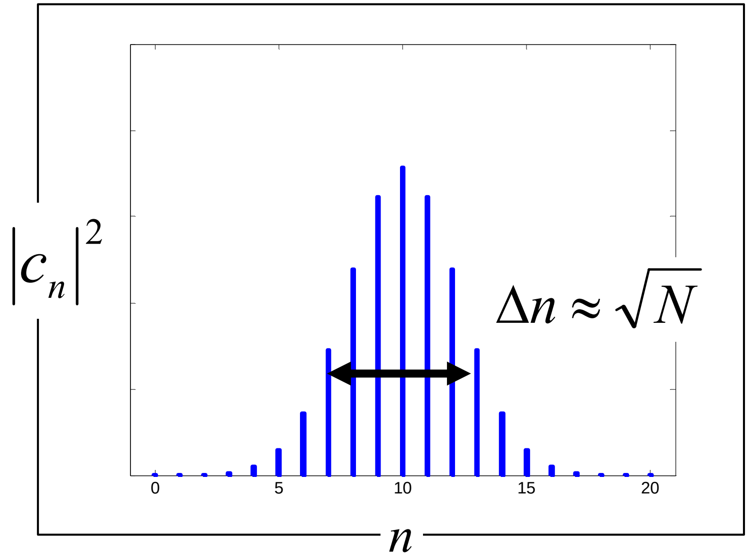
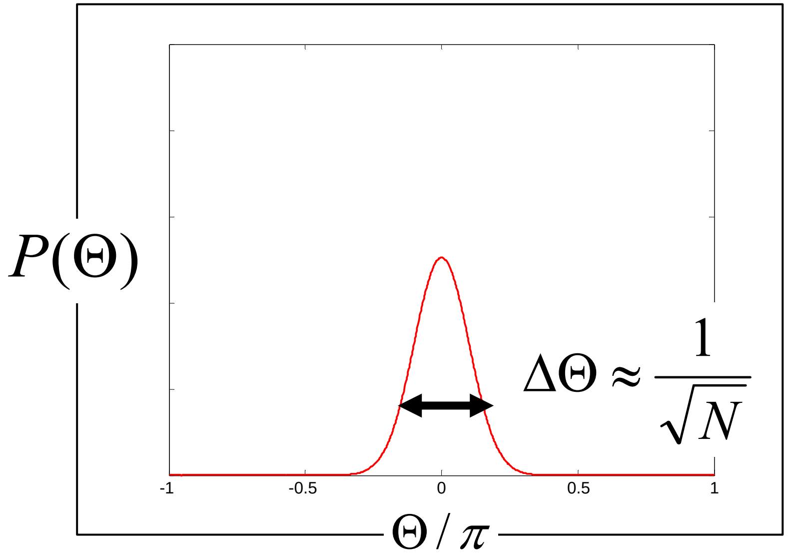
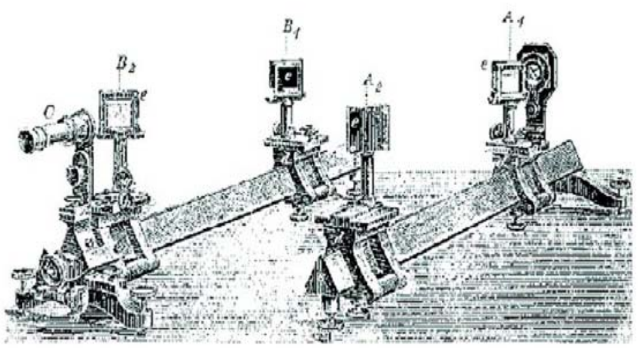


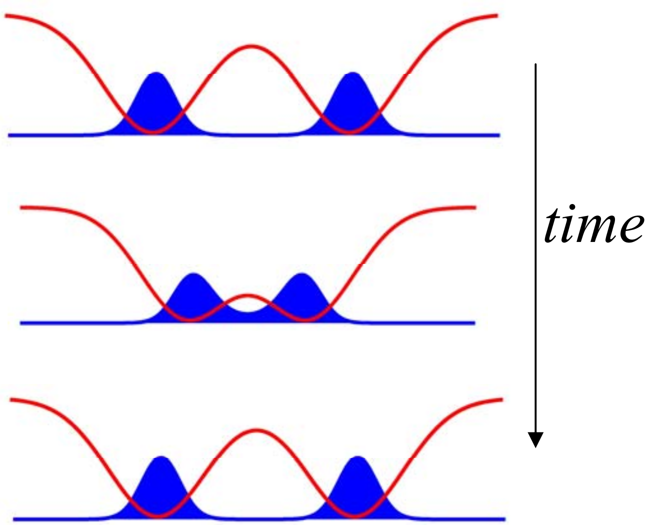


Nonlinear Beam Splitter with BEC

A key component of a BEC interferometer is the beam splitter.



$$\left|\psi_{bs}\right\rangle = e^{-i(\pi/2)(\hat{a}^{+}\hat{b}+\hat{b}^{+}\hat{a})/2}\left|\psi_{inp}\right\rangle$$



$$\left|\psi_{bs}\right\rangle = e^{-i\left(E_c\,(\hat{a}^{+}\hat{a}\,\hat{b}^{+}\hat{b})/2+K(t)(\hat{a}^{+}\hat{b}+\hat{b}^{+}\hat{a})\right)t}\left|\psi_{inp}\right\rangle$$

There are different ways to create a BEC beam splitter.
The common problem is to evaluate the role of nonlinearity

