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Path integrals for fermions and supersymmetric quantum mechanics

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Fermions at the classical level can be described by Grassmann variables, also known as anticommuting numbers or fermionic variables. Grassmann variables allow to define "classical" models whose quantization produce degrees of freedom that satisfy a Pauli exclusion principle. Here we use them to parametrize the degrees of freedom associated to spin. Models with Grassmann variables are often called "pseudoclassical", as the spin at the classical level is a formal construction (the value of spin vanish for $\hbar \to 0$ and cannot be measured classically).

In worldline approaches to quantum field theories, one describes relativistic point particles with spin by their worldline coordinates that identify the particle position in space-time, and by Grassmann variables that take into account the additional spin degrees of freedom. From the worldline point of view Grassmann variables have equations of motion that are first order in time. They satisfy the equivalent of a Dirac equation in one dimension (the worldline can be considered as a 0+1 dimensional space-time, with the time coordinate corresponding to the only dimension while space is just a point). They can be interpreted as worldline fermions since they obey the Pauli exclusion principle. The latter arises as the Grassmann variables are quantized with anticommutators. Of course there is no concept of spin in $0 + 1$ dimensions, as there is no rotation group in a zero dimensional space. On the other hand, the spin of the point particle as an observable living in the $3 + 1$ minkowskian spacetime emerges precisely form the quantization of the Grassmann variables.

In this chapter we study the path integral quantization of models with Grassmann variables, and refer to them as path integrals for fermions, or fermionic path integrals. In a hypercondensed notation the resulting formulae describe the quantization of higher dimensional fermionic quantum field theories as well, including Dirac fields.

We start introducing Grassmann variables and develop canonical quantization for mechanical models containing Grassmann variables. Then we derive a path integral representation of the transition amplitude for fermionic systems starting from its operatorial expression, using a suitable definition of fermionic coherent states. To exemplify the use of Grassmann variables, we present a class of supersymmetric mechanical models, where bosonic and fermionic variables are mixed together by the supersymmetry transformation rules. These models allow to introduce supersymmetry in one of its simplest realization. As we shall see in later chapters, worldline supersymmetry is a guiding principle for describing relativistic spinning particles. Eventually, we present the construction of the $N = 2$ and $N = 1$ superspaces in one dimension. Superspaces are useful as they allow to formulate supersymmetric theories in a manifestly supersymmetric way. The $N = 2$ superspace is used to reproduce the mechanical models described previously, while the $N = 1$ superspace is used in the worldline treatment of Dirac fields.

1 Grassmann algebras

A n-dimensional Grassmann algebra $\mathcal{G}_n$ is generated by a set of generators θ_i with $i = 1, ..., n$ that satisfy

$$\theta_i \theta_j + \theta_j \theta_i = 0 \tag{1}$$

or, equivalently, in terms of the anticommutator

$$\{\theta_i, \theta_j\} = 0 \ . \tag{2}$$

In particular any fixed generator squares to zero

$$\theta_i^2 = 0 \tag{3}$$

suggesting already at the classical level the essence of the Pauli exclusion principle, according to which one cannot put two identical fermions in the same quantum state. Physicists often call these generators anticommuting numbers.

One can multiply these generators and their products by real or complex numbers, and form polynomials that are used to define functions of the Grassmann variables (i.e. the elements of the Grassmann algebra). For example, for $n = 1$ there is only one Grassmann variable θ and an arbitrary function is given by

$$f(\theta) = f_0 + f_1 \theta \tag{4}$$

where f_0 and f_1 are taken to be either real or complex numbers. Similarly, for $n = 2$ one has

$$f(\theta_1, \theta_2) = f_0 + f_1 \theta_1 + f_2 \theta_2 + f_3 \theta_1 \theta_2 \ . \tag{5}$$

A term with $\theta_2 \theta_1$ is not written as it is not independent of $\theta_1 \theta_2$, as $\theta_2 \theta_1 = -\theta_1 \theta_2$. Terms with an even number of θ's are called Grassmann even (or equivalently even, commuting, bosonic). Terms with an odd number of θ's are called Grassmann odd (or equivalently odd, anticommuting, fermionic). Generic functions are always defined in terms of their Taylor expansions, which contain a finite number of terms because of the Grassmann property. For example, the exponential function e^θ means just $e^\theta = 1 + \theta$ because $\theta^2 = 0$.

