Quantum Information, geometry, critical phenomena

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Joint work with

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(Quantum) Phase Transition: dramatic change of the *(ground) state* properties of a quantum system with respect A smooth change of some control parameter e.g., *temperature, external field, coupling constant*,...

Characterization of PTs:

"Traditional":

Local Order parameter (**OP)**, symmetry breaking (**SB**)

(correlation functions/length) Landau-Ginzburg framework

Smoothness of the (GS) Free-energy (I, II,.. order QPTs)

Quantum Information views:

Quantum Entanglement (concurrence, block entanglement,..)

Geometrical Phases

Question: How to map out the phase diagram of a system with no a priori knowledge of its *symmetries* and **op**s?

Answer(?): distinguishability =distance in the information space

Distinguishability of Exps ⟺ Distance in Prob space

Qualitative: the difference between the means should be (much) bigger than the finite size variances....

d(P,Q)= number of (asymptotically) distinguishable preparations between *P* and *Q*: *ds=d**X**/Variance(**X**)*

Quantitative: Fisher Information metric

$$d_F(P,Q=P+dP)=\sum_{i=1}^{E}\frac{(p_i-q_i)^2}{p_i}=\sum_{i=1}^{E}p_i(d\log p_i)^2$$

Problem: Quantumly each Q-preparation $|P\rangle$ defines infinitely many prob distributions, one for each observable: $p_i=|\langle i|P\rangle|^2$
one has to maximize over all possible experiments!

Surprise!

$$\max_{Exps} d_F(P,Q) =$$ *Projective Hilbert space distance!*

$$D_{FS}(P,Q) \equiv \cos^{-1}\frac{|\langle Q|P\rangle|}{\|Q\|\cdot\|P\|} \equiv \cos^{-1}(F)$$

***Q-fidelity*!**

(Wootters 1981)

Statistical distance and geometrical one collapse:

Hilbert space geometry ***is (quantum) information geometry.....***

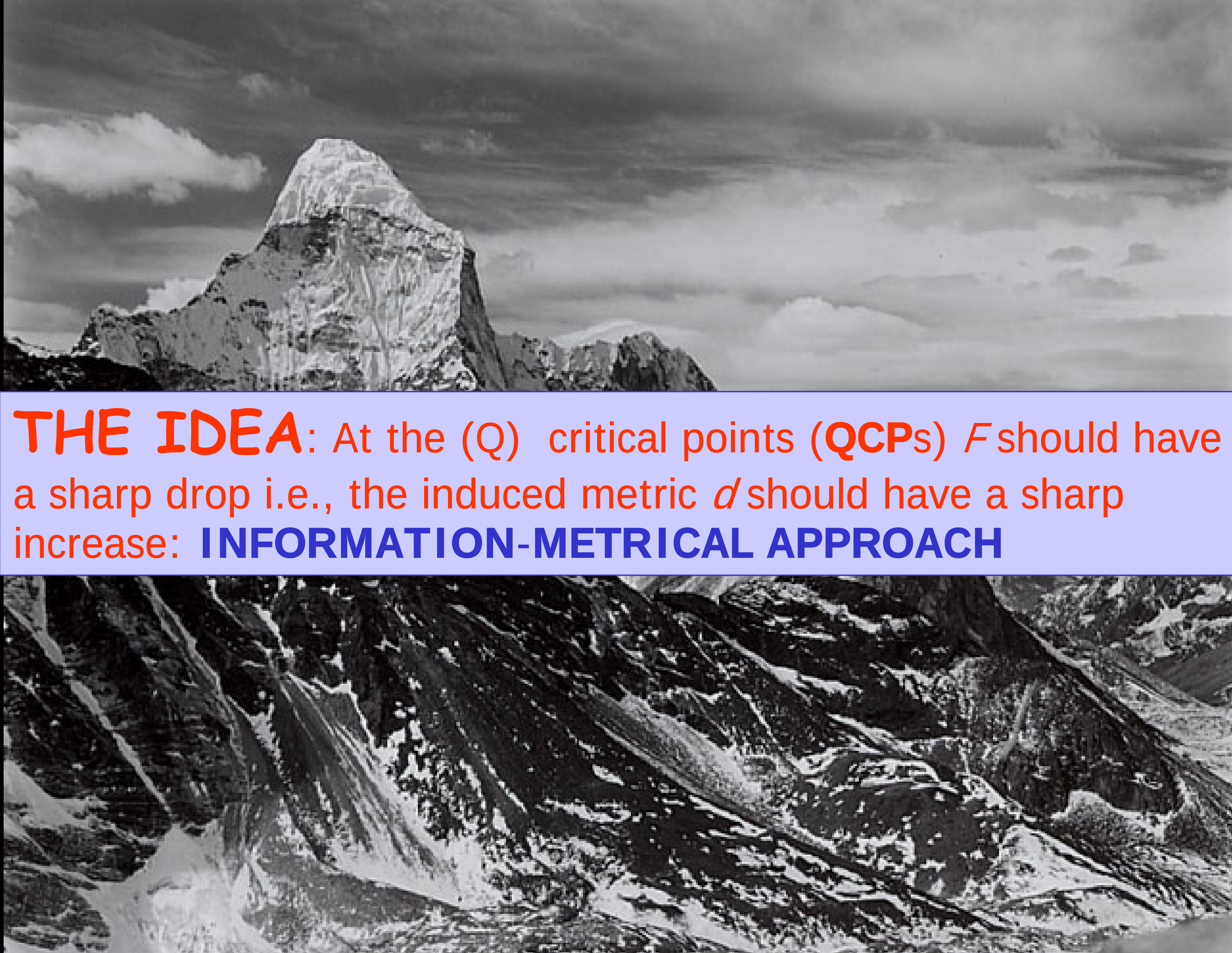
Question: What about non pure preparations ρ_P ?!?

Answer: Bures metric & Uhlmann fidelity!
(Braunstein Caves 1994)

$$d(\rho_0,\rho_1) = \cos^{-1}(F)$$

$$F(\rho_0,\rho_1) = tr\sqrt{\rho_1^{1/2}\rho_0\rho_1^{1/2}}$$

Remark: In the classical case one has commuting objects, simplification ....



THE IDEA: At the (Q) critical points (**QCP**s) *F* should have a sharp drop i.e., the induced metric *d* should have a sharp increase: **INFORMATION-METRICAL APPROACH**

Differential Geometry of QPTs

$H(\lambda) \in L(H)$ Hamiltonians

Control parameters $\lambda \in M$

$|\Psi(\lambda)\rangle \in H$ Ground states

$$ds^2 = \| \Psi_\lambda - \Psi_{\lambda+\delta\lambda} \|^2 = \sum_{\mu\nu} g_{\mu\nu} d\lambda_\mu d\lambda_\nu$$

Perturbation theory

$$|\Psi_0(\lambda+\delta\lambda)\rangle \propto |\Psi_0(\lambda)\rangle + \sum_n |\Psi_n(\lambda)\rangle \frac{\langle \Psi_n(\lambda)|\delta H|\Psi_0(\lambda)\rangle}{E_0(\lambda)-E_n(\lambda)}$$

$$\delta H \equiv H(\lambda+\delta\lambda) - H(\lambda) = \sum_\mu (\partial_\mu H) d\lambda_\mu$$

$$g_{\mu\nu} = \mathrm{Re} \sum_n \frac{\langle \Psi_0(\lambda)|\partial_\mu H|\Psi_n(\lambda)\rangle\langle \Psi_n(\lambda)|\partial_\nu H|\Psi_0(\lambda)\rangle}{(E_0(\lambda)-E_n(\lambda))^2}$$

A Rienmannian metric g on the parameter space is pulled back from the metric on *H*: **QPTs do correspond to singularities of g !**

