Boundary Field Theory approach to Superconducting Quantum Circuits

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Outline

- Circuits involving Josephson junctions;
- Superconducting flux (persistent current) qubits
- Boundary field theory describing the rf-SQUID;
- Boundary field theory describing a dc-SQUID;
- Final Remarks.

Circuits involving Josephson junctions.

- The inhomogeneous Josephson junction chain with a weak link

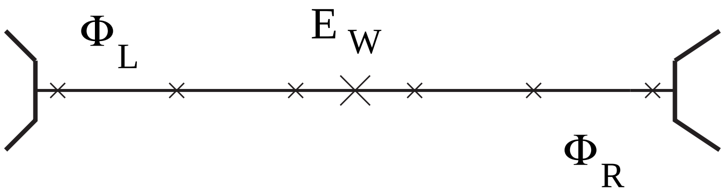

Figure 1: The chain.

1. Mapping onto a spin-chain Hamiltonian : operators

$$Pe^{\pm i\phi_j}P = S_j^{\pm} \quad ; \; P\left(-i\frac{\partial}{\partial\phi_j} - n - \frac{1}{2}\right)P = S_j^z$$

2. Hamiltonian

$$H_{\text{eff}} = -\frac{J}{2}\sum_{j=1}^{L/a}[S_j^+S_{j+1}^- + S_{j+1}^+S_j^-] - E_C h\sum_{j=1}^{L/a}S_j^z - \frac{3}{16}\frac{J^2}{E_C}\sum_{j=1}^{L/a}S_j^zS_{j+1}^z$$

3. Jordan-Wigner Fermions

$$S_j^+ \equiv a_j^\dagger \exp[i\pi\sum_{l=1}^{j-1}a_l^\dagger a_l] \quad ; \; S_j^z \equiv a_j^\dagger a_j - \frac{1}{2}$$

4. Weak link + Bosonization

$$H_{\text{JJ}} = \frac{g}{4\pi}\sum_{a=L,R}\int_0^L dx\left[\frac{1}{u}\left(\frac{\partial\Phi_a}{\partial t}\right)^2 + u\left(\frac{\partial\Phi_a}{\partial x}\right)^2\right] + H_W$$

5. Parameters

$$v = \sqrt{v_F(v_F+2\gamma)} \quad ; \; g = \sqrt{\frac{v_F+2\gamma}{v_F}} \qquad \gamma = 16\pi(a\Delta)\sin^2(k_F a))$$

6. Boundary Hamiltonian

$$H_W = -\bar{E}_W : \cos\left[\phi_R(0) - \phi_L(0))\right] :$$

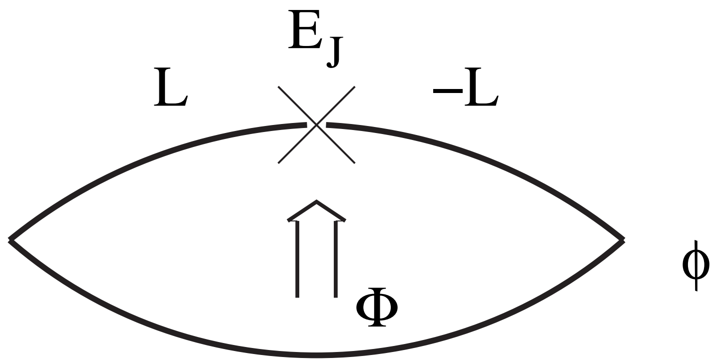

Figure 2: The rf-SQUID.

0.1 The rf-SQUID

- The rf-SQUID

  1. Lagrangian:

$$L = \int_{-L}^{L} dx \left\{ \frac{\hbar^2}{2e_c} \left(\frac{\partial \Phi}{\partial t} \right)^2 - \frac{\hbar^2 n_s S}{4m} \left(\frac{\partial \Phi}{\partial x} - \frac{\Phi}{2L} \right)^2 \right\}$$

$$+E_J \cos[\Phi(L,t) - \Phi(-L,t)]$$

  2. n_s is the superfluid density, m is the electron mass, $1/e_c$ is the characteristic inverse charging energy per unit length of the loop, $S = r \times r$ is the cross section of the wire, ϵ is the dielectric constant of the medium the loop is embedded within, R is the distance from a metallic screen, E_J is the Josephson energy of the junction.

  3. Charging energy of the lead + inductive energy of the lead + Josephson energy of the junction $\Rightarrow$

$$H_{\rm rf} = \frac{g}{4\pi} \int_{-L}^{L} dx \left[\frac{1}{u} \left(\frac{\partial \Phi(x,t)}{\partial t} \right)^2 + u \left(\frac{\partial \Phi(x,t)}{\partial x} \right)^2 \right]$$

$$-E_J \cos[\Phi(L,t) - \Phi(-L,t) + \varphi]$$

4. Luttinger parameters and spinless Luttinger Hamiltonian: $g = \pi\sqrt{n_s S/(m e_c)}$, $u = \sqrt{n_s S e_c/(4m)}$, $\varphi = \Phi/\Phi_0^*$, $\Phi_0^* = 2e/h$, $\Phi(x,t) \to \Phi(x,t) - \varphi x/(2L)$.
5. Comment:$1/g$ is the dimensionless zero-frequency impedance of the superconducting wire: when $g > 1$ one has a Tomonaga-Luttinger liquid (TLL) with an attractive interaction; for $g < 1$, the interaction is repulsive. $g = 1$ describes an essentially free theory. Clean aluminum wires allow for $g > 1$ and, thus, the dynamics of the phase fluctuations is described by an attractive TLL.