Derivatives with respect to Grassmann variables are very simple. As any function can be at most linear with respect to a fixed Grassmann variable, its derivative is straightforward, and one has to keep track just of signs. Left derivatives are defined by removing the variable from the left of its Taylor expansion: for example for the function $f(\theta_1, \theta_2)$ given above

$$\frac{\partial_L f(\theta_1, \theta_2)}{\partial \theta_1} = f_1 + f_3 \theta_2 \ . \tag{6}$$

Similarly, right derivatives are obtained by removing the variable from the right

$$\frac{\partial_R f(\theta_1, \theta_2)}{\partial \theta_1} = f_1 - f_3 \theta_2 \tag{7}$$

where a minus sign emerges because one has first to commute θ_1 past θ_2. Equivalently, using Grassmann increments $\delta\theta$, one may write

$$\delta f = \delta\theta \frac{\partial_L f}{\partial \theta} = \frac{\partial_R f}{\partial \theta} \delta\theta \tag{8}$$

which makes evident how to keep track of signs. If not specified otherwise, we use left derivatives and omit the corresponding subscript.

Integration can be defined, according to Berezin, to be identical with differentiation

$$\int d\theta \equiv \frac{\partial_{L}}{\partial\theta} \; . \tag{9}$$

This definition has the virtue of producing a translational invariant measure, that is

$$\int d\theta f(\theta+\eta) = \int d\theta f(\theta) \; . \tag{10}$$

This statement is easily proven by a direct calculation

$$\int d\theta f(\theta+\eta) = \int d\theta \; (f_0 + f_1\theta + f_1\eta) = f_1 = \int d\theta f(\theta) \; . \tag{11}$$

Grassmann variables can be defined to be either real or complex. A real variable satisfies

$$\bar{\theta} = \theta \tag{12}$$

with the bar indicating complex conjugation. For products of Grassmann variables the complex conjugate is defined to include an exchange of their position

$$\overline{\theta_1\theta_2} = \bar{\theta}_2\bar{\theta}_1 \; . \tag{13}$$

Thus the complex conjugate of the product of two real variables is purely imaginary

$$\overline{\theta_1\theta_2} = -\theta_1\theta_2 \; . \tag{14}$$

It is $i\theta_1\theta_2$ that is real, as the complex conjugate of the imaginary unit carries the additional minus sign to obtain a formally real object

$$\overline{i\theta_1\theta_2} = i\theta_1\theta_2 \; . \tag{15}$$

Complex Grassmann variables η and $\bar{\eta}$ can always be decomposed in terms of two real Grassmann variables θ_1 and θ_2 by setting

$$\eta = \frac{1}{\sqrt{2}}(\theta_1 + i\theta_2) \; , \qquad \bar{\eta} = \frac{1}{\sqrt{2}}(\theta_1 - i\theta_2) \; . \tag{16}$$

These are the definitions that are most useful for physical applications, and one requires that real variables become hermitian operators upon quantization.