Non-interacting limit

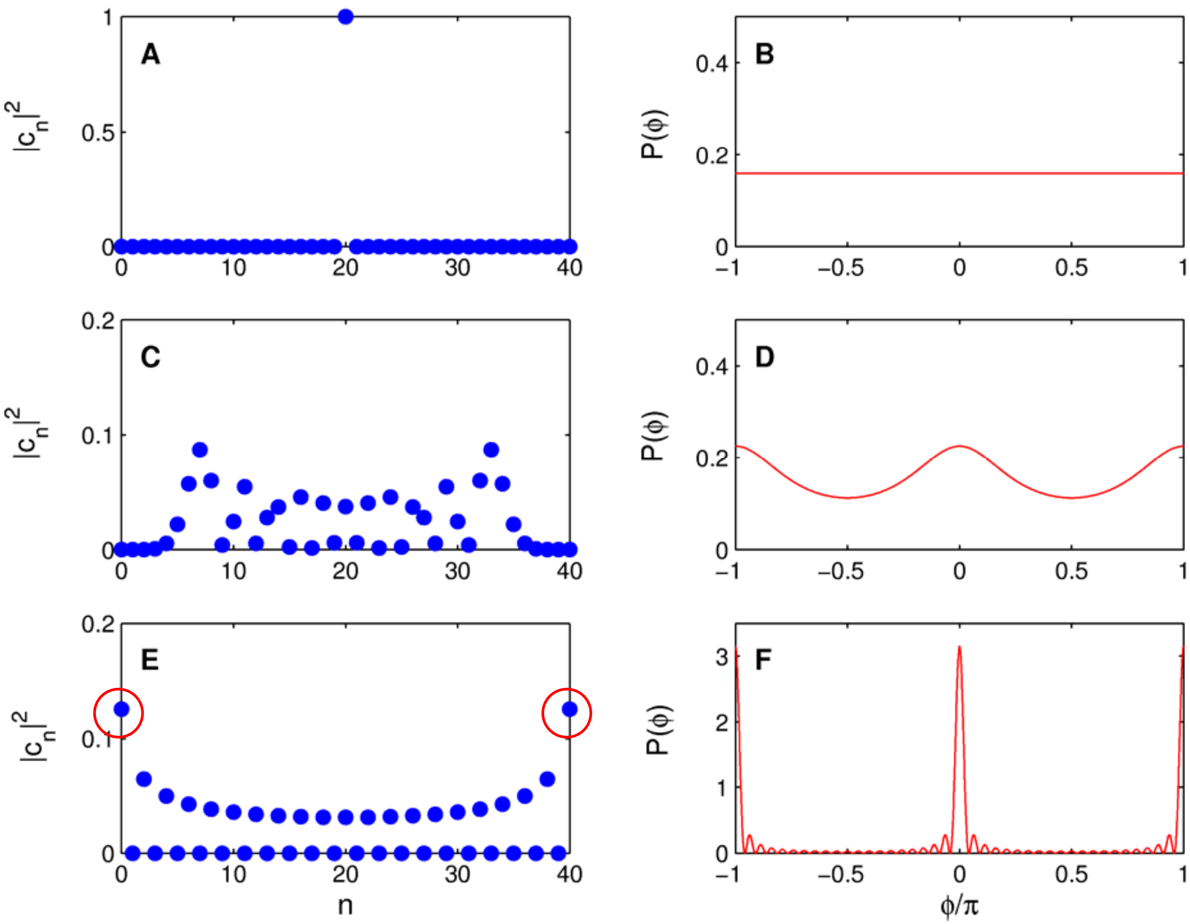
$$E_c = 0$$

$$\left|\psi_{bs}\right\rangle = e^{-i\left(K(t)\,(\hat{a}^{+}\hat{b}+\hat{b}^{+}\hat{a})\right)t}\left|\psi_{inp}\right\rangle = \sum_{m=0}^{N} c_n \left|m\right\rangle$$

$$\left|\Phi\right\rangle = \frac{1}{\sqrt{N+1}}\sum_{m=0}^{N} e^{i\Phi m}\left|m\right\rangle$$

$$P(\Phi) = \left|\left\langle \psi_{bs} \middle| \Phi\right\rangle\right|^2$$

$$\left|\psi\right\rangle_{inp} = \left|N/2, N/2\right\rangle$$



50/50 beam-splitter

$$\int_0^{t_{bs}} 2K(t)dt = \frac{\pi}{2}$$

Role of interaction $E_c \neq 0$

$$\left|\psi_{bs}\right\rangle = e^{-i\left(E_c\,(\hat{a}^+\hat{a}\,\hat{b}^+\hat{b})/2 + K(t)(\hat{a}^+\hat{b}+\hat{b}^+\hat{a})\right)t}\left|\psi_{inp}\right\rangle$$

Rabi Regime :

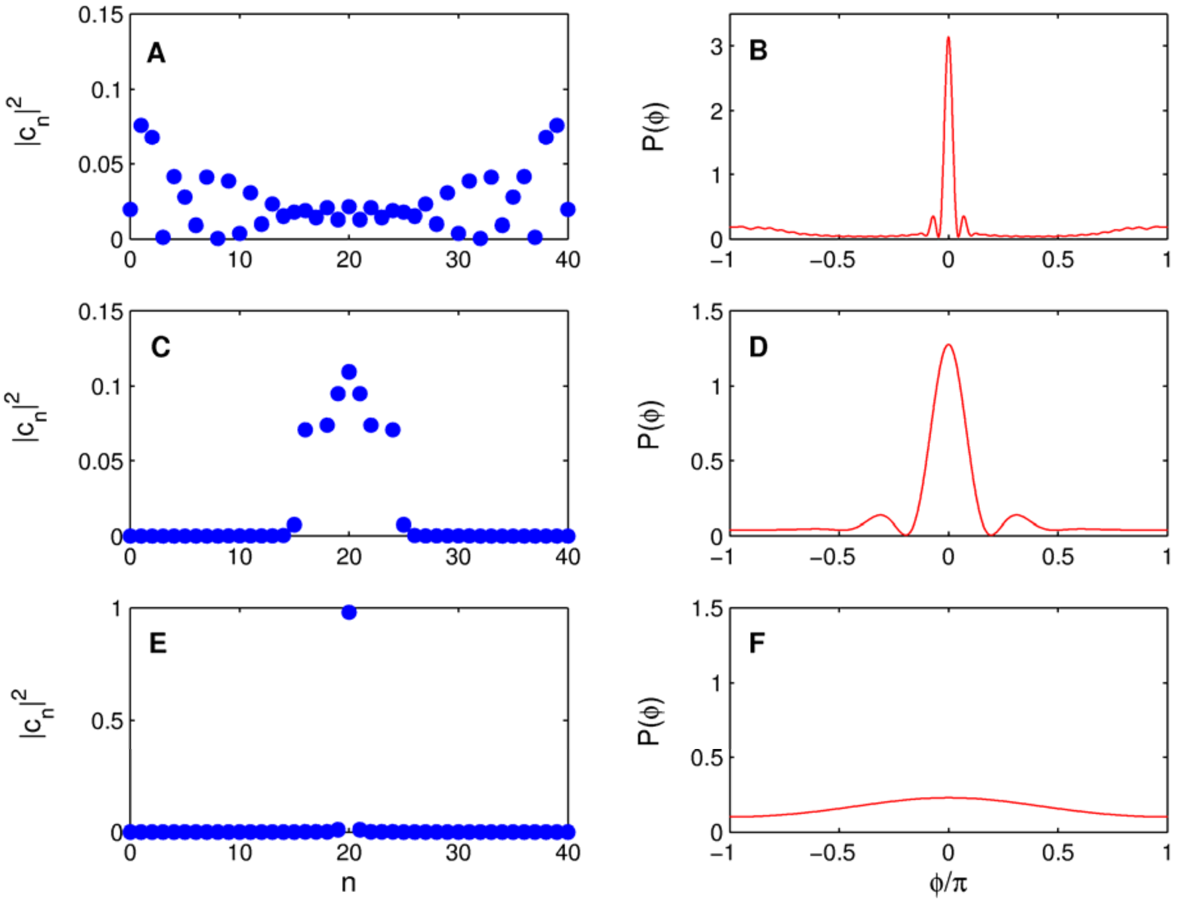
$$\frac{\mathrm{K}}{\mathrm{E_c}} >> N \Rightarrow \Delta\Theta \approx 1/N$$