Free Fermi Systems

$$H_Z = \sum_{ij} c^+{}_i A_{ij} c_j + \frac{1}{2} \sum_{ij} (c^+{}_i B_{ij} c_j{}^+ + h.c)$$

$$A = A^T, B = -B^T$$

L x L Matrix of coupling constants

$$Z \equiv A - B \in M_L(R)$$

$$|\Psi_Z\rangle \propto \exp(\sum_{ij} c_i^+ G_{ij} c_j^+)|0\rangle$$

Gaussian ground-state
(even number of particles)

$$c_i|0\rangle = 0$$

Problem: given *Z* find the antisymmetric matrix *G*

Diagonalization (Lieb-Schulz-Mattis 61)

$$\{c_i\}_{i=1}^L \xrightarrow{Fourier} \{c_k\}_{i=1}^L \xrightarrow{Bogoliubov} \eta_k = \sum_i (g_{ki} c_i + h_{ki} c^+{}_i)$$

$$[H, \eta_k] = -\Lambda_k \eta_k, H = \sum_{k>0} \Lambda_k \eta^+{}_k \eta_k + E_0, \eta_k |\Psi_0\rangle = 0$$

Ground-State Fidelity

The polar decomposition of the physical relevant data $Z = \Lambda_\Phi T$ contains

$$G = -g^{-1}h = \frac{T-1}{T+1}$$

$Sp(\Lambda_\Phi) \subset R_0^+$ — Quasi-particle spectrum

$T \in O_L(R)$ — Many-body **GS** structure

$M_L(R) \to O_L(R) : Z \to T$ Continuous map for Z non-singular

$O_{2L}(R)$ Coherent states (Perelemov)

$$F(Z_1, Z_2) \equiv |\langle \Psi(Z_1) | \Psi(Z_2) \rangle| = \sqrt{|\det \frac{T_1 + T_2}{2}|}$$

$$T_1^{-1}T_2 \in O_L(R) \Rightarrow Sp(T_1^{-1}T_2) = \{\exp(\pm i\Theta_\nu)\}_{\nu=1}^{L/2} \Rightarrow$$

$$F(Z_1, Z_2) = \prod_{\mu=1}^{L} |\frac{1+e^{i\Theta_\mu}}{2}| = \prod_{\nu=1}^{L/2} |\cos(\Theta_\nu)|$$

$$[T_1, T_2] = 0 \Rightarrow F(Z_1, Z_2) = \prod_{\nu=1}^{L/2} |\cos\frac{\theta_1 - \theta_2}{2}|$$

Differential Geometry of QPTs: QuasiFree Fermions

M — Control parameters → $Z \in M_L(R)$ — Coupling constants → $H_Z \in L(H)$ — Hamiltonians

M → $T = e^K = Z|Z|^{-1} \in SO_L(R)$ — Unitary part → $|\Psi_Z\rangle \in H$ — Ground states

$$ds^2 := \| \Psi_Z - \Psi_{Z+\delta Z} \|^2 = 2(1 - F(Z, Z+\delta Z)) \approx \frac{1}{8} tr(dK)^2$$

$$dK = \sum_\mu \frac{\partial K}{\partial \lambda_\mu} d\lambda_\mu$$

$$K = i \oplus_k \theta_k(\lambda) \sigma_y^{(k)}$$

$$dK = \oplus_k (\partial_\mu \theta_k d\lambda_\mu) \sigma_y^{(k)}$$

$$g_{\mu\nu} = \sum_k \partial_\mu \theta_k \partial_\nu \theta_k$$

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} d\lambda_\mu d\lambda_\nu \propto \sum_{\mu\nu} Tr\left(\frac{\partial K}{\partial \lambda_\nu} \frac{\partial K}{\partial \lambda_\mu} \right) d\lambda_\nu d\lambda_\mu$$

$$g_{\mu\nu} = \sum_k \partial_\mu \theta_k \partial_\nu \theta_k$$

If moving from $T_1 = T(\lambda)$ to $T_2 = T(\lambda + \delta\lambda)$ makes $T_1^{-1}T_2$ to develop an eigenvalue close *–1* one has a sharp fidelity drop $\Longleftrightarrow$ $d(T_1, T_2) \equiv \| T_1 - T_2 \|_\infty = \| 1 - T_1^{-1} T_2 \|_\infty$ has a sharp increase

$$T(\lambda) = e^{K(\lambda)}, K(\lambda) \in o_L(R)$$

$$Z_1 = Z(\lambda), Z_2 = Z(\lambda + \delta\lambda)$$

$$F(Z_1, Z_2) = \exp\left(\frac{\delta\lambda^2}{16} Tr\left(\frac{\partial K}{\partial \lambda}\right)^2 + o(\delta\lambda^4)\right)$$

$$K(\lambda) = i \oplus_{\nu=1}^{L/2} \theta_\nu(\lambda) \sigma_{(\nu)}^y \Rightarrow Tr(K')^2 = -2 \sum_{\nu=1}^{L/2} \left(\frac{\partial \theta_\nu}{\partial \lambda}\right)^2$$

$$\theta_\nu = \tan^{-1} \frac{\mathrm{Im}(z_\nu)}{\mathrm{Re}(z_\nu)} \Rightarrow F = \exp\left(-\frac{1}{8} \delta\lambda^2 S_2 + o(\delta\lambda^4)\right)$$

$$Sp(Z) = \{z_\nu\}^L_{\nu=1}$$

$$\Lambda_\nu = | z_\nu |$$

$$S_2 = \sum_{\nu=1}^{L/2} \left(\frac{\mathrm{Re}(z_\nu)\,\mathrm{Im}(z_\nu') - \mathrm{Im}(z_\nu)\,\mathrm{Re}(z_\nu')}{| z_\nu |^2}\right)^2$$

When the quasi-particle spec has a zero $| z_\nu | \to 0$

S_2 has a sharp increase and *F* a sharp decrease

Finite-size scaling and scaling behaviour
The Cyclic complete graph

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} d\lambda_\mu d\lambda_\nu \propto \sum_{\mu\nu} Tr\left(\frac{\partial K}{\partial \lambda_\nu}\frac{\partial K}{\partial \lambda_\mu}\right) d\lambda_\nu d\lambda_\mu$$

$$Eig(g) = \{0, h\}$$

$$h(\mu \neq 1, \gamma \neq 0) = -\frac{L}{16} \frac{(\mu-1)^2 + \gamma^2}{|\mu-1||\gamma|(|\mu-1|+|\gamma|)^2}$$

$$h(\mu = 1, \gamma \neq 0) = -\frac{1}{24}\left(\frac{L}{\gamma}\right)^2$$

$$h(\mu \neq 1, \gamma = 0) = -\frac{1}{8}\left(\frac{L}{\mu-1}\right)^2$$

Quantum Fidelity and Thermodynamics

Uhlmann fidelity between Gibbs states of the same Hamiltonian at different temperatures $Z(\beta) = tr\exp(-\beta H) = \exp(-\beta A(\beta))$

= *partition function*

$$F(\beta_0, \beta_1) = \frac{Z(\beta_0/2 + \beta_1/2)}{\sqrt{Z(\beta_0)Z(\beta_1)}}$$

$$F(\beta_0, \beta_1) = \exp(\beta_0 A(\beta_0)/2 + \beta_1 A(\beta_1)/2 - (\beta_0 + \beta_1)A(\beta_0/2 + \beta_1/2)/2)$$

$$F(T, T + \delta T) \approx \exp(-\delta T^2 \frac{c_V(T)}{8T^2})$$

Singularities of the specific heat $c_V(T)$ shows up at the fidelity level:
Divergencies e.g., lambda points, results in a singular drop of fidelity

Temperature driven phase transitions can be detected by mixed state fidelity in both classical and quantum systems!