Having defined integration over Grassmann variables, we consider in more details the gaussian integration, which is at the core of fermonic path integrals. For the case of a single real Grassmann variable θ the gaussian function is trivial, $e^{-a\theta^2} = 1$, since $\theta^2 = 0$ as θ anticommutes with itself. One needs at least two real Grassmann variables θ_1 and θ_2 to have a nontrivial exponential function with an exponent quadratic in Grassmann variables

$$e^{-a\theta_1\theta_2} = 1 - a\theta_1\theta_2 \tag{17}$$

where a is either a real or complex number. With the above definitions the corresponding "gaussian integral" is computed straghtforwardly

$$\int d\theta_1 d\theta_2\, e^{-a\theta_1\theta_2} = a\ . \tag{18}$$

Defining the antisymmetric 2×2 matrix A^{ij} by

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \tag{19}$$

one may rewrite the result of the Grassmann gaussian integration as

$$\int d\theta_1 d\theta_2\, e^{-\frac{1}{2}\theta_i A^{ij}\theta_j} = \det{}^{\frac{1}{2}} A\ . \tag{20}$$

The square root of the determinant of an antisymmetric matrix A is called the pfaffian, and is often indicated by $\mathrm{Pfaff}\, A$. Note indeed that the determinant is always positive definite for real antisymmetric matrices, and its square root is well defined (by analytic extensions it is also well defined for antisymmetric matrices with complex entries). It is easy to see that the above formula extends to an even number $n = 2m$ of real Grassmann variables, so that one may write in general

$$\int d^n\theta\, e^{-\frac{1}{2}\theta_i A^{ij}\theta_j} = \det{}^{\frac{1}{2}} A \tag{21}$$

with the measure normalized as $d^n\theta \equiv d\theta_1 d\theta_2 \ldots d\theta_n$. Indeed with an orthogonal transformation one can skew-diagonalize the antisymmetric matrix A^{ij} and put it in the form

$$\begin{pmatrix} 0 & a_1 & & & & & & \\ -a_1 & 0 & & & & & & \\ & & 0 & a_2 & & & & \\ & & -a_2 & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 0 & a_m \\ & & & & & & -a_m & 0 \end{pmatrix} . \tag{22}$$

The orthogonal transformation leaves the integration measure invariant and one immediately gets the above result with $\det^{\frac{1}{2}} A = a_1 a_2 \cdots a_m$. As a cautionary note, to compute correctly the jacobian under a change of variables one should recall the definition of the integration in terms of derivatives (the Berezin integration).

In an analogous way, one finds that gaussian integration over complex Grassmann variables $(\eta_i, \bar\eta_i)$ produce a determinant

$$\int d^n\bar\eta d^n\eta\, e^{-\bar\eta_i A^{ij}\eta_j} = \det A \tag{23}$$

where the measure is now defined by $d^n\bar\eta d^n\eta \equiv d\bar\eta_1 d\eta_1 d\bar\eta_2 d\eta_2 \ldots d\bar\eta_n d\eta_n$.

For applications to dynamical models and subsequent path integral quantization, it is useful to consider infinite dimensional Grassmann algebras ($n\to\infty$). Then one may use Grassmann valued functions of time, i.e. $\theta_i \rightsquigarrow \theta(t)$. For different values of t one has different generators of the algebra, so that properties such as $\theta^2(t) = 0$ and $\theta(t_1)\theta(t_2) = -\theta(t_2)\theta(t_1)$ hold. They are used to describe interesting mechanical systems.

2 Pseudoclassical models and canonical quantization

To gain some familiarity with the use of Grassmann numbers as dynamical variables we consider the fermionic harmonic oscillator, an example that by itself contains the essence of all fermionic systems.

Classically, it is described by the function $\psi(t)$ and its complex conjugate $\bar\psi(t)$ that take values in a Grassmann algebra (t denotes the time). The Grassmann property implies generic relations like $\psi(t)\psi(t) = 0$, $\psi(t)\psi(t') = -\psi(t')\psi(t)$, $\dot\psi(t)\dot\psi(t) = 0$, $\psi(t)\dot\psi(t) = -\dot\psi(t)\psi(t)$, etc., where dots denote time derivatives, $\dot\psi = \frac{d}{dt}\psi$. These relations can be used in extremizing the action, for testing the presence of symmetries, and so on.