Josephson Regime

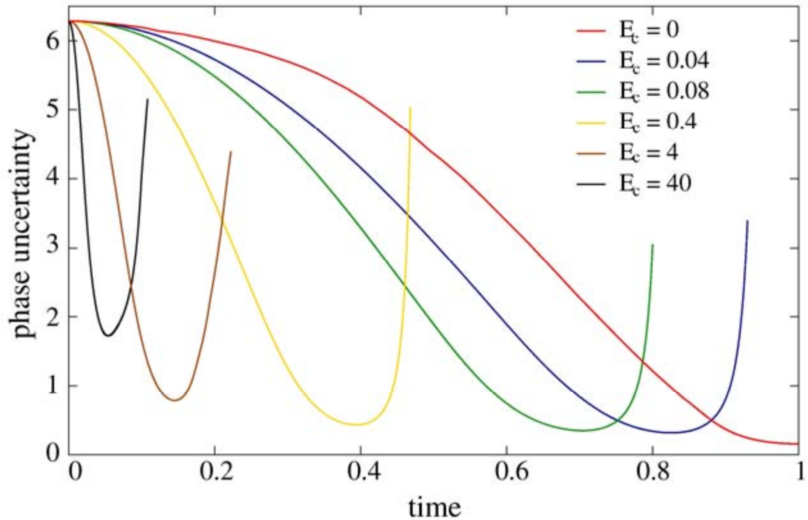
$$\frac{1}{N} << \frac{K}{E_c} << N \Rightarrow \Delta\Theta = 1/\sqrt{N}$$

Fock Regime

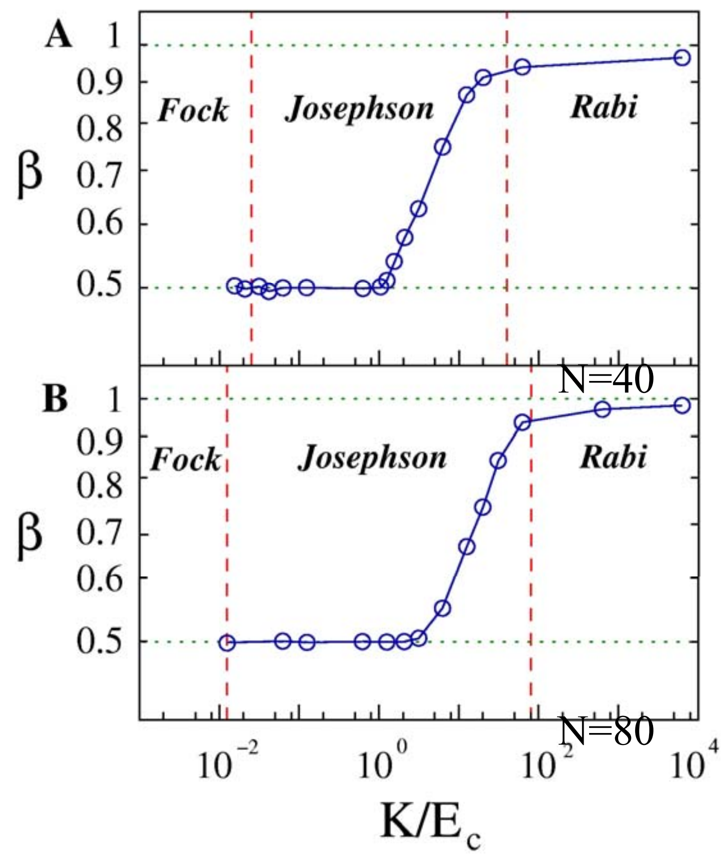
$$\frac{K}{E_c} << \frac{1}{N} \Rightarrow \Delta\Theta = 2\pi$$

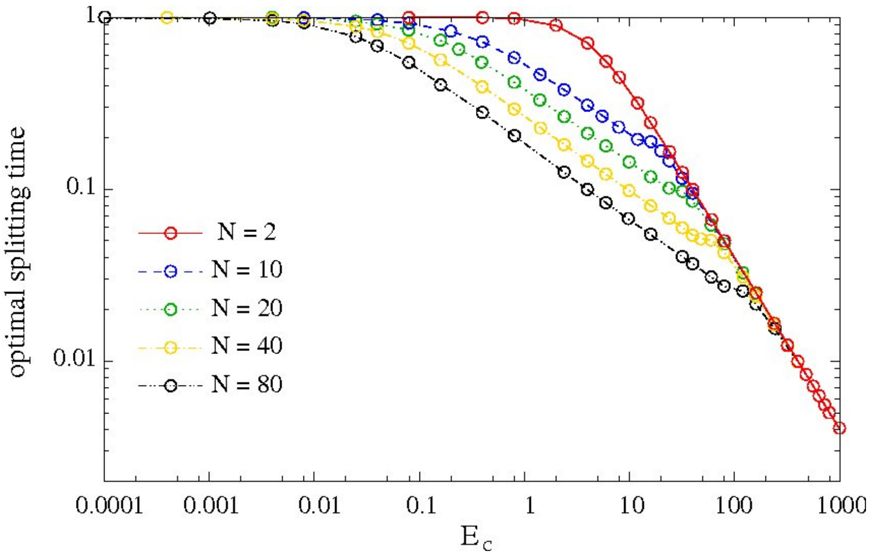


Role of interaction



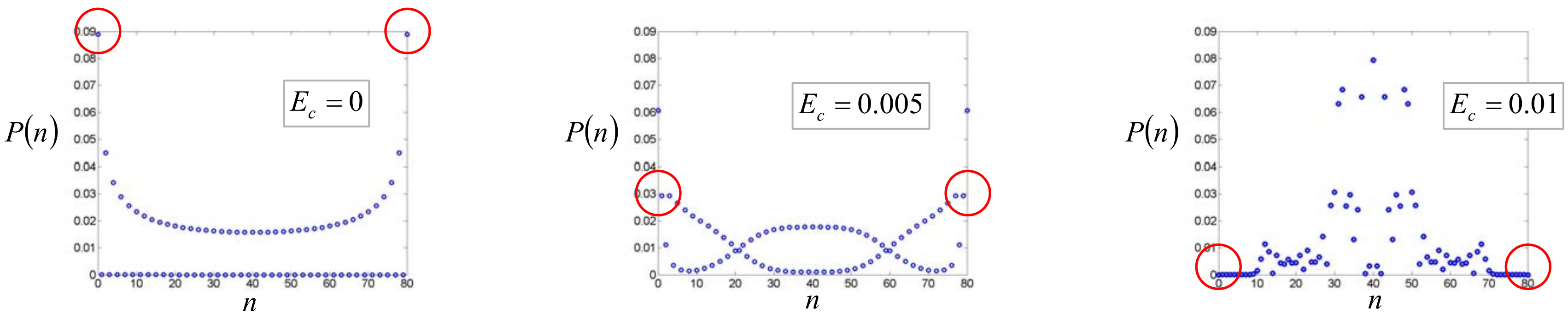
$$\Delta\Theta = \frac{\alpha}{N^{\beta}}$$