0.2 The dc-SQUID

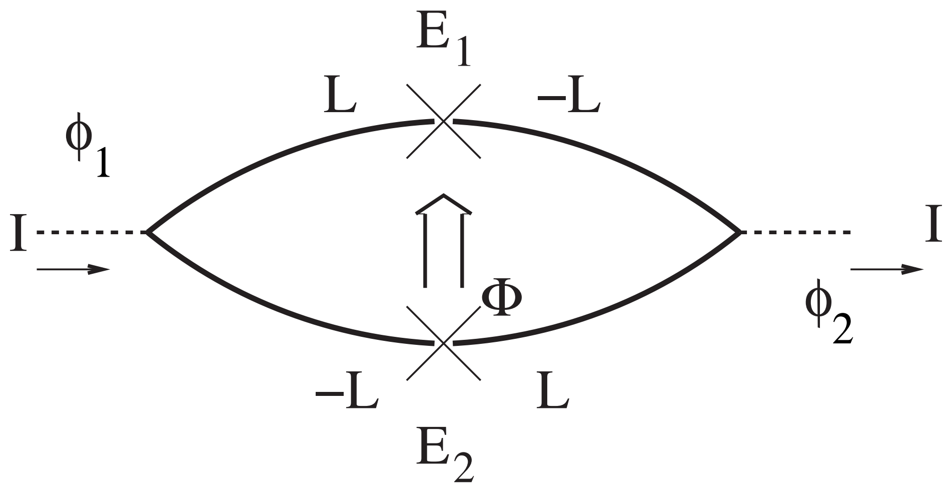

Figure 3: The dc-SQUID.

- The dc-SQUID

1. Two-field, two-junction Luttinger-like Hamiltonian:

$$H_{\rm dc} = \frac{g}{4\pi}\int_{-L}^{L} dx \sum_{j=1,2}\left[\frac{1}{u}\left(\frac{\partial\Phi_j(x,t)}{\partial t}\right)^2 + u\left(\frac{\partial\Phi_j(x,t)}{\partial x}\right)^2\right]$$

$$-E_1\cos\left[\Phi_1(L) - \Phi_2(L) + \frac{\varphi}{2}\right] - E_1\cos\left[\Phi_1(-L) - \Phi_2(-L) - \frac{\varphi}{2}\right]$$

where $\Phi_a(x,t) \to \Phi_a(x,t) - \varphi x/(4L)$, E_1, E_2 are the Josephson energies of the junctions.

2. Usually: current driven.

- Classical description: not enough

  1. rf-SQUID

  $$\Phi(L,t) - \Phi(-L,t) \approx 0 \mod 2\pi \Rightarrow E[\varphi] \approx -E_J \cos(\varphi) \Rightarrow$$

  Josephson current

  $$I[\varphi] = \frac{2e}{c}\frac{\partial E[\varphi]}{\partial \varphi} \equiv I_J \sin(\varphi)$$

  2. dc-SQUID

  $$[\Phi_1(L,t) - \Phi_2(-L,t)] + [\Phi_2(L,t) - \Phi_1(L,t)] = -\varphi \mod 2\pi$$

  Thus

  $$[\Phi_1(L,t) - \Phi_2(-L,t)] = \frac{\Delta}{2} - \frac{\varphi}{2} \ ; \ [\Phi_2(L,t) - \Phi_1(-L,t)] = -\frac{\Delta}{2} + \frac{\varphi}{2}$$

  Current conservation at the junction fixes Δ

  $$I = I_1 \cos\left[\Delta + \frac{\varphi}{2}\right] + I_2 \cos\left[\Delta - \frac{\varphi}{2}\right]$$

  3. $u(\Phi_0^*)^2/L/E_j \leq 1 \Rightarrow$ strong effect of phase fluctuations of the order parameter $\Rightarrow$ <u>field theory approach</u>

Josephson flux Qu-bits

- Josephson flux qubit $\equiv$ superconducting ring interrupted by one or several JJ

   1. persistent currents flow
   2. magnetic fluxes are enclosed

- simplest design is rf-SQUID: when $g > 1$, it behaves as a stable two level quantum system if the superconducting loop is very large and the applied flux is close to $\Phi_0^*/2$.

- BUT, large loops means large sensitivity to noise!

- WAY OUT: small loops with more (3,4,..) junctions

- **Boundary Field Theory provides a fully quantum analysis of the relationship between loop size and emergence of a two level quantum system in a superconducting circuit made with Josepson junctions**

Boundary Field Theory describing the rf-SQUID

- Field Theory for the bulk

1. Even/odd parity combinations

$$\Phi_{e/o}(x) = \frac{1}{\sqrt{2}}[\Phi(x,t) \pm \Phi(-x,t)]$$

Only $\Phi_o(x) \to \Phi(x)$ matters for the interaction.

2. Mode expansion of the relevant fields
$\Phi(x,t) = \phi_0 - \frac{2\pi}{L}Px + \frac{2\pi}{L}\frac{\tilde{P}}{g}ut +$

$$\frac{i}{\sqrt{2g}} \sum_{n \neq 0} \left[\frac{\alpha_R(n)}{n} e^{ik_n(x-ut)} + \frac{\alpha_L(n)}{n} e^{ik_n(x+ut)} \right] \quad .$$

3. $\Theta(x,t) = \theta_0 - \frac{2\pi}{L}\tilde{P}x + \frac{2\pi}{L}gPut +$

$$i\sqrt{\frac{g}{2}} \sum_{n \neq 0} \left[\frac{\alpha_R(n)}{n} e^{ik_n(x-ut)} - \frac{\alpha_L(n)}{n} e^{ik_n(x+ut)} \right] \quad .$$