Before introducing the action of the fermionic harmonic oscillator, and for gaining some intuition, let us rewrite the action of the usual bosonic harmonic oscillator in phase space by using complex combinations of the coordinate and momentum (x, p), as defined by

$$a = \frac{1}{\sqrt{2\omega}}(\omega x + ip) \ , \qquad \bar a = \frac{1}{\sqrt{2\omega}}(\omega x - ip) \ . \tag{24}$$

Up to boundary terms, one finds

$$S[x,p] = \int dt\Big(p\dot x - \frac{1}{2}(p^2 + \omega^2 x^2)\Big) \quad \to \quad S[a,\bar a] = \int dt\,(i\bar a\dot a - \omega\bar a a) \ . \tag{25}$$

Upon quantization the complex variables $(a, \bar a)$ give rise to the annhilation/creation operators $(\hat a, \hat a^\dagger)$ that satisfy the algebra $[\hat a, \hat a^\dagger] = \hbar$. They are used in the Fock construction of the Hilbert space of the harmonic oscillator. For convenience, it is reviewed later on in sec. 2.1.

The dynamics of the fermionic harmonic oscillator is similarly described by the complex Grassmann valued functions $\psi(t)$ and $\bar\psi(t)$ and is fixed by the action

$$S[\psi,\bar\psi] = \int dt\,(i\bar\psi\dot\psi - \omega\bar\psi\psi) \ . \tag{26}$$

The action is formally real (up to boundary terms), just like its bosonic cousin in (25). The equation of motion is obtained by extremizing the action, and easily solved

$$i\dot\psi - \omega\psi = 0 \qquad \Longrightarrow \qquad \psi(t) = \psi_0\, e^{-i\omega t} \tag{27}$$

where ψ_0 a suitable initial datum. This equation of motion may be called the Dirac equation in a 0+1 dimensional space time. Indeed one may rewrite it as $(\gamma^0\partial_0 + \omega)\psi = 0$, with $\gamma^0 = -i$, $x^0 = t$, and with ω playing the role of the Dirac mass.

Canonical quantization is achieved by considering the hamiltonian structure of the model. We sketch it now, postponing briefly a general discussion of the phase space structure associated to Grassmann variables. The momentum π conjugate to ψ is defined by

$$\pi \equiv \frac{\partial_{_L} L}{\partial\dot\psi} = -i\bar\psi \tag{28}$$

which shows that the systems is already in a hamiltonian form, the conjugate momenta being $\bar\psi$ up to a factor. The classical Poisson bracket $\{\pi,\psi\}_{_{PB}} = -1$ is rewritten as $\{\psi,\bar\psi\}_{_{PB}} = -i$, and has the property of being symmetric for fermionic systems (this will be discussed in a short while).

Quantizing with anticommutators (fermionic system must be treated that way) one obtains

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = \hbar \,, \quad \{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0 \tag{29}$$

that is, the classical variables ψ and $\bar{\psi}$ are promoted to linear operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ satisfying anticommutation relations that are set to be equal to $i\hbar$ times the value of the classical Poisson brackets. Setting $\hbar = 1$ for simplicity, one finds the fermionic creation/annihilation algebra

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = 1 \,, \quad \{\hat{\psi}, \hat{\psi}\} = \{\hat{\psi}^\dagger, \hat{\psi}^\dagger\} = 0 \tag{30}$$

that can be realized in a two dimensional Hilbert space, and with the correct hermiticity properties. The Hilbert is explicitly constructed à la Fock, considering $\hat{\psi}$ as destruction operator and $\hat{\psi}^\dagger$ as creation operator[1]. One starts defining the Fock vacuum $|0\rangle$, fixed by the condition $\hat{\psi}|0\rangle = 0$. A second state is obtained acting with $\hat{\psi}^\dagger$

$$|1\rangle = \hat{\psi}^\dagger|0\rangle \,. \tag{31}$$

No other states can be obtained acting again with the creation operator $\hat{\psi}^\dagger$ as $(\hat{\psi}^\dagger)^2 = 0$. Normalizing the Fock vacuum to unity, $\langle 0|0\rangle = 1$, with $\langle 0| = |0\rangle^\dagger$, one finds that these two states are orthonomal