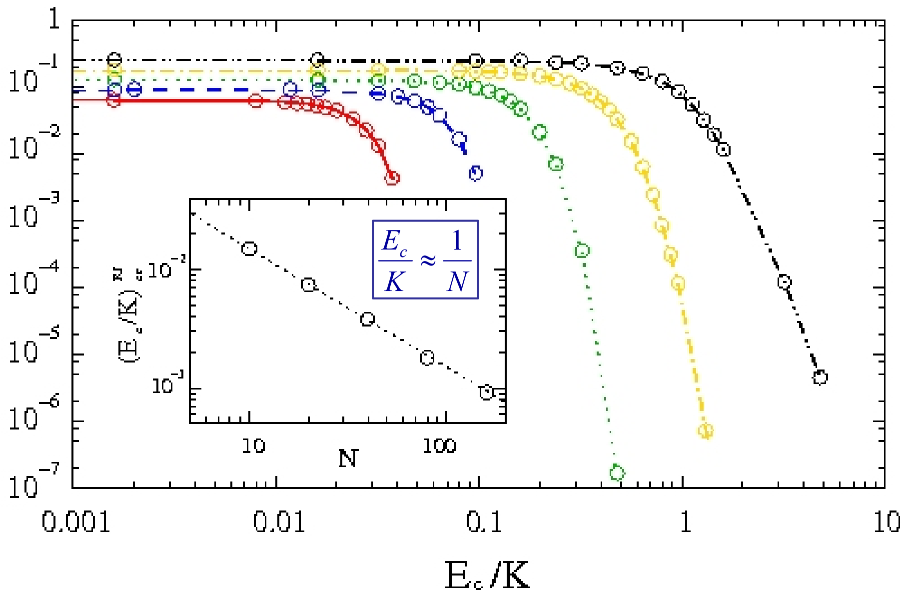
Rabi-Josephson transition

$$\left|C_n(t)\right|^2 = \left|\left\langle N-n,n\right|e^{-i\hat{H}(t)t}\left|\psi_{inp}\right\rangle\right|^2 \equiv P(n,t)$$



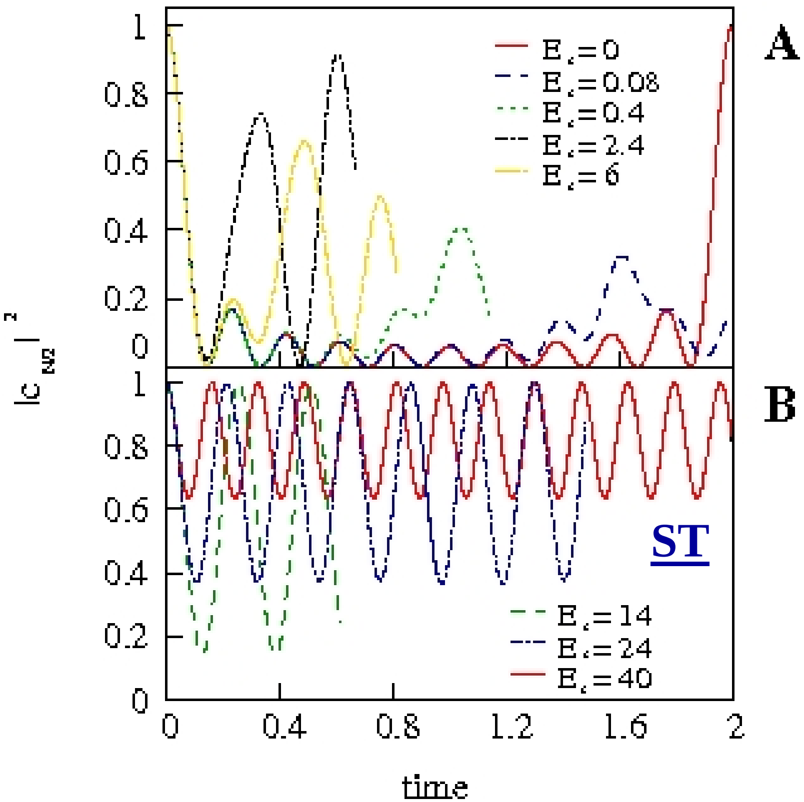
Height of the NOON

$C_{NOON}(t) \equiv \left|C_0(t)\right|^2 + \left|C_N(t)\right|^2$



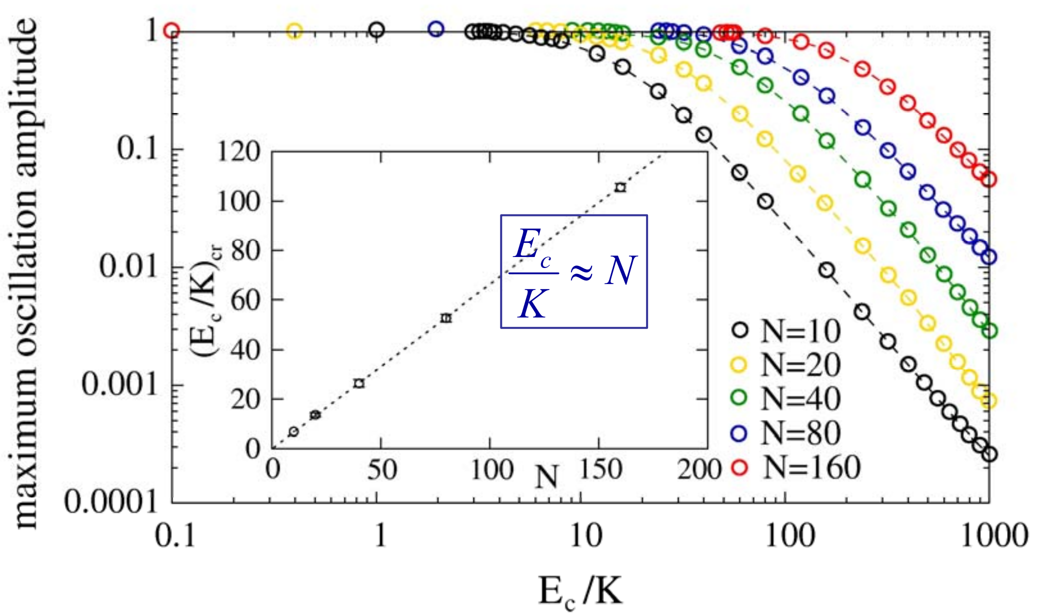
Josephson-Fock Transition

Self-Trapping of particle number fluctuations



initially: $C_n(0)=1$ for $n=N/2$

$C_n(0)=0$ otherwise



Self-Trapping: the initial energy does not redistribute over different modes–

ineffective beam splitter

Summary

1. I discussed how to reach the ultimate limit on phase sensitivity imposed by Quantum Mechanics requires the creation of “noon” (maximally entangled) states.