Summary & Conclusions

- We analyze the phase diagram of a systems in terms of the induced metric *g* on the parameter space

- Boundaries bewteen different phases can be detected by singularities of *g*

- *g* and related functions show critical scaling behaviour and Universality

- Extensions to finite temperature and to classical systems

Information-Metric approach to (Q)PTs:
Geometrical, ***QI***-theoretic, Universal

The "Hey *dude*! *If you have the GS you have EVERYTHING*..." Slide

Ok cool. Let's suppose you're somehow given the **GS** *e.g.,a MPS*

$$|\Psi(g)\rangle = \sum_{i_1,\ldots,i_N=1}^{d} Tr(A_{i_1}\cdots A_{i_N})\,|\,i_1,\cdots,i_N\rangle$$

$$A_{i_k}(g)\in M_D(C), \qquad g=(g_1,\cdots,g_Q)\in R^Q$$

Please, now tell me where the **QCP**s are.... You may answer:

"Great! Now I compute the ***OP*** and its correlations! ...."

Well... you have infinitely many candidates: Which one?!?

and perhaps ***none*** *of them is gonna work out (e.g.,* ***TOP****order)...*

"Ok, fair enough, then I compute the energy!"

I see, so you want also the H? You didn't ask for that so far...

Anyways, here we go: identically vanishing E(g)=0....

Matrix Product States: the fidelity

$$|\Psi(g)\rangle = \sum_{i_1,\ldots,i_N=1}^{d} Tr(A_{i_1}\cdots A_{i_N})\,|i_1,\cdots,i_N\rangle$$

$$A_{i_k}(g) \in M_D(C), \qquad g=(g_1,\cdots,g_Q)\in R^Q$$

$$E(g_1,g_2) := \sum_{i=1}^{d} A_i(g_1)\otimes A^*{}_i(g_2)$$

Generalized transfer operator

$$\langle\Psi(g_1)|\Psi(g_2)\rangle = \frac{trE(g_1,g_2)^N}{\sqrt{trE(g_1,g_1)^N\, trE(g_2,g_2)^N}}$$

$$F(g-\delta,g+\delta) \approx_{\delta\to 0} 1-\frac{\delta^2}{2}S(g) \approx \exp(-\frac{\delta^2}{2}S(g))$$

Finite-size behaviour and scaling of *S(g)* contain quantitative Information about the **QPT**s

Finite-size scaling

If $spec(E)=\{\lambda_1,\ldots,\lambda_{D^2}\}$ with $|\lambda_i|\ge|\lambda_{i+1}|\,(i=1,\ldots,D^2-1)$

$$S(g)\approx_{N\to\infty}\lim_{g\to g_c}\left[\left(\frac{N\partial\vartheta}{\cosh(N\vartheta/2)}\right)^2+4N\sum_{i=1}^{2}\frac{\lambda^N{}_i}{\lambda^N{}_1+\lambda^N{}_2}\frac{\partial^2}{\partial g_1\partial g_2}\log\lambda_i\right]_{g_1=g_2=g}$$

$\vartheta:=\log(\lambda_2/\lambda_1)$

1st term drops

Regular points: $|\vartheta|>0$ ⇒ $S(g)\propto N$

1st term blows up

Critical points: $\lim_{g\to g_c}\vartheta=0$ ⇒ $S(g_c)\propto N^2$

S(g) has an (super) extensive scaling at regular (critical) Points: enhanced orthogonalization rate a ***QPT*s**

Example: spin chain

$$H = \sum_i [(2+g^2)\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + 2(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 + +2(4-g^2)(S_i^z)^2 + (g+2)^2(S_i^z S_{i+1}^z)^2 + +g(g+2)\{S_i^z S_{i+1}^z, \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}\}] \,,$$

Matrices in $|\Psi(g)\rangle$:

$$\{A_j(g)\} = \{\sigma_z, \sigma_-, g\sigma_+\}$$

Generalized transfer operator:

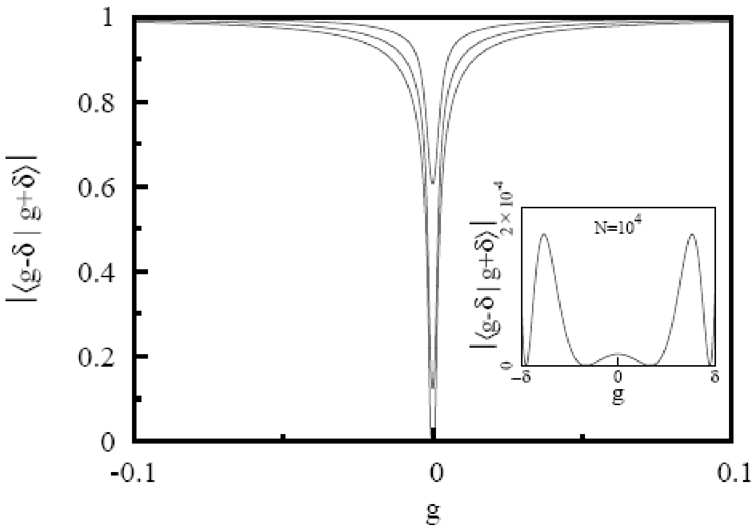
$$E(g_1, g_2) = \sigma_z \otimes \sigma_z + \sigma_- \otimes \sigma_- + g_1 g_2 \sigma_+ \otimes \sigma_+$$

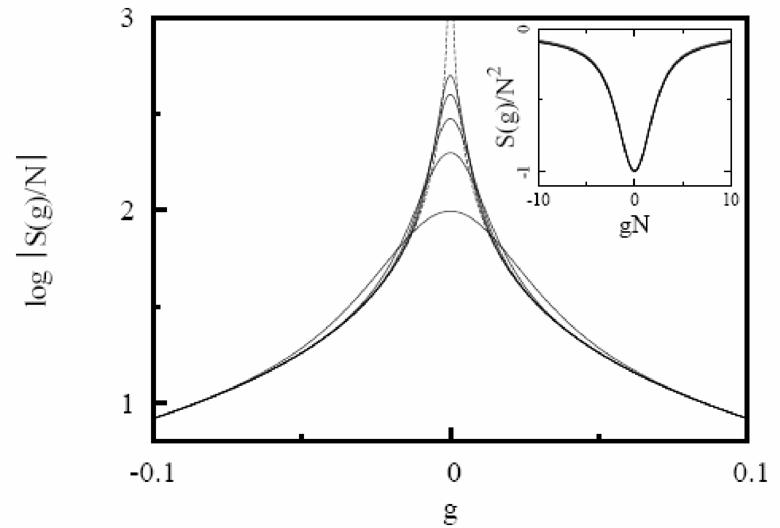
Generalized transfer operator eigenvalues:

$$\{\lambda_k(g_1, g_2)\} = \{-1, -1, 1 \pm \sqrt{g_1 g_2}\}$$

Degeneracy (in modulus) at $g = 0$

Fidelity drop and finite size scaling





$$S(g \neq 0) \;\sim\; -\frac{N}{|g|(1+|g|)^2}$$

$$S(g=0) \;=\; \begin{cases} -N(N-1) & (N \text{ even}) \\ -(N-2)(N-3)/3 & (N \text{ odd}) \end{cases}$$

Second derivative $S(g)$: scaling ~ N everywhere except at the critical point $g = 0$, where $S(g) \sim N^2$

THE XY Model (I)

$$H(\lambda,\gamma) = \sum_{i=1}^{L} \left(\frac{1+\gamma}{2}\sigma_i^x \sigma_{i+1}^x + \frac{1-\gamma}{2}\sigma_i^y \sigma_{i+1}^y + \lambda\sigma_i^z\right)$$

γ =anysotropy parameter, λ =external magnetic field

QCPs:

- $\gamma = 0$ XX line III-order QPT
- $\lambda = \pm 1$ Ferro/para-magnetic II order QPT

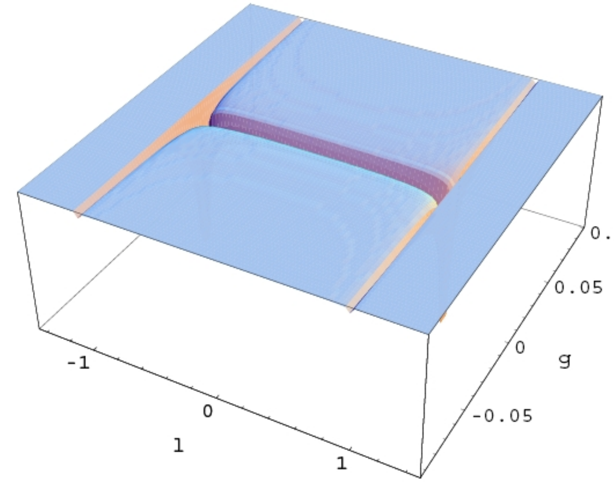
Jordan-Wigner mapping *H* ➡ Free-Fermion system:
EXACTLY SOLVABLE!