4. Cross relations between the derivatives

$$g\frac{\partial \Phi(x,t)}{\partial x} = \frac{1}{u}\frac{\partial \Theta(x,t)}{\partial t} \quad ; \quad \frac{\partial \Phi(x,t)}{\partial x} = \frac{1}{gu}\frac{\partial \Theta(x,t)}{\partial t}$$

5. Current/charge density

$$j(x,t) = \sqrt{2}e^* gu \frac{\partial \Phi(x,t)}{\partial x} \quad , \; \rho(x,t) = \frac{\sqrt{2}e^* g}{u}\frac{\partial \Phi(x,t)}{\partial t}$$

6. Hamiltonian

$$H = \frac{\pi u}{L}\left[\frac{P^2}{g} + g(\tilde{P})^2\right] + \frac{2\pi u}{L}\sum_{n=1}^{\infty}[\alpha_R(-n)\alpha_R(n) + \alpha_L(n)\alpha_L(-n)]$$

7. Primary fields

$$V_{Q,\tilde{Q}}(x,t) =: \exp[i\tilde{Q}\Phi(x,t) + iQ\Theta(x,t)] :$$

- Junctions ⇒ Boundary Interactions

1. Basic commutators

$$\left[-\frac{\partial\Phi(x,t)}{\partial x}, V_{Q,\tilde{Q}}(x',t)\right] = 2\pi Q\delta(x-x')V_{Q,\tilde{Q}}(x',t)$$

and

$$\left[-\frac{\partial\Theta(x,t)}{\partial x}, V_{Q,\tilde{Q}}(x',t)\right] = 2\pi\tilde{Q}\delta(x-x')V_{Q,\tilde{Q}}(x',t)$$

⇒ $V_{Q,\tilde{Q}}(x,t)$ changes the eigenvalue of $\tilde{P}$ by an amount $\propto \tilde{Q}$, the eigenvalue of P by an amount $\propto Q$.

2. Construction of boundary operators ("Delayed Evolution of Boundary Conditions" - DEBC).

$$x \to \pm x_{\text{Boundary}} \Rightarrow V_{Q,\tilde{Q}}(x,t) \longrightarrow V^{(B)}(t)$$

3. Energy conservation ⇒ (two)-boundary conditions

$$\frac{gu}{2\pi}\frac{\partial\Phi(L,t)}{\partial x} + \sqrt{2}\bar{E}_J\sin[\sqrt{2}\Phi(L,t)+\varphi] = 0 \;;\; \Phi(0,t)=0$$

- Neumann boundary conditions

1. Two boundary conditions at weak coupling

$$\bar{E}_J = 0 \Rightarrow \frac{\partial\Phi(L,t)}{\partial x} = \Phi(0,t) = 0$$

$$\Phi(L,t) = i\sqrt{\frac{2}{g}}\sum_n(-1)^n\,\frac{\alpha(n)}{n+\frac{1}{2}}e^{-i\frac{\pi}{L}\left(n+\frac{1}{2}\right)ut}$$

2. Boundary operators

$$V_n^{(N)}(t) =: \exp\left[i\sqrt{2}n\Phi(L,t)\right] : \;\;;\;\; (\tilde{Q}=\sqrt{2}n)$$

3. Scaling dimensions $h_n^{(N)}$

$$\langle V_{-n}^{(N)}(t) V_n^{(N)}(t') \rangle \sim_{u(t-t')/L \ll 1} \frac{1}{\left[\frac{u}{L}(t-t')\right]^{\frac{4n^2}{g}}} \Rightarrow h_n^{(N)} = \frac{2n^2}{g}$$

4. Higher-harmonics operators involved in the O.P.E.'s of boundary fields

$$V_n^{(N)}(t) W_{n'}^{(N)}(t') \approx_{t' \to t} \left[\frac{\pi u}{L}(t-t')\right]^{-h_n^{(N)} - h_{n'}^{(N)} + h_{n+n'}^{(N)}} V_{n+n'}^{(N)}(t')$$

5. Effective boundary interactions and running couplings

$$\tilde{H}_J = -\frac{1}{2} \sum_{n=1}^{\infty} \left\{ E_n e^{in\varphi} V_n^{(N)} + E_n e^{-in\varphi} V_{-n}^{(N)} \right\} \quad ; \quad g_n = \left(\frac{L}{a}\right)^{1 - \frac{2n^2}{g}} E_n$$

6. Renormalization group equations

$$\frac{dg_1}{d \ln(L/L_0)} = \beta_1(g_1, g_2) = \left(1 - \frac{2}{g}\right) g_1 + g_1 g_2$$

$$\frac{dg_2}{d \ln(a/a_0)} = \beta_2(g_1, g_2) = \left(1 - \frac{8}{g}\right) g_2 + (g_1)^2$$

7. Solutions

$$g_1(L) \approx g_1(L_0) \left(\frac{L}{L_0}\right)^{\left(1 - \frac{2}{g}\right)} \quad ; \quad g_2(L) \approx \frac{(g_1^2(L))}{1 + 4/g} \left[1 - \left(\frac{L}{L_0}\right)^{-1 - \frac{4}{g}}\right]$$

8. Conclusion: $g < 2 \Rightarrow$ irrelevant interaction (stable weakly coupled fixed point): the theory is perturbative in E_J. On the other hand, for $g > 2$ all the higher harmonics become relevant (for instance $(g_2(L)/g_1(L)) \propto \left(\frac{L}{L_0}\right)^{1 - \frac{2}{g}}$).