2. The problem of phase estimation in interferometry is not trivial. Optimal phase sensitivity of the “classical” Mach-Zehnder with Bayesian analysis.

3. Conditions for the creations of efficient BEC beam-splitters.

The Standard Quantum Limit

1) Consider an ensemble of N states $\left|\Psi_{inp}\right\rangle \approx \left(|0\rangle|1\rangle + |1\rangle|0\rangle\right)^N$

2) Phase shift : $e^{-i\hat{N}\Theta}\left|\Psi_{inp}\right\rangle = \left(e^{i\Theta}|0\rangle|1\rangle + e^{-i\Theta}|1\rangle|0\rangle\right)^N$

3) A projective measurement over the initial state gives

$$\left|\left\langle\Psi_{inp}\right|e^{-i\Theta\hat{\mathrm{N}}}\left|\Psi_{inp}\right\rangle\right|^2 \approx \cos^{2N}(\Theta) \approx \exp[-\Theta^2/4N]$$

$\Longrightarrow$ Orthogonality is reached at $\Delta\Theta \approx \dfrac{1}{\sqrt{N}}$

Interferometry with N uncorrelated particles and/or p independent measurements is bounded by the SQL (shot - noise) sensitivity $1/\sqrt{N\,p}$

The Heisenberg limit

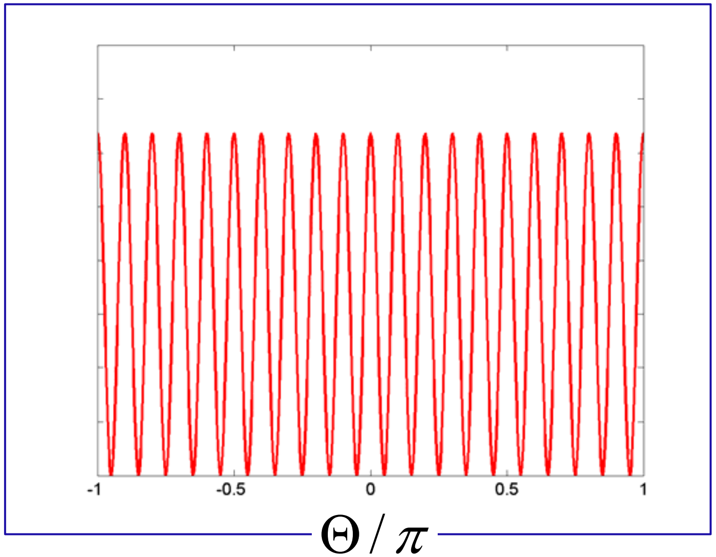
1) Schroedinger cat (NOON) state : $|\psi_N\rangle \approx |N,0\rangle + |0,N\rangle$

2) Phase shift : $e^{-i\hat{N}\Theta}|\psi_N\rangle = e^{-iN\Theta}|N,0\rangle + e^{iN\Theta}|0,N\rangle$

3) A projective measurement over the initial state gives

$$\left|\langle\psi_N|e^{-i\hat{N}\Theta}|\psi_N\rangle\right|^2 \approx \cos^2(N\Theta/2) \quad \Longrightarrow \quad \text{Orthogonality is reached for } \Delta\Theta \approx \frac{1}{N}$$

Entanglement (quantum correlations) can provide sensitivity at the Heisenberg limit $\frac{1}{N}$



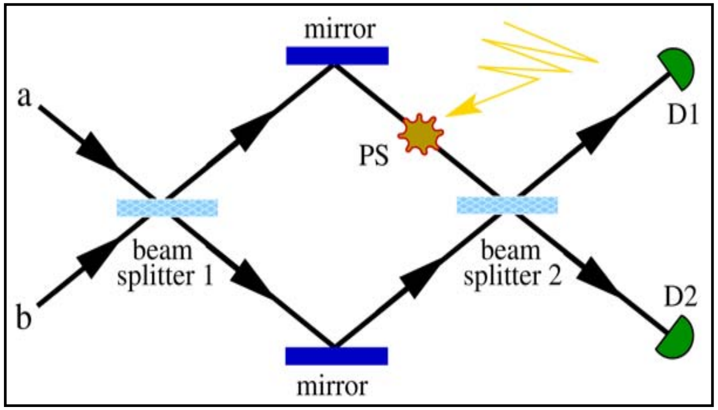
The "classical" Mach-Zehnder

Input state

$$\left|\Psi_{inp}\right\rangle = \left|\alpha\right\rangle_a \left|0\right\rangle_b$$

Output state

$$\left|\Psi_{out}\right\rangle = e^{\Theta(\hat{a}^+\hat{b}-\hat{b}^+\hat{a})/2}\left|\Psi_{inp}\right\rangle$$

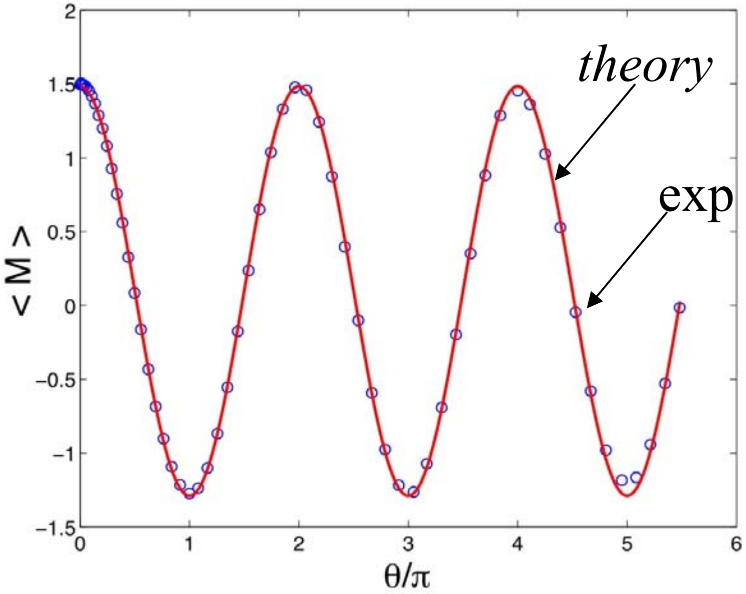


Measurement : N_{D1}, N_{D2}

with probability $\left|\left\langle N_{D1}, N_{D2} \middle| \Psi_{out}\right\rangle\right|^2$

$$\left\langle \hat{M} \right\rangle = \left\langle \hat{N}_{D1} - \hat{N}_{D2} \right\rangle = \left|\alpha\right|^2 \cos(\Theta)$$

$$\left\langle \hat{N} \right\rangle = \left\langle \hat{N}_{D1} + \hat{N}_{D2} \right\rangle = \left|\alpha\right|^2$$



“classical” phase estimation

as estimator, *choose* :

$$\overline{M} \equiv \frac{1}{p}\sum_{i=1}^{p}(N_{D1} - N_{D2}) = |\alpha|^2 \cos(\Theta_{est})$$

The estimated value Θ_{est} of the true phase shift Θ is *defined* as the average of the relative number of particles in p independent measurements.