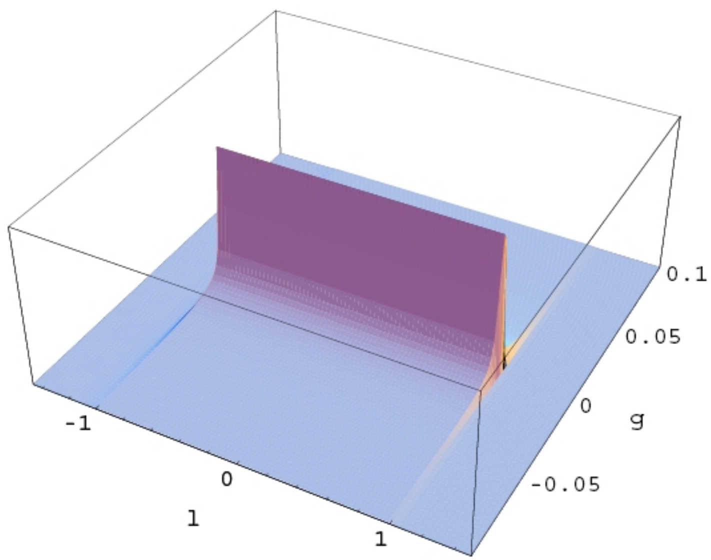
$$\Lambda_k = \sqrt{(\cos\frac{2\pi k}{L} - \lambda)^2 + \gamma^2 \sin^2 \frac{2\pi k}{L}}$$

Quasi-particle spectrum: zeroes in the **TDL** in all the **QCP**s ⬌ Gaplessness of the many-body spectrum

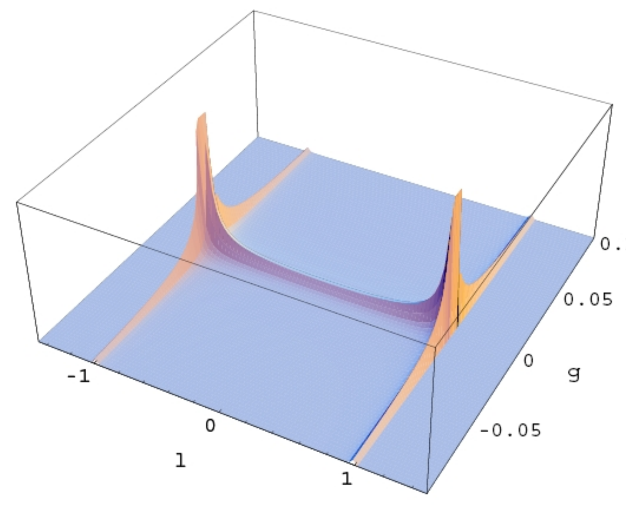
XY Model (III): Overlap Functions

$$F(\lambda+\delta\lambda, \gamma+\delta\gamma) \approx \exp(-\delta\lambda^2 S_2^{\lambda} + \delta\gamma^2 S_2^{\gamma})$$





$S_2^{\gamma}(\lambda, \gamma)$

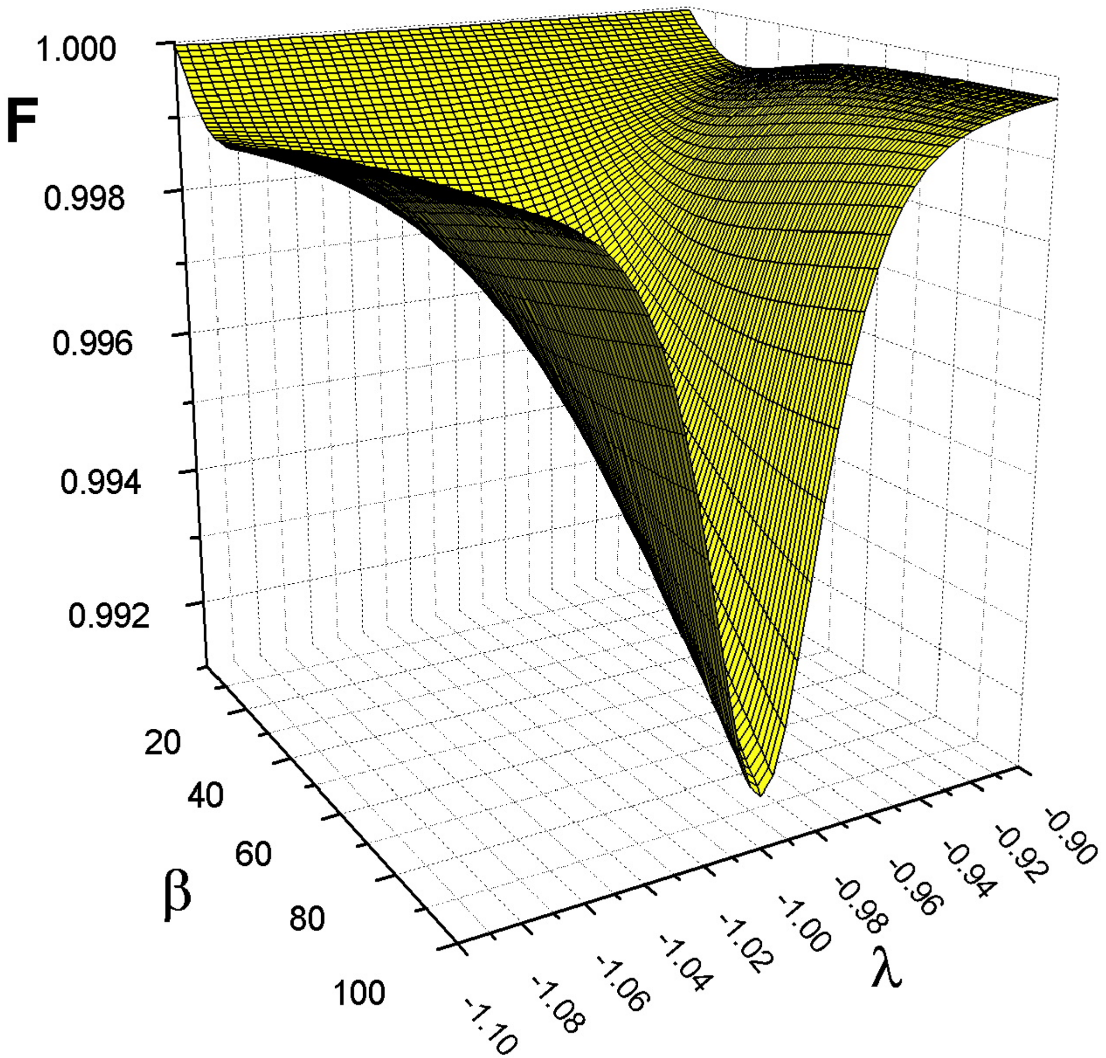


$S_2^{\lambda}(\lambda, \gamma)$

Finite Temperature Fidelity: XY Model

$L = 2001$

$\gamma = 1/2$



Statistical distance

Experiment with E outcomes $\{e_i\}_{i=1}^{E}$ with probabilities $\{p_i\}_{i=1}^{E}$

$$p_i \geq 0, \sum_{i=1}^{E} p_i = 1$$

Different "preparation": $P \equiv \{p_i\}_{i=1}^{E} \rightarrow Q \equiv \{q_i\}_{i=1}^{E}$

Problem: *probs are asymptotyc concepts; with finite samples one has just frequencies with finite statistical fluctuations*

$$\left\{ f_i \equiv \frac{n_i}{N} \right\}_{i=1}^{E} \xrightarrow{N \to \infty} \{p_i\}$$

(Large number law)

How to discriminate different preparations P and Q with a finite size sample*?*

How to quantify the distinguishability *of* P and Q ?