9. The scaling ceases to be valid at $\bar{L}_*$ at which $g_1(\bar{L}_*) \sim 1$, that is

$$\bar{L}_* = 2\pi a \left(\frac{u}{a E_J}\right)^{\frac{g}{g-2}}$$

- Dirichlet boundary conditions

  1. Two boundary conditions at strong coupling

  $$E_J \to \infty \Rightarrow \Phi(0,t) = 0 \quad ; \quad \sqrt{2}\Phi(L,t) + \varphi = 2\pi k$$

  2. Eigenvalues of P and dual fields

  $$P_k = -k - \frac{\varphi}{2\pi} \quad ; \quad \frac{\partial\Theta(L,t)}{\partial x} = \frac{\partial\Theta(L,t)}{\partial x} = 0$$

  3. Partition function

  $$\mathcal{Z}[\varphi] = \frac{1}{\eta(e^{-\beta\frac{\pi u}{L}})} \sum_{k\in Z} \exp\left[-\beta\frac{g\pi u}{L}\left(-\frac{\varphi}{2\pi} + k\right)^2\right]$$

  4. Josephson current

  $$I[\varphi] = -\lim_{\beta\to\infty} \frac{1}{\beta}\frac{\partial \ln \mathbf{Z}_D[\varphi]}{\partial\varphi} \propto \varphi - [\varphi]$$

  5. Allowed boundary perturbations: must change P_k by $n/\sqrt{2}$

  $$V_n^{(D)}(t) =: \exp\left[i\frac{n}{\sqrt{2}}\Theta(L,t)\right] : \quad ; \quad h_n^{(D)} = \frac{gn^2}{2}$$

  6. Most relevant boundary perturbation

  $$H^{(D)} = -Y\{: e^{\frac{i}{\sqrt{2}}\Theta(L,t)} : + : e^{-\frac{i}{\sqrt{2}}\Theta(L,t)} :\}$$

  7. Physical meaning of $H^{(D)}$: $P = P_k \Rightarrow$ "zero-mode" (inductive) contribution to the total energy $E_k^{(0)} = \frac{\pi u}{L}\left(k + \frac{\varphi}{2\pi}\right)^2$

  8. At $\varphi = \pi$, $E_0^{(0)} = E_1^{(0)} \Rightarrow H^{(D)}$ representing "jumps" back and fro the minima, i.e., real-time version of instanton solutions interpolating between the minima.

  9. From the renormalization group equations

  $$Y(L) = Y(L_0)\left(\frac{L}{L_0}\right)^{1-\frac{g}{2}}$$

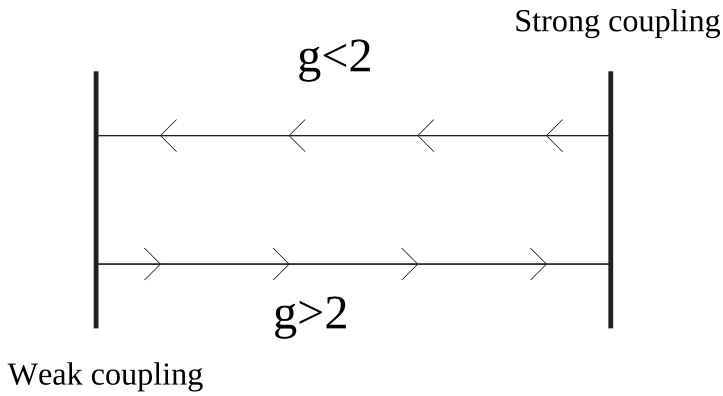

Figure 4: Phase diagram

10. $g > 2 \Rightarrow \tilde{H}_B$ is an irrelevant interaction (stable strongly coupled fixed point): the theory is perturbative in Y.

- Application: transport properties

1. $g < 2 \Rightarrow$ perturbative calculations in $\bar{E}_J$

$$I[\varphi] = 2e^*\bar{E}_J \frac{\langle \mathbf{T}_\tau[: \sin[\sqrt{2}\Phi(\tau) + \varphi] : e^{\left[\bar{E}_J \int_0^\beta \, d\tau \, :\cos[\sqrt{2}\Phi(\tau)+\varphi]:\right]}\rangle}{\langle \mathbf{T}_\tau[\exp\left[\bar{E}_J \int_0^\beta \, d\tau \ : \cos[\sqrt{2}\Phi(\tau) + \varphi] :\right]\rangle}$$

$$\approx 2e^*\bar{E}_J \sin(\varphi) + 2e^*(\bar{E}_J)^2 A \sin(2\varphi)$$

2. $g < 2 \Rightarrow \tilde{H}_B$ is an irrelevant interaction; the theory is perturbative in λ. For $g > 2$ $\tilde{H}_B$ is a relevant interaction, which drives the system out of the strongly coupled fixed point.
3. Normal component of the dc-current $\Rightarrow$ calculation at imaginary times (it is time independent)
4. Applying a bias voltage $V \Rightarrow$ "opening" the SQUID $\Rightarrow$ equivalence to the junction chain: mode expansion

$$\Phi(L,t) = q + \frac{2\pi}{L}\tilde{P}ut + i\sqrt{\frac{2}{g}}\sum_{n\neq 0}(-1)^n \frac{\alpha(n)}{n} e^{-i\frac{\pi}{L}nut}$$