From error propagation…

$$\Delta\Theta \underset{p>>1}{=} \frac{\Delta\hat{M}}{\left|\partial\langle\hat{M}\rangle / \partial\Theta\right|}\frac{1}{\sqrt{p}} = \frac{1}{\sqrt{p|\alpha|^2}}\frac{1}{\sin(\Theta)}$$

➡ Optimal phase sensitivity at $\Theta = \pi/2$

Is it possible to reach the SQL for <u>any</u> value of the phase shift ?

ADVANTAGES:

1) Rigorous analysis without statistical assumptions

2) Possibility to consistently include classical noise and detector efficiency

3) Phase estimation with a single measurement. The sensitivity $\Delta\Theta = \frac{1}{\sqrt{N_{D1}+N_{D2}}}$

4) Asymptotically in the number of measurements $\Delta\Theta \underset{p>>1}{=} \frac{1}{\sqrt{N_{ave}}}$

cfr. the "classical" theory $\Delta\Theta \underset{p>>1}{=} \frac{1}{\sqrt{N_{ave}}}\frac{1}{\sin(\Theta)}$

These predictions can be tested experimentallyc

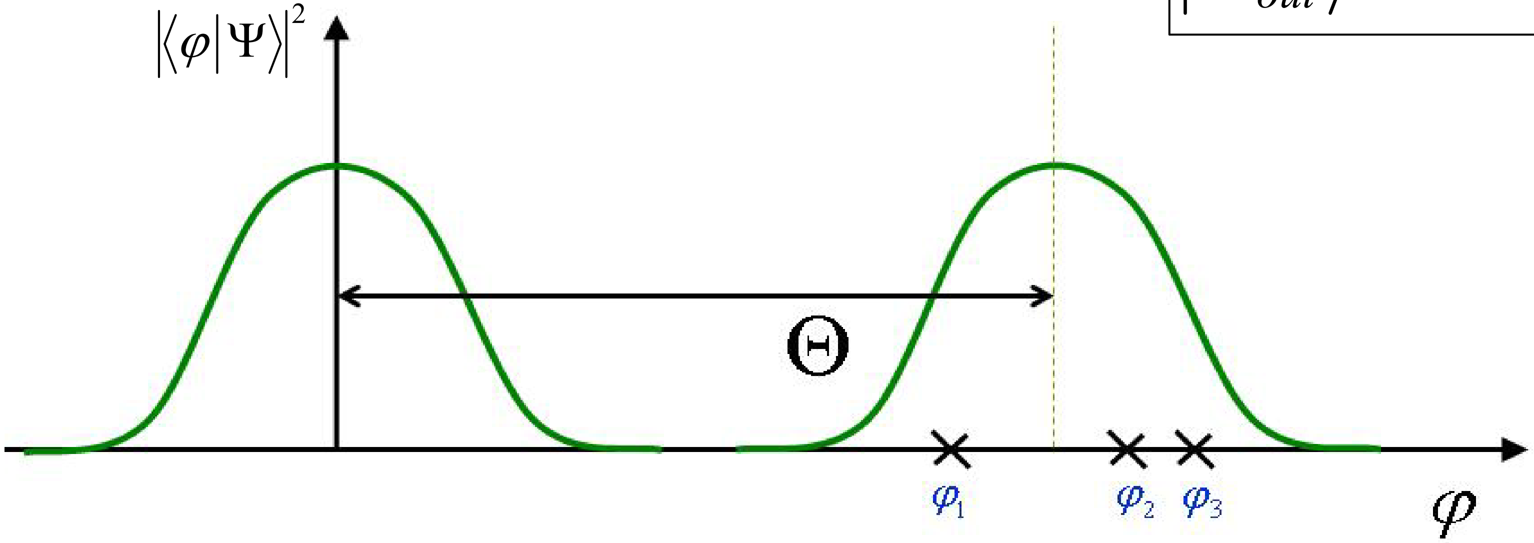
Quantum Interferometry

Luca Pezze'
Augusto Smerzi

CNR-INFM BEC, Trento, Italy

The phase inference problem

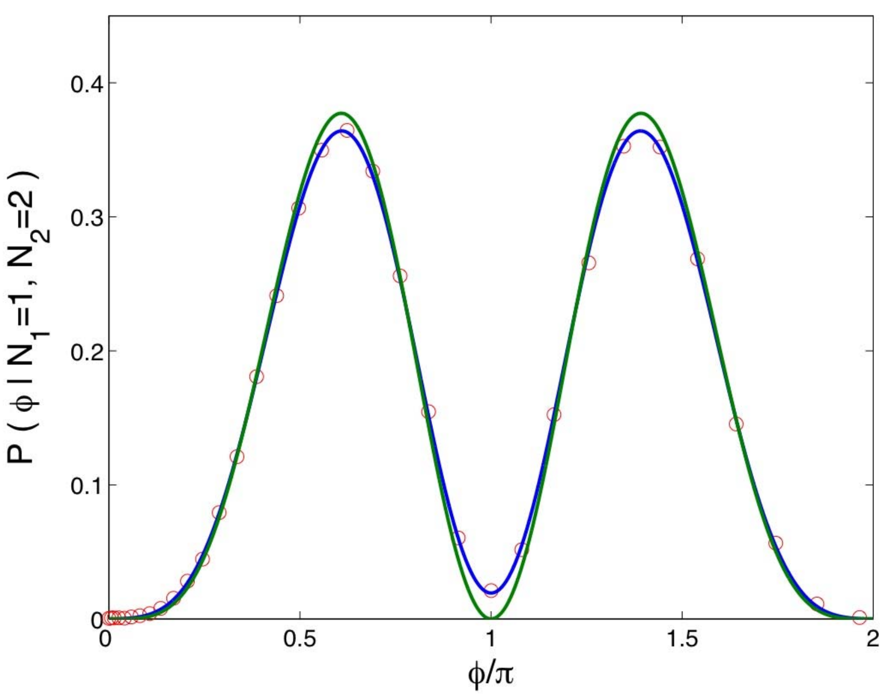
$$\left|\Psi_{out}\right\rangle = e^{-i\,\Theta\,\hat{N}}\left|\Psi_{inp}\right\rangle$$