THE XY Model (II)

Ground State

$$|\Psi_0(\lambda,\gamma)\rangle = \otimes_{k>0} \cos\frac{\theta_k}{2}|00\rangle_{k,-k} - i\sin\frac{\theta_k}{2}|11\rangle_{k,-k}$$

$$\theta_k = \cos^{-1}\frac{\cos\frac{2\pi k}{L}-\lambda}{\Lambda_k}$$

$$c_k c_{-k}|00\rangle_{k,-k} = 0$$

$$F = |\langle\Psi_0|\Psi_0^{'}\rangle| = \prod_{k=1}^{L/2}|\cos\frac{\theta_k-\theta_k^{'}}{2}|$$

Fidelity

$$F(\lambda,\lambda+\delta\lambda) = \exp(-\delta\lambda^2 S^{\lambda}{}_2(\lambda) + o(\delta\lambda^4))$$

$$S_2^{\ L}(\lambda,\gamma) = \sum_{k=1}^{M}\Lambda^{-4}{}_k\gamma^2\sin^2\frac{2\pi k}{L}$$

Strong L-dependent peak at $\lambda_c = \pm 1$ universality features in the **TDL**

$$S_2^L(\lambda,\gamma) \xrightarrow{L\to\infty} \begin{cases} O(L^2) & (\lambda=\lambda_c) \\ O(L) & (\lambda\neq\lambda_c) \end{cases}$$

$$S_2^{\ \infty}(\lambda,\gamma) \propto |\lambda-\lambda_c|^{-\alpha(\gamma)} \propto$$

$$\alpha(\gamma) \approx const \approx 1$$

Orthogonalization in the TDL: enhanced @ **QCP**s!

PZ, N. Paunkovic, Phys. Rev E (2006)

Finite-temperature extensions

$$|\Psi_0^\alpha\rangle\langle\Psi_0^\alpha| \mapsto \rho_\alpha(\beta) = Z^{-1}\exp(-\beta H^\alpha)$$

$$F(\rho_0,\rho_1) = tr\sqrt{\rho_1^{1/2}\rho_0\rho_1^{1/2}}$$ Mixed-state fidelity

Example

$$H^\alpha = \sum_\nu \varepsilon_\nu^\alpha (n_\nu + n_{-\nu}) + \Delta_\nu^\alpha (c_\nu^+ c_{-\nu}^+ + h.c.)$$

$$F = \prod_\nu \frac{1 + 1/\sqrt{2}\sqrt{1+\cosh(\beta\Lambda_\nu^0)\cosh(\beta\Lambda_\nu^1) + \sinh(\beta\Lambda_\nu^0)\sinh(\beta\Lambda_\nu^1)\cos(\theta_\nu^0 - \theta_\nu^1)}}{\sqrt{(1+\cosh(\beta\Lambda_\nu^0))(1+\cosh(\beta\Lambda_\nu^1))}}$$

Finite *T* signatures of **QPT**s

Question: can we detect classical I.e.,T-driven PTs by comparing thermal states with (slightly) different Ts?

Work in progress....

A Remark: specific heat and metrics

$$\beta \in R \longrightarrow \rho_\beta := \frac{e^{-\beta H}}{Z(\beta)} \in L(H)$$

Parameter — Thermal state

Note that:

$$c_V(\beta)\beta^{-2} = \langle H^2 \rangle_\beta - \langle H \rangle_\beta^{\ 2} = \| H - \langle H \rangle_\beta \|_\beta^{\ 2}$$

where

$$\langle O \rangle_\beta = tr(O\rho_\beta)$$

$$\| O \|_\beta := \sqrt{\langle O^+ O \rangle_\beta}$$

$$ds \propto \| H - \langle H \rangle_\beta \|_\beta \, d\beta$$

The metric induced on the β line is the ***GNS*** distance between H and $\langle H \rangle_\beta$ associated to the Gibbs state ρ_β


THANKS!

Yet another one: Ground State Fidelity !

PZ, N. Paunkovic, PRE **74** 031123 (2006)

$|\Psi_0(\lambda)\rangle$ Ground State of $H(\lambda)$

$$F(\lambda,\lambda') = |\langle \Psi_0(\lambda) | \Psi_0(\lambda')\rangle|$$

F is a universal (Hilbert space) *geometrical* quantity:
No *a priori* understanding of the **SB** pattern or **OP** is needed

$$d^2(\lambda,\lambda') := \| \Psi_0(\lambda) - \Psi_0(\lambda') \|^2 = 2 - 2\,\mathrm{Re}\langle \Psi_0(\lambda) | \Psi_0(\lambda')\rangle \approx_{\lambda\to\lambda'} \left(\frac{\partial^2 F}{\partial \lambda^2} \right) d^2\lambda$$

THE KEY (naive) IDEA

At the quantum critical points (**QCP**s) *F* should have a sharp drop i.e., the induced metric *d* should have a sharp increase. Weak topology *vs* norm topology: a **METRIC APPROACH**

Free Fermion on Complete graph

$$H = \sum_{i,j=1}^{L} c_i^\dagger A_{i,j} c_j + 1/2 \sum_{i,j=1}^{L} (c_i^\dagger B_{i,j} c_j^\dagger + \text{h.c.})$$

$$A(\mu)_{ij} = 1 + (\mu - 1)\delta_{ij} \,, \quad B(\gamma)_{ij} = \gamma\, \text{sign}(j - i)$$

$$(i, j = 1, \ldots, L)$$

- phase diagram symmetric for $\pm\gamma$
- analytical results for $\gamma = 0$
- I and II order QPT*s*

Boundary conditions I

Long range version of the free-ends chain:

$$B_{ij}(\gamma) = \gamma\,\mathrm{sign}(j-i)$$

Long range version of the cyclic chain (translational invariance):

$$B_{ij}(\gamma) = \gamma\,\mathrm{sign}(j-i)\left[\theta\left(\left\lfloor\frac{L-1}{2}\right\rfloor - |i-j|\right) - \theta\left(|i-j| - L + \left\lfloor\frac{L-1}{2}\right\rfloor\right)\right]$$

(convention: $\theta(0) = 1, \ \mathrm{sign}(0) = 0$)

Cyclic case

Periodic boundary conditions and translational invariance (modulo L)

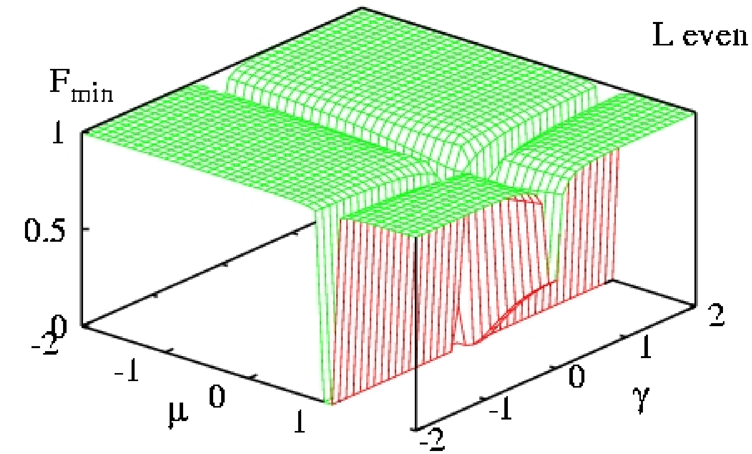
→ A, B are circulant matrices

Circulant matrices are diagonalized by the same unitary:

- sums and products are still circulant
- circulant commute
- inverse (if exists) is circulant

$$\psi^{(\mathrm{circ})}_{jk} = \frac{1}{\sqrt{L}} e^{-i2\pi(j-1)(k-1)/L}$$

Z=A-B, T=Z/|Z| are circulant
and can be easily analyzed analytically
(T and $\tilde{T}$ commute)


L even
F min
1
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-2
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μ
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2
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-1
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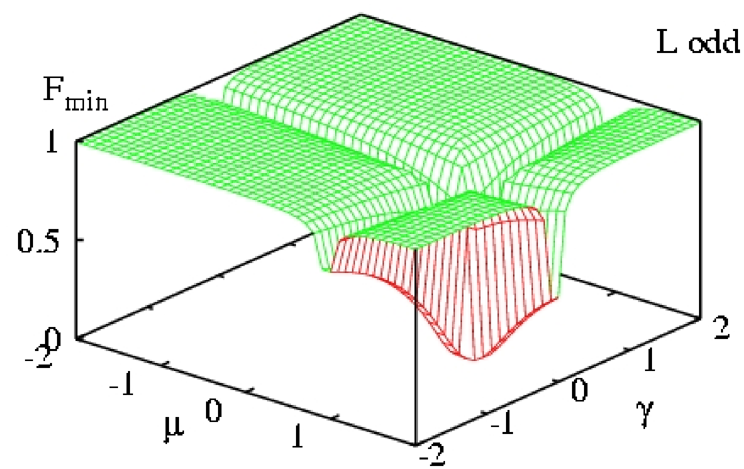
L odd
F min
1
0.5
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-2
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1
0
-1
γ
-2

Loschmidt Echo: Finite T

$$\rho_0(\beta) := 1/Z_0 \exp(-\beta H_0)$$

$$U_1(t) := \exp(-itH_1)$$

$$L_\beta(t) := F(\rho_0(\beta), U_1(t)\rho_0(\beta)U_1^+(t))$$

$$H_\alpha = \sum_\nu \varepsilon_\nu^\alpha (n_\nu + n_{-\nu}) + (\Delta_\nu^\alpha c_\nu^+ c_{-\nu}^+ + h.c.)$$

$$L_\beta(t) = \prod_\nu \frac{1 + 1/\sqrt{2}\sqrt{(1 - \sin^2(2\alpha_\nu)\sin^2(\Lambda_\nu^1 t))\cosh(2\beta\Lambda_\nu^0) + \sin^2(2\alpha_\nu)\sin^2(\Lambda_\nu^1 t) + 1}}{1 + \cosh(\beta\Lambda_\nu^0)}$$

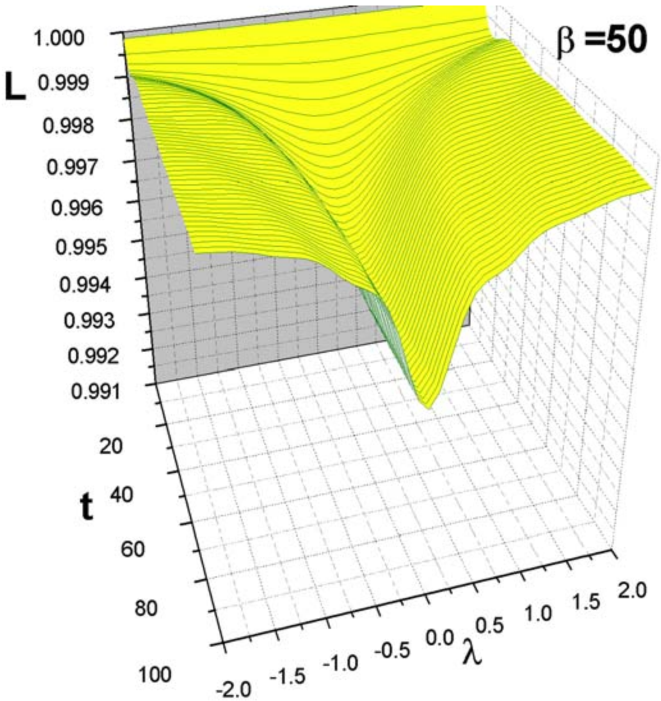
$\Lambda_\nu^\alpha := \sqrt{\varepsilon_\nu^{\alpha 2} + \Delta_\nu^{\alpha 2}}$ =quasi-particle spectrum $\qquad \alpha_\nu = \theta_\nu^0 - \theta_\nu^1$

$$L_\beta(t) \xrightarrow{\beta \to \infty} \prod_\nu \sqrt{1 - \sin^2(2\alpha_\nu)\sin^2(\Lambda_\nu^1 t)} \qquad T=0$$

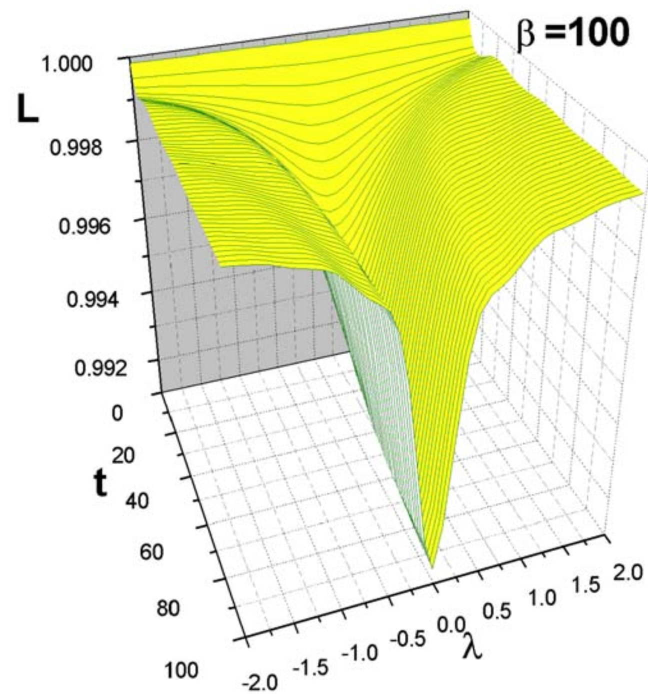
H.T Quan et al, Phys. Rev. Lett. 96, 140604 (2006)