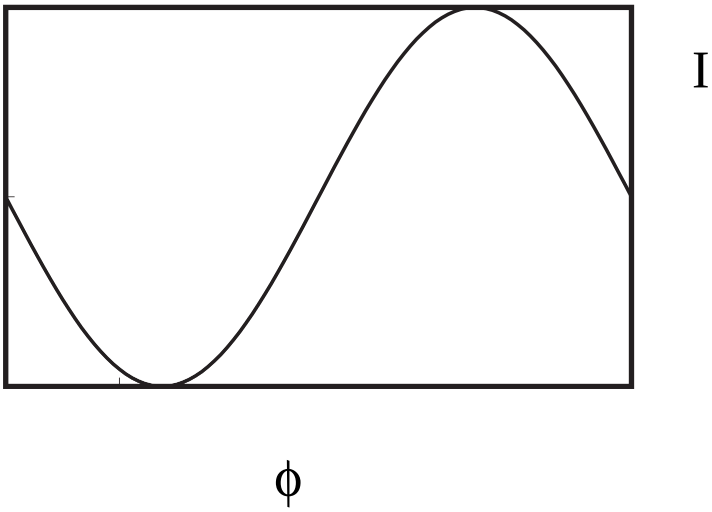

Figure 5: Sinusoidal current: first harmonics

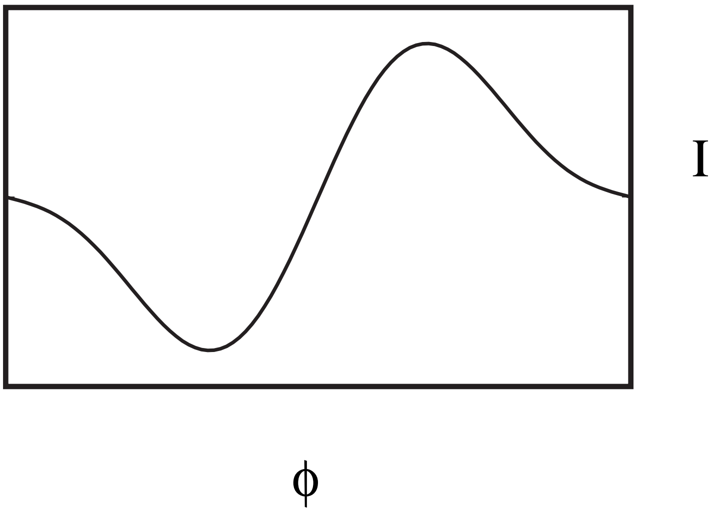

Figure 6: Josephson current: second harmonics

5. dc-current

$$I \approx 2e^{*}(\bar{E}_J)^2 \int_0^{\beta} d\tau' \, \langle \mathbf{T}_\tau [: \sin[\sqrt{2}\Phi(i\tau) + ie^{*}V\tau] : \times$$

$$: \cos[\sqrt{2}\Phi(i\tau') + ie^* V\tau'] :]\rangle$$

6. "Linear response" limit: $u/L \gg e^* V$

$$I = \frac{e^* \pi (\bar{E}_J)^2 L \Gamma[1 - \frac{2}{g}]}{\pi u} B\, V$$

7. Kane-Fisher limit: $u/L \ll e^* V$

$$I = \frac{2e^* (\bar{E}_J)^2 L}{\pi u} \sin\left[\frac{\pi}{g}\right] \Gamma\left[1 - \frac{2}{g}\right] \left[\frac{2Le^* V}{\pi u}\right]^{\frac{2}{g}-1}$$

8. Josephson current at finite voltage bias: time dependent $\Rightarrow$ real time calculation.

$$I_J = \langle \bar{E}_J : \sin[\sqrt{2}\Phi(t) + e^* V t] :\rangle = \bar{E}_J \sin[e^* V t]$$

Negligible, as $u/L \ll e^* V$.

9. Current operator in dual representation

$$j(t) = \frac{e^*}{2\pi} \frac{\partial \Theta(L,t)}{\partial t}$$

10. $\varphi \neq 2\pi k + \pi \Rightarrow$ non degenerate ground state

$$I[\varphi] \approx \frac{e^* g u}{2\pi L}\varphi - e^* \frac{2L\Gamma[1-2g]}{2\pi u} Y^2 \frac{\partial}{\partial \varphi}\left\{ \frac{\Gamma[g(1-\frac{\varphi}{2\pi})]}{\Gamma[1-g(1+\frac{\varphi}{2\pi})]}\right.$$

$$\left. + \frac{\Gamma[g(1+\frac{\varphi}{2\pi})]}{\Gamma[1-g(1-\frac{\varphi}{2\pi})]}\right\} = \left[\frac{e^* g u}{2\pi L} - Y^2 G[g]\right]\varphi$$

11. $\varphi = \pi +$ instantons $\Rightarrow$ removal of the degeneracy

$$\delta E_{\mathrm{GS}}[\varphi - \pi] = -Y - \left(\frac{gu}{2L}\right)^2 \frac{\beta(\varphi-\pi)^2}{2Y} + \frac{gu}{4\pi L}(\varphi - \pi)^2$$