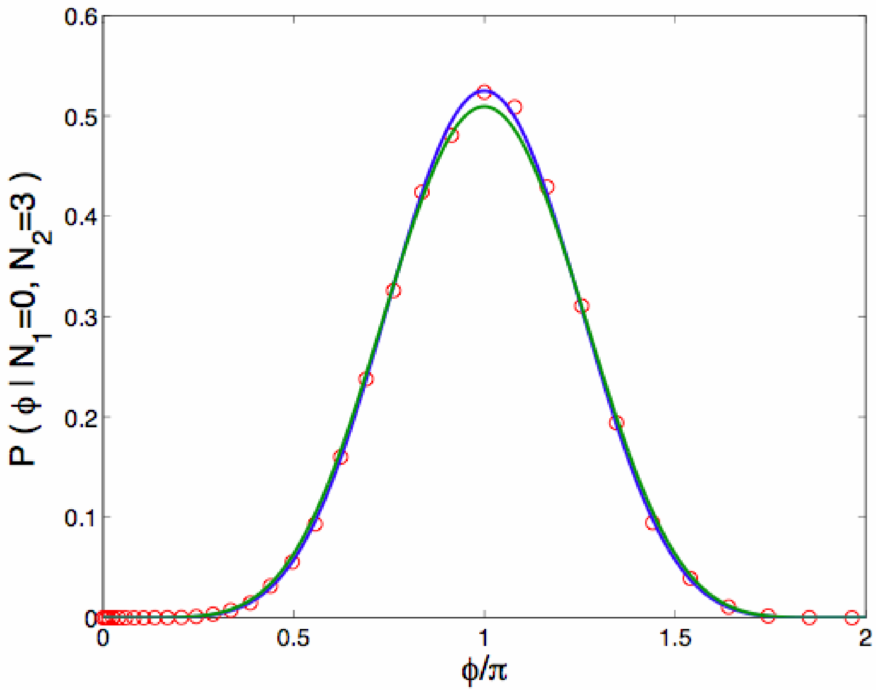


How precisely the unknown phase shift Θ can be infered from the results of p *independent* measurements $\{\varphi_1, \varphi_2, \ldots \varphi_p\}$?

Exp. vs. theor. Bayesian distributions

$|\alpha|^2 = 1.38$





Theor. $P(\Phi \mid N_{D1}, N_{D2}) \propto \sin^{2N_{D2}}(\Phi/2)\cos^{2N_{D1}}(\Phi/2)$

Exp. (with optical MZ and number counting photodetectors @ UCLA)

Fit of the exp. data including the finite efficiency of the detectors

L. Pezze', A. Smerzi, G. Khoury, Juan Hodelin, D. Bouwmeester, submitted

Bayesian phase estimation

Calibration.

Calculate $P(N_1, N_2 | \Phi)$

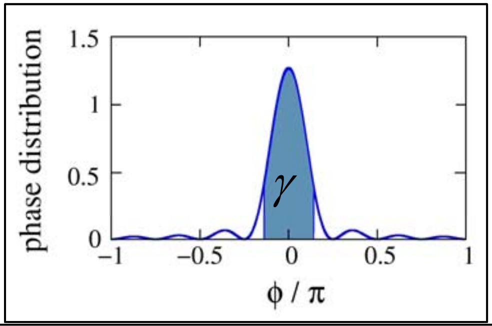
Measurement.

Unknown Θ

N_1, N_2

Bayes theorem & estimation.

$$P(\phi \,|\, N_1, N_2) = \frac{P(N_1, N_2 \,|\, \phi)}{\int_{-\pi}^{+\pi} P(N_1, N_2 | \phi)\, \mathrm{d}\phi}$$



Estimator Θ_{est} :

maximum of the phase distribution

Uncertainty $\Delta\Theta$:

γ - confidence $\int_{\Theta_{est}-\Delta\Theta}^{\Theta_{est}+\Delta\Theta} d\phi\, P(\phi | N_1, N_2) = \gamma$

Phase estimation with a single measurement. The sensitivity $\Delta\Theta = \frac{1}{\sqrt{N_{D1} + N_{D2}}}$

Asymptotically in the number of measurements $\Delta\Theta \underset{p>>1}{=} \frac{1}{\sqrt{N_{ave}}}$

cfr. the "classical" theory :

$$\Delta\Theta \underset{p>>1}{=} \frac{1}{\sqrt{N_{ave}}} \frac{1}{\sin(\Theta)}$$

These predictions can be tested experimentally

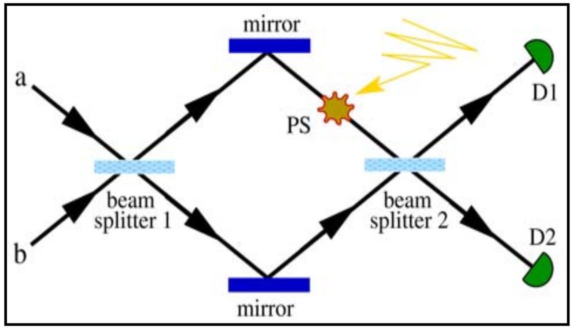
Phase estimation experiment

$\Theta / \pi = 0.69$

$|\alpha|^2 = 1.38$

$|\alpha\rangle_a$

$|0\rangle_b$



Measurements :

$N_{D1}^{(1)}, N_{D2}^{(1)}$

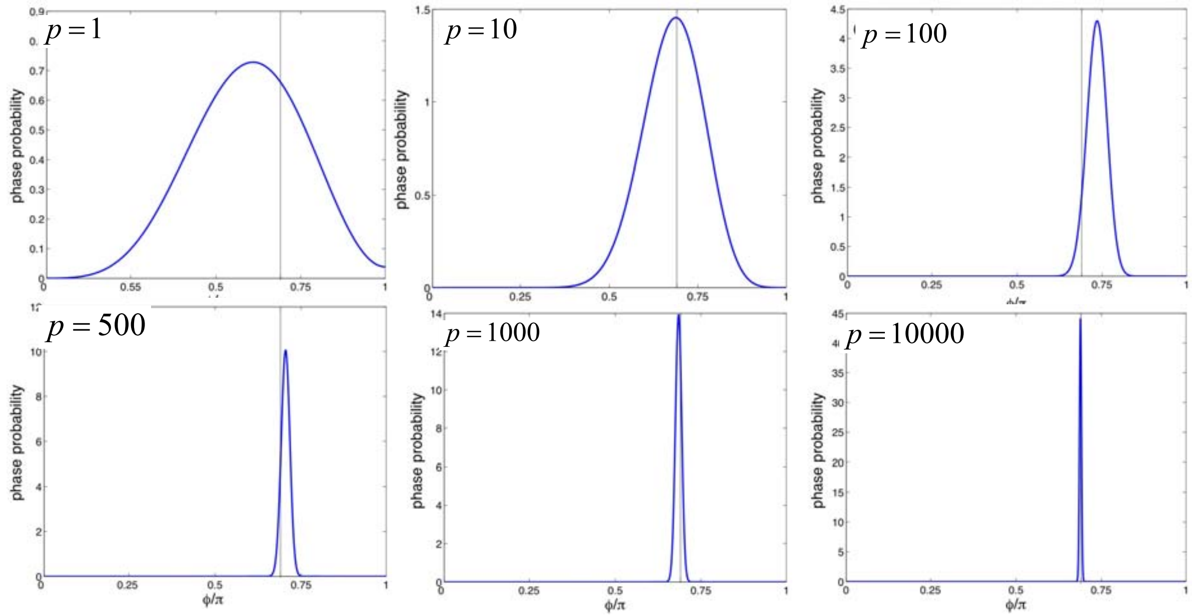
$N_{D1}^{(2)}, N_{D2}^{(2)}$

....