Loschmidt Echo Finite T: XY model

$$L = 2001$$
$$\gamma = 1/2$$
$$\beta = 50$$



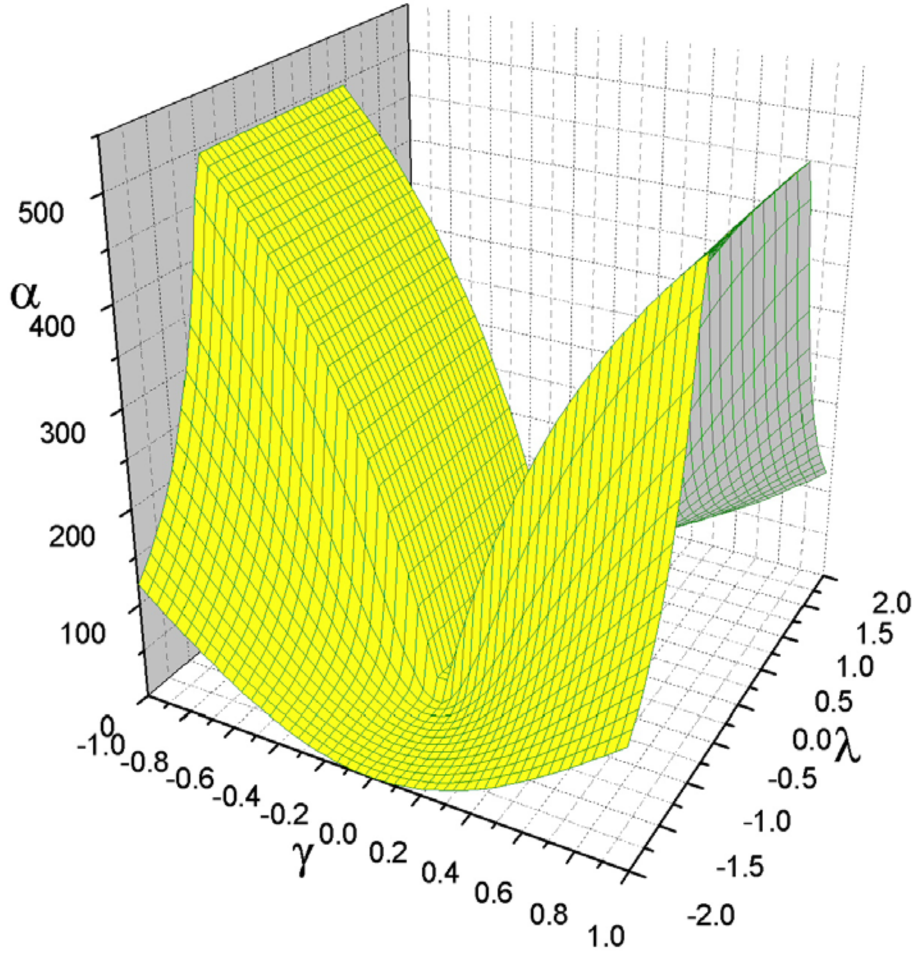
$$L = 2001$$
$$\gamma = 1/2$$
$$\beta = 100$$



Loschmidt Echo: short time Gaussian decay rate

$$L_\infty(t) \approx_{t\to 0} \exp(-\alpha_\infty t^2)$$

L=2001



$$\Phi = (g+h)/2,$$
$$\Lambda = diag(\Lambda_1, \cdots, \Lambda_L) \geq 0$$

$$Z^+ Z = \Phi^{-1}\Lambda^2\Phi \equiv \Lambda^2{}_\Phi$$

The polar decomposition of $Z = \Lambda_\Phi T$ contain the physical relevant data

$$G = -g^{-1}h = \frac{T-1}{T+1}$$

$$T \equiv \Lambda^{-1}{}_\Phi Z$$

$$T^+ = T^{-1}$$

$Sp(\Lambda_\Phi) \subset R_0^+$ Quasi-particle spectrum

$T \in O_L(R)$ Many-body **GS** structure

$M_L(R) \rightarrow O_L(R) : Z \rightarrow T$ Continuous map for Z non-singular

$\det T = -1$ ➡ GS in the odd sector

$$T = e^K, K = U(\oplus_\nu i\theta_\nu \sigma^y{}_{(\nu)})U^T \in o_L(R) \qquad Sp(T) = \{\exp(\pm i\theta_\nu)\}_{\nu=1}^{L/2}$$

$$|\Psi_Z\rangle = \otimes_{\nu=1}^{L/2}(\cos\frac{\theta_\nu}{2}|00\rangle_{\nu,-\nu} - \sin\frac{\theta_\nu}{2}|11\rangle_{\nu,-\nu})$$

$$\Phi = (g+h)/2,$$

$$\Lambda = diag(\Lambda_1, \cdots, \Lambda_L) \geq 0$$

$$Z^+ Z = \Phi^{-1} \Lambda^2 \Phi \equiv \Lambda^2{}_\Phi$$

The polar decomposition of $Z = \Lambda_\Phi T$ contain[s] the physical relevant data

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Boundary conditions II

Free-ends:

$$B=\gamma\begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ -1 & \ddots & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & -1 & 0 \end{pmatrix}$$

nearest neighbour
XY model

Cyclic chain:

$$B=\gamma\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & -1 \\ -1 & \ddots & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & 0 & -1 & 0 \end{pmatrix}$$

Boundary conditions II

Free-ends:

$$B = \gamma \begin{pmatrix} 0 & 1 & \dots & \dots & \dots & 1 \\ -1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 1 \\ -1 & \dots & \dots & \dots & -1 & 0 \end{pmatrix}$$

Cyclic chain:

$$B = \gamma \begin{pmatrix} 0 & \overbrace{1 \quad 1}^{\lfloor (L-1)/2 \rfloor} & & -1 & -1 \\ -1 & \ddots & \ddots & \ddots & & -1 \\ -1 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & & \ddots & \ddots & \ddots & 1 \\ 1 & 1 & & -1 & -1 & 0 \end{pmatrix}$$

fully coordinated
fermionic model

Fidelity and the Loschdmidt Echo

1)

$$L(t) = |\langle \Psi_0(\lambda) | e^{-iH(\lambda+\delta\lambda)t} | \Psi_0(\lambda)\rangle| = |\int d\omega D(\omega) e^{-i\omega t}|$$

$$D(\omega) := \sum_n |\langle \Psi_n(\lambda+\delta\lambda) | \Psi_0(\lambda)\rangle|^2 \; \delta(\omega - E_n(\lambda+\delta\lambda))$$

$$= F^2(\lambda, \lambda+\delta\lambda)\delta(\omega - E_0(\lambda+\delta\lambda)) + \ldots$$

L-echo is the Fourier transform of **LDOS** whose leading Term (for $\delta\lambda << 1$) is the fidelity $F \approx 1 - o(\delta\lambda)$:

the smaller *F* → the broader *D* → the faster the decay of *L*

H.T Quan et al, Phys. Rev. Lett. 96, 140604 (2006)

2)

$$F(Z_1, Z_2) = |\langle \Psi(Z_1) | e^{-iH(W)} | \Psi(Z_1)\rangle| = \langle \Psi(Z_1) | \Psi(Z_2)\rangle|$$

$$H(Z_2) = e^{-iH(W)} H(Z_1) e^{iH(W)}$$

L-echo is the FFs can be seen as fidelity between two GSs

(FF Hs close in Lie-algebra) **NB** Z_2 might be complex

Examples

1)

$$H = c_1^+ c_2 + c_2^+ c_1 + +\mu(c_1^+ c_1 + c_2^+ c_2) + \Delta(c_1^+ c_2^+ + c_1 c_2)$$

$$Z = \sigma^x + \mu 1 - i\sigma^y$$

$$Sp(|Z|) = \{1 \pm \sqrt{\mu^2 + \Delta^2}\}$$

$$T = \begin{cases} \sigma^x \; for \sqrt{\mu^2 + \Delta^2} \le 1 & \text{(odd sector)} \\ e^{i\theta\sigma^y} \; for \sqrt{\mu^2 + \Delta^2} \ge 1 & \text{(even sector)} \end{cases}$$

At $\sqrt{\mu^2 + \Delta^2} = 1$ Z becomes singular and T gets discontinuous:
1st order QPT!