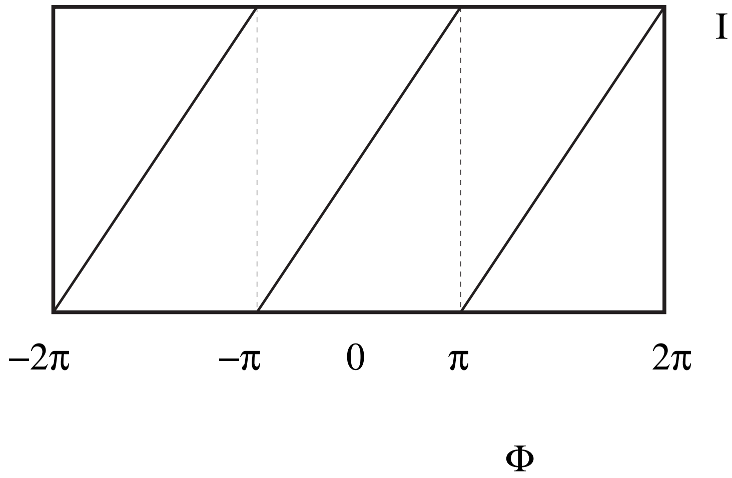

Figure 7: Josephson current: sawtooth

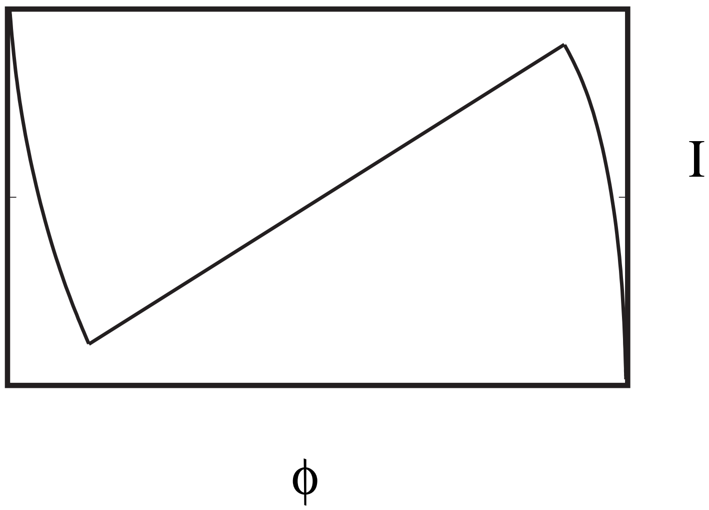

Figure 8: Josephson current: smoothed sawtooth

12. Josephson current for $\varphi \sim \pi$

$$I \approx e^* \frac{gu}{2\pi L}(\varphi - \pi) - \left(\frac{gu}{2L}\right)^2 \frac{(\varphi - \pi)}{Y}$$

Smoothing down of the sawtooth $I[\varphi]$.

13. Current operator in the dual representation

$$\tilde{j}(\tau) = \frac{g(e^*)^2}{2\pi}V + 4Y\frac{e^* g u}{L}\sin(\frac{\Theta(i\tau)}{\sqrt{2}} - ie^* V\tau) :$$

14. Dual formula for the current

$$I = \frac{g(e^*)^2}{2\pi}V + \frac{2e^* Y^2 L}{\pi}\sin[g\pi]\,\Gamma[1-2g]\left[\frac{2Le^* V}{\pi u}\right]^{2g-1}$$

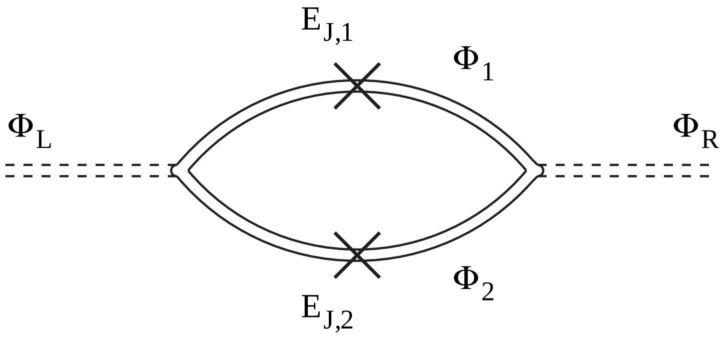

Figure 9: dc SQUID with the contacts

Boundary Field theory describing a dc-SQUID

- The theory

1. Even/odd combinations of the basic fields

$$\Phi_{o/e,j}(x) = \frac{1}{\sqrt{2}}[\Phi_j(x) \pm \Phi_j(-x)] \;\; ; \;\; 0 \leq x \leq L$$

2. Effective fields

$$X(x) = \frac{1}{\sqrt{2}}[\Phi_{e,1}(x) - \Phi_{e,2}(x)] \;\; ; \;\; \xi(x) = \frac{1}{\sqrt{2}}[\Phi_{o,1}(x) - \Phi_{o,2}(x)]$$

3. Boundary conditions for $X(x,t)$
   Energy conservation $\Rightarrow$ Boundary conditions at $x = L$

$$\frac{gu}{2\pi}\frac{\partial X(L,t)}{\partial x} + \bar{E}_1 \sin\left[X(L,t) + \xi(L,t) + \frac{\varphi}{2}\right]$$

$$+\bar{E}_2 \sin\left[X(L,t) - \xi(L,t) - \frac{\varphi}{2}\right] = 0 \;\; ; \;\; \frac{\partial X}{\partial x}(0,t) = 0$$

4. Boundary conditions for $\xi(x,t)$
   Energy conservation $\Rightarrow$ Boundary conditions at $x = L$

$$\frac{gu}{2\pi}\frac{\partial \xi(L,t)}{\partial x} + \bar{E}_1 \sin\left[X(L,t) + \xi(L,t) - \frac{\varphi}{2}\right]$$

$$+\bar{E}_2 \sin\left[X(L,t) - \xi(L,t) - \frac{\varphi}{2}\right] = 0 \;\; ; \;\; \xi(0,t) = 0$$

5. Dual fields: $\Theta(x,t), \Theta_\xi(x,t)$.
6. Primary fields

$$V_{n_X,n_\xi,\tilde{n}_X,\tilde{n}_\xi}(x,t) =: \exp[i(n_X X(x,t) + n_\xi \xi(x,t)$$