$N_{D1}^{(p)}, N_{D2}^{(p)}$

Phase probability distribution :

$$P(\phi) = \prod_{i=1}^{p} P(\phi \mid N_{D1}^{(i)}, N_{D2}^{(i)})$$



Exp. with optical MZ and number counting photodetectors @ UCSB

L. Pezze', A. Smerzi, G. Khoury, Juan Hodelin, D. Bouwmeester, submitted

NOON states with Beryllium ions

Creation:

$$\left|N\downarrow\right\rangle \equiv \left|\downarrow\right\rangle_1 \ldots \left|\downarrow\right\rangle_N \quad \text{initial state (gs)}$$

$$\hat{U}_N \equiv e^{-i\frac{\xi\pi}{2}\hat{J}_x} e^{-i\frac{\pi}{2}\hat{J}_x^2} \quad \text{NLBS operator}$$

$$\hat{U}_N\left|N\downarrow\right\rangle = \frac{\left|N\downarrow\right\rangle + i^{\xi+N+1}\left|N\uparrow\right\rangle}{\sqrt{2}}$$

$\xi = 0$ when N odd, $\xi = 1$ when N is even

Molmer & Sorensen, PRL 2000

Phase shift:

$$e^{-i\Theta\hat{J}_z}\,\hat{U}_N\left|N\downarrow\right\rangle = \frac{\left|N\downarrow\right\rangle + e^{-i\frac{N\Theta}{2}}\,i^{\xi+N+1}\left|N\uparrow\right\rangle}{\sqrt{2}}$$

Decoding:

$$\hat{U}_N\,e^{-i\Theta\hat{J}_z}\,\hat{U}_N\left|N\downarrow\right\rangle = \cos\left(\frac{N\Theta}{2}\right)\left|N\downarrow\right\rangle + \sin\left(\frac{N\Theta}{2}\right)\left|N\uparrow\right\rangle$$

Projective measurement:

$$P\left(N\downarrow\middle|N,\Theta\right) = \left|\left\langle N\downarrow\middle|\hat{U}_N e^{-i\Theta\hat{J}_z}\hat{U}_N\middle|N\downarrow\right\rangle\right|^2 = \cos^2\left(N\Theta/2\right)$$

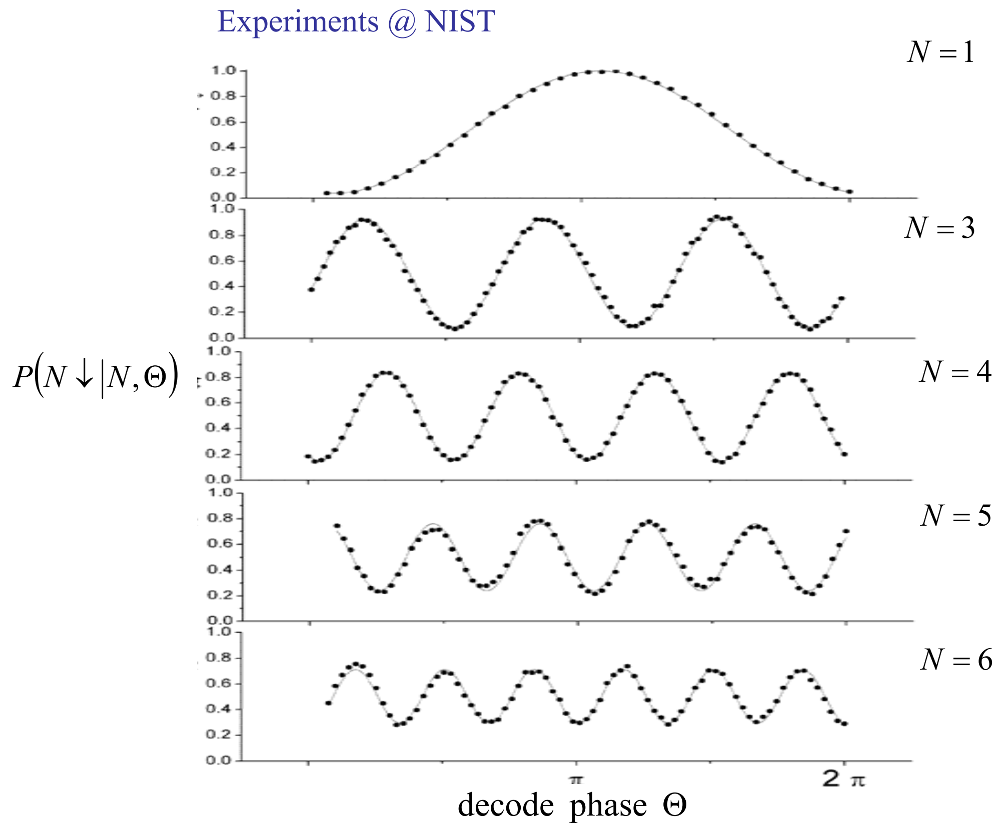
$$P\left(N\uparrow\middle|N,\Theta\right) = \left|\left\langle N\uparrow\middle|\hat{U}_N e^{-i\Theta\hat{J}_z}\hat{U}_N\middle|N\downarrow\right\rangle\right|^2 = \sin^2\left(N\Theta/2\right)$$

The two states $\left|N\downarrow\right\rangle$ and $\left|N\uparrow\right\rangle$ give a different fluorescent signal.

NOON states with Beryllium ions

The probability distributions oscillate with period $2\pi/\mathrm{N}$

Is the $2\pi/N$ period enough to conclude that we have a phase sensitivity at the HL?



$$P(N\downarrow|N,\Theta)=\left|\left\langle N\downarrow\right|\hat{U}_N e^{-i\Theta\hat{J}_z}\hat{U}_N\left|N\downarrow\right\rangle\right|^2=\cos^2(N\Theta/2)$$

$$P(N\uparrow|N,\Theta)=\left|\left\langle N\uparrow\right|\hat{U}_N e^{-i\Theta\hat{J}_z}\hat{U}_N\left|N\downarrow\right\rangle\right|^2=\sin^2(N\Theta/2)$$