2)

$$H = \sum_\nu \varepsilon_\nu (n_\nu + n_{-\nu}) + \Delta_\nu (c_\nu^+ c_{-\nu}^+ + h.c.)$$

$$Z = \oplus_\nu \varepsilon_\nu 1_\nu - i\Delta_\nu \sigma_{(\nu)}^y,$$

$$Sp(|Z|) = \{\sqrt{\varepsilon_\nu^2 + \Delta_\nu^2}\}_\nu$$

$$T = \oplus_\nu \exp(i\theta_\nu \sigma_{(\nu)}^y)$$

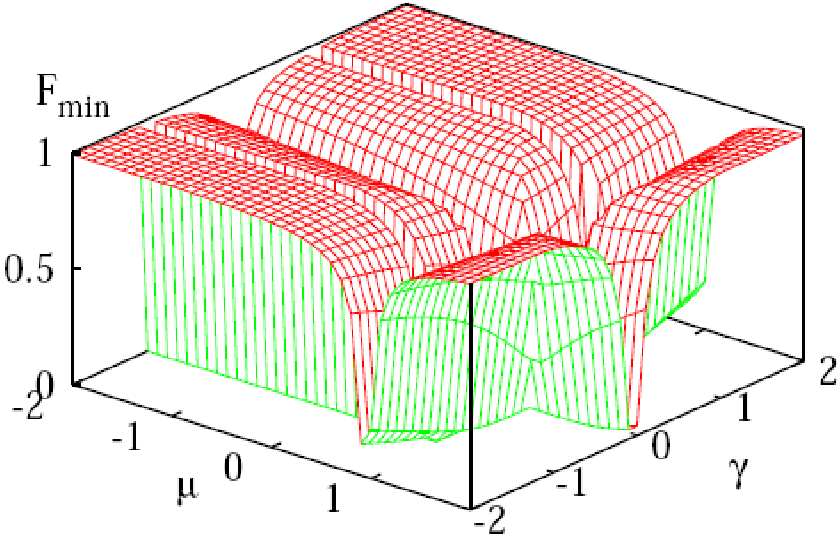
$$\theta_\nu = \tan^{-1}\frac{\Delta_\nu}{\varepsilon_\nu}$$

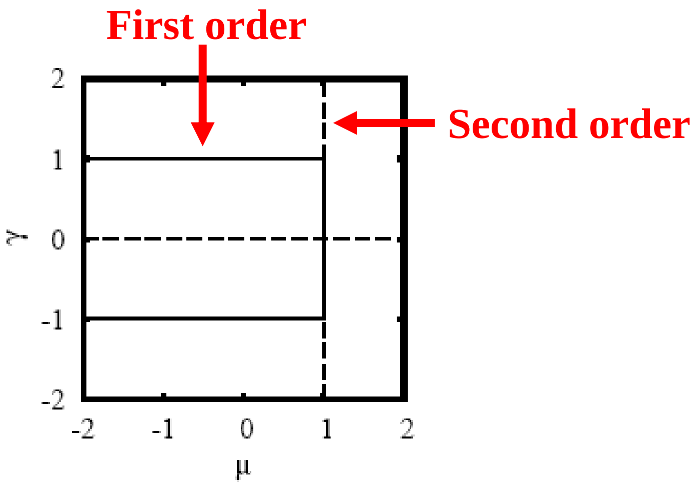
No zeros in the $(\varepsilon_\nu, \Delta_\nu)$ space but the origin, T analytic

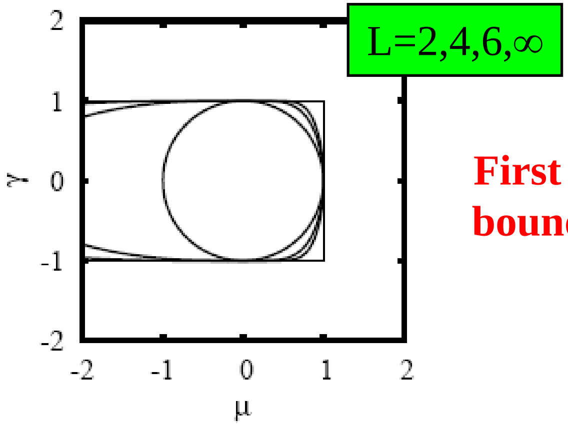
Ground State diagram

$$\mathcal{F}_{\min} = \min[\mathcal{F}(\tilde{Z}_{\delta\mu}, Z), \mathcal{F}(\tilde{Z}_{\delta\gamma}, Z)]$$

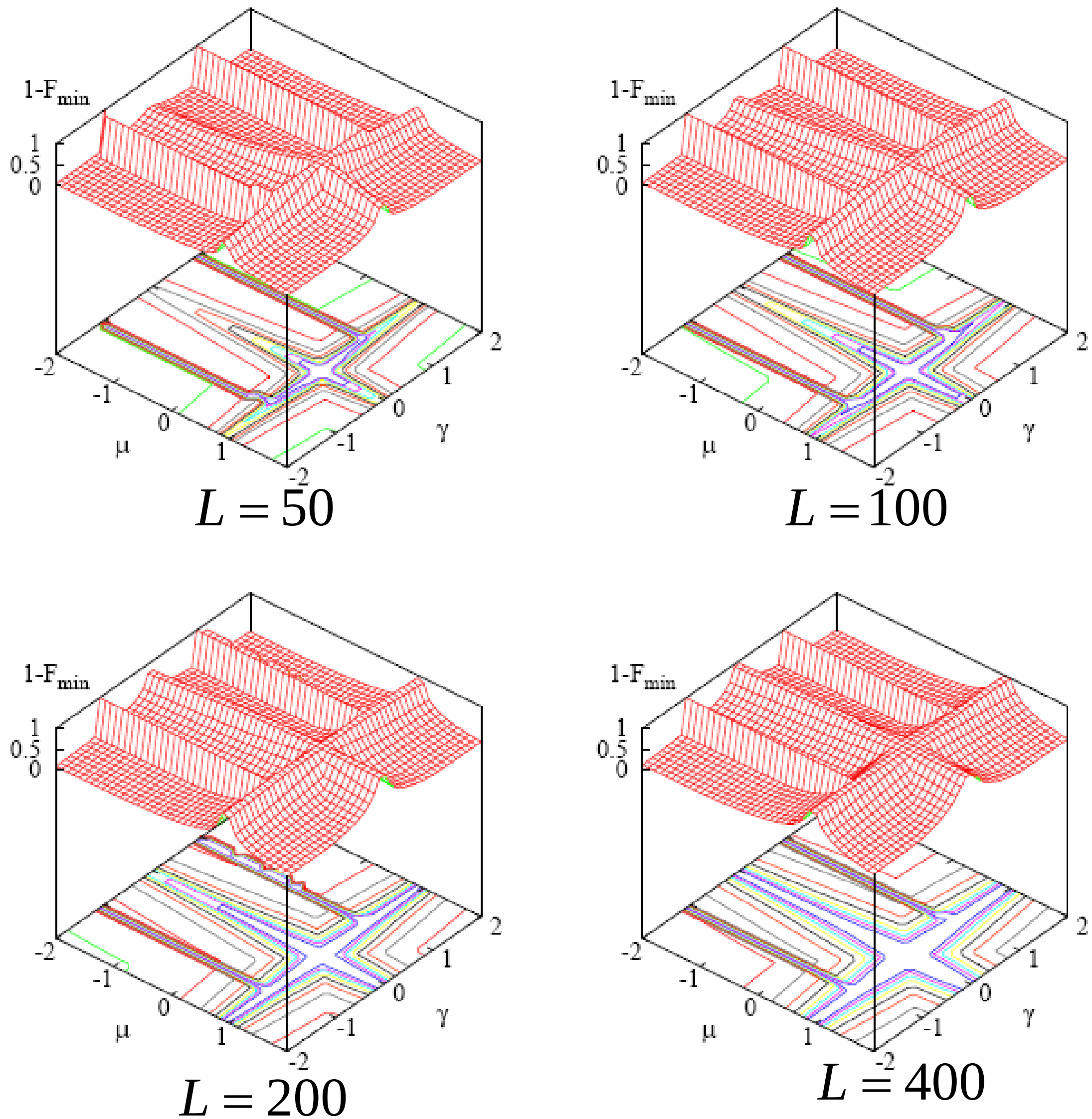
$$[\tilde{Z}_{\delta\mu} = Z(\mu + \delta\mu, \gamma) \text{ and } \tilde{Z}_{\delta\gamma} = Z(\mu, \gamma + \delta\gamma)]$$







First order boundaries


1-F_min
1
0.5
0
-2
-1
0
1
μ
2
1
0
-1
-2
γ
L = 50
L = 100
L = 200
L = 400

Scaling around $\mu = 1$