$$+\tilde{n}_X \Theta_X(x,t) + \tilde{n}_\xi \Theta_\xi(x,t))] :$$

- Neumann boundary conditions
  1.
  $$\bar{E}_1 = \bar{E}_2 = 0 \Rightarrow \frac{\partial X(L,t)}{\partial x} = \frac{\partial \xi(L,t)}{\partial x} = 0$$
  2. Boundary interaction
  $$H_B = -\frac{\bar{E}_1}{2} \sum_{a=\pm 1} V^{(N)}_{a,a}(t) - \frac{\bar{E}_1}{2} \sum_{a=\pm 1} V^{(N)}_{a,-a}(t)$$
  3. Boundary vertex and scaling dimensions
  $$V_{a,b}(i\tau) =: \exp\left\{ i\left[bX(i\tau) + a\xi(i\tau)\right] - a\frac{\varphi}{2}\right\}$$
  $$\langle \mathbf{T}_\tau [V_{a,b}(i\tau) V_{a',b'}(i\tau')]\rangle \approx \left[\frac{\pi u}{L}|\tau - \tau'|\right]^{\frac{2(aa'-1)}{g}} \delta_{b+b',0} \Rightarrow h_{a,b} = \frac{2}{g}$$
  4. Running couplings
  $$G_j(L) = \left(\frac{L}{2\pi a}\right)^{1-\frac{2}{g}} \frac{aE_j}{u}$$
  5. Renormalization group equations
  $$\frac{d \ln G_j(L)}{d \ln (L/L_0)} = \beta_j(G_1, G_2) = \left[1 - \frac{2}{g}\right] G_j(L)$$
  Irrelevant couplings for $g < 2$ (Neumann fixed point I.R. stable). Relevant couplings for $g > 2 \Rightarrow$ nonperturbative theory for $L \sim 2\pi a \left(\frac{u}{aE_{J,\min}}\right)^{\frac{g}{g-2}}$.
  6. Mode expansion of $X(L,t)$
  $$X(L,t) = q_X + \frac{\pi u t}{L}\frac{\tilde{P}_X}{g} + i\sqrt{\frac{2}{g}} \sum_{n \neq 0} (-1)^n \frac{\alpha_X(n)}{n} e^{-i\frac{\pi n}{L} ut}$$

7. Mode expansion of $\xi(L,t)$

$$\xi(L,t) = -\sqrt{\frac{2}{g}}\sum_{n\neq 0}(-1)^n \frac{a_\xi(n)}{n} e^{-i\frac{\pi}{L}(n+\frac{1}{2})ut}$$

8. Partition function

$$\prod_{n=0}^{\infty}\left\{\left[\frac{\sum_{m\in Z} e^{-\beta\frac{2g\pi u}{L}m^2}}{1-q^{n+\frac{1}{2}}}\right]\left[\frac{1}{1-q^{n+1}}\right]\{1+\frac{2L}{\pi u}\beta \bar{E}_{J,1}\bar{E}_{J,2}B\cos(\varphi)\}\right.$$

9. Josephson current

$$I[\varphi] = e^*\frac{2L}{\pi u}B\bar{E}_{J,1}\bar{E}_{J,2}\sin(\varphi)$$

- Dirichlet boundary conditions

1. $\bar{E}_j \to \infty \Rightarrow$

$$X(L,t) = \pi(n_1+n_2) \quad ; \quad \xi(L,t) - \frac{\varphi}{2} = \pi(n_1-n_2)$$

2. Zero-mode eigenvalues

$$P_\xi = \left[\frac{\varphi}{4\pi} + \frac{m}{2}\right]$$

3. Dual fields: $\frac{\partial\Theta_X(L,t)}{\partial x} = \frac{\partial\Theta_\xi(L,t)}{\partial x} = 0 \Rightarrow$ dual Hamiltonian

$$H_D[\Theta_X,\Theta_\xi] = \frac{2\pi u g}{L}(P_\xi)^2 + \sum_{n\neq 0}[\alpha_X(-n)\alpha_X(n-1) + \alpha_\xi(-n)\alpha_\xi(n)]$$

4. Dual boundary Hamiltonian

$$\tilde{H}_{dc,J} = Y_1[: e^{\frac{i}{2}[\Theta_X+\Theta_\xi]} : + : e^{-\frac{i}{2}[\Theta_X+\Theta_\xi]} :] + Y_2[: e^{\frac{i}{2}[\Theta_X-\Theta_\xi]} : +$$

$$: e^{-\frac{i}{2}[\Theta_X-\Theta_\xi]} :] + Y_X[: e^{i\Theta_X} : + : e^{-i\Theta_X} :] + Y_\xi[: e^{i\Theta_\xi} : + : e^{-i\Theta_\xi} :]$$

5. Scaling of the fugacities

$$\frac{Y_{X(\xi)}(L)}{Y_{X(\xi)}(uT_x)} = \left(\frac{L}{uT_x}\right)^{1-4g} \quad ; \quad \frac{Y_{1(2)}(L)}{Y_{1(2)}(uT_x)} = \left(\frac{L}{uT_x}\right)^{1-2g}$$

6. Instanton size

$$T_x \sim \sqrt{gL/[\pi u(E_{J,1}+E_{J,2})]}$$

7. Partition function

$$\mathbf{Z}_D = \prod_{n=0}^{\infty}\left[\left(\frac{1}{1-q^{n+\frac{1}{2}}}\right)\left(\frac{1}{1-q^{n+1}}\right)\right]\sum_{k\in Z} e^{\left[-\beta\frac{\pi u g}{2L}\left(k-\frac{\varphi}{2\pi}\right)^2\right]}$$

8. Josephson current

$$I[\varphi] = \frac{e^* g u}{4\pi L}\{\varphi - [\varphi]\}$$

9. Conclusions

   $g < 2 \Rightarrow$, phase slips at the two junctions as the most relevant perturbation to the Dirichlet fixed point. The strongly coupled fixed point is not infrared stable. $g > 2 \Rightarrow$, irrelevant perturbation that, smooths down the edges of the sawtooth shape of the Josephson current.

- Transport properties

1. Boundary conditions at the Y-junctions I

$$\frac{\partial}{\partial x}[\Phi_a(x,t)+\Phi_{u,a}(x,t)+\Phi_{d,a}(x,t)]\Big|_{x=0} = 0$$

2. Boundary conditions at the Y-junctions II

$$-2\Phi_a(0,t)+\Phi_{u,L}(0,t)+\Phi_{d,a}(0,t) = \Phi_{u,a}(0,t)-\Phi_{d,a}(0,t) = 0$$

3. Linear combinations

$$\Phi_{u/d}(x,t) = \frac{1}{\sqrt{2}}[\Phi_{u/d,R}(x,t)-\Phi_{u/d,L}(x,t)]$$

4. Relation to X, ξ

$$\Phi_u(x,t) = \frac{1}{2}[X(x,t)+\xi(x,t)] \quad ; \quad \Phi_d(x,t) = \frac{1}{2}[X(-x,t)-\xi(-x,t)]$$

5. Weak coupling: Adding the voltage bias V

$$\Phi_{u/d}(x,t) \to \Phi_{u/d}(x,t) - \frac{e^*V}{2u}x \Rightarrow X(x,t) \to X(x,t) - \frac{e^*V}{u}x$$

6. Current operators across each junction

$$j_{1/2}(i\tau) = 2e^*E_1 \sin\left[X(i\tau) \pm \xi(i\tau) \pm \frac{\varphi}{2} + ie^*V\tau\right]$$

7. dc current

$$I = I_1 + I_2 = \frac{4e^*L}{\pi u}\Gamma[1-\frac{2}{g}]\Big\{[(\bar{E}_1)^2 + (\bar{E}_{J,2})^2]\sin\left[\frac{\pi}{g}\right]$$

$$+2\bar{E}_1\bar{E}_{J,2}\cos(\varphi)\cos\left[\frac{2e^*VL}{u} + \frac{\pi}{g}\right]\Big\}\left[\frac{4Le^*V}{\pi u}\right]^{\frac{2}{g}-1}$$

8. Josephson current

$$I_J = \frac{8e^*\bar{E}_1\bar{E}_{J,2}L}{\pi u}\Gamma[1-\frac{2}{g}]\sin(\varphi)\sin\left[\frac{2e^*VL}{u} + \frac{\pi}{g}\right]\Big\}\left[\frac{4Le^*V}{\pi u}\right]^{\frac{2}{g}-1}$$

9. Comments: Interference contribution, depending on both φ and V derived by treating the plasmon modes as modes of a dynamical fields. Similar effects in nonequilibrium (finite V) Josephson current.

10. Strong coupling: Current operators in the dual representation

$$j_{1/2}(i\tau) = -i\frac{e^*}{2\pi}\left[\frac{\partial\Theta_X(i\tau)}{\partial\tau} \pm \frac{\partial\Theta_\xi(i\tau)}{\partial\tau}\right] = j_X(i\tau) \pm j_\xi(i\tau)$$

11. Source term

$$\tilde{H}_V = -\frac{e^*Vg}{2\pi u}\int_0^L dx\,\frac{\partial\Theta_X(x)}{\partial\tau}$$

12. Corrections to the current

$$\delta j_X = -\frac{4e^*g(Y_1)^2}{\pi}\Big\{\Gamma[1-2g]\Big[\frac{\Gamma[-\frac{a_-}{2}+g]}{\Gamma[1-\frac{a_-}{2}-g]} - \frac{\Gamma[\frac{a_-}{2}+g]}{\Gamma[1+\frac{a_-}{2}-g]}\Big]\Big\}$$

$$+\frac{4e^*g(Y_2)^2}{\pi}\Big\{\Gamma[1-2g]\Big[\frac{\Gamma[\frac{a_+}{2}+g]}{\Gamma[1+\frac{a_+}{2}-g]} - \frac{\Gamma[-\frac{a_+}{2}+g]}{\Gamma[1-\frac{a_+}{2}-g]}\Big]\Big\}$$

13. Corrections to the current

$$\delta j_\xi = -\frac{4e^* g(Y_1)^2}{\pi}\left\{\Gamma[1-2g]\left[\frac{\Gamma[-\frac{a_-}{2}+g]}{\Gamma[1-\frac{a_-}{2}-g]} - \frac{\Gamma[\frac{a_-}{2}+g]}{\Gamma[1+\frac{a_-}{2}-g]}\right]\right\}$$

$$-\frac{4e^* g(Y_2)^2}{\pi}\left\{\Gamma[1-2g]\left[\frac{\Gamma[\frac{a_+}{2}+g]}{\Gamma[1+\frac{a_+}{2}-g]} - \frac{\Gamma[-\frac{a_+}{2}+g]}{\Gamma[1-\frac{a_+}{2}-g]}\right]\right\}$$

14.

$$a_\mp = \frac{2ge^* VL}{\pi u} \mp \frac{u\varphi}{\pi}$$

15. Strong coupling: Small V, φ expansion

$$j_X \approx \frac{(e^*)^2 g}{2\pi}\left\{1 + \frac{16Y^2 gL}{\pi u} C\Gamma[1-2g]\right\} V$$

16.

$$j_\xi \approx \frac{e^* gu}{4\pi L}\left\{1 + \frac{16Y^2 gL}{\pi u} C\Gamma[1-2g]\right\}\varphi$$

Final Remarks

- BFT approach to SQUID devices $\equiv$ 1+1-dimensional boundary field theories for quantum wires with impurities/boundaries;
- Phase diagram of the devices: cross-over between weak and strong coupling fixed points, role of instantons and vertex operators.
- Transport properties: remarkable interference effects in transport phenomena and Josephson current.
- More complex boundary interaction (i.e. considering loops with more than 2 junctions) $\Rightarrow$ finite-coupling IR fixed points? Relevance for stability (insensitivity to noise) of the operational point of flux qu-bit?