$$\langle m|n\rangle = \delta_{mn} \qquad m, n = 0, 1 \tag{32}$$

and thus span a two-dimensional Hilbert space, $\mathcal{F}_2 = \mathrm{Span}\{|0\rangle, |1\rangle\}$. In terms of matrices one finds the realization

$$\begin{aligned}
\hat{\psi} \quad &\longrightarrow \quad \begin{pmatrix} \langle 0|\hat{\psi}|0\rangle & \langle 0|\hat{\psi}|1\rangle \\ \langle 1|\hat{\psi}|0\rangle & \langle 1|\hat{\psi}|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & \langle 0|0\rangle \\ 0 & \langle 1|0\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
\hat{\psi}^\dagger \quad &\longrightarrow \quad \begin{pmatrix} \langle 0|\hat{\psi}^\dagger|0\rangle & \langle 0|\hat{\psi}^\dagger|1\rangle \\ \langle 1|\hat{\psi}^\dagger|0\rangle & \langle 1|\hat{\psi}^\dagger|1\rangle \end{pmatrix} = \begin{pmatrix} \langle 0|1\rangle & 0 \\ \langle 1|1\rangle & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \,.
\end{aligned} \tag{33}$$

The Fock vacuum is indeed the ground state of the fermionic oscillator, whose quantum Hamiltonian $\hat{H} = \omega(\hat{\psi}^\dagger\hat{\psi} - \frac{1}{2})$ is obtained from the classical one by choosing a suitable ordering of the operators upon quantization. Indeed the classical hamiltonian is given by the Legendre transform

$$H = \dot{\psi}\pi - L = \omega\bar{\psi}\psi = \frac{\omega}{2}(\bar{\psi}\psi - \psi\bar{\psi}) \,. \tag{34}$$

The last form is a classically equivalent way of writing it, and it is the one that is quantized to resolve the ordering ambiguities

$$\hat{H} = \frac{\omega}{2}(\hat{\psi}^\dagger\hat{\psi} - \hat{\psi}\hat{\psi}^\dagger) = \omega\Big(\hat{\psi}^\dagger\hat{\psi} - \frac{1}{2}\Big) \,. \tag{35}$$

The last expression is obtained by using the first relation in (30). Note also that in the Legendre transform the order of $\dot{\psi}$ and π matters, and we have used the one that follows from having defined the conjugate momentum (28) with left derivatives.

Hamiltonian structure and canonical quantization

Path integrals for fermions can be derived from the canonical formalism, just as in the

[1]Their role could also be reversed for fermionic systems.

bosonic case. For that we need first to review the hamiltonian formalism and the canonical quantization of mechanical systems with Grassmann variables.

The hamiltonian formalism aims at producing equations of motion that are first order differential equations in time. For a simple bosonic model with phase space coordinates (x, p), the phase space action is usually written in the form

$$S[x,p] = \int dt \, \Big(p\dot{x} - H(x,p)\Big) \, . \tag{36}$$

The first term with derivatives (the $p\dot{x}$ term) is called the symplectic term, and fixes the Poisson bracket structure of phase space. Up to total derivatives it can be written in a more symmetrical form, with the time derivatives shared equally by x and p

$$S[x,p] = \int dt \, \Big(\frac{1}{2}(p\dot{x} - x\dot{p}) - H(x,p)\Big) = \int dt \, \Big(\frac{1}{2} z^a (\Omega^{-1})_{ab} \dot{z}^b - H(z)\Big) \tag{37}$$

where we have denoted collectively the phase space coordinates by $z^a = (z^1, z^2) = (x, p)$. The symplectic term contains the constant invertible matrix

$$(\Omega^{-1})_{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{38}$$

with inverse

$$\Omega^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \, . \tag{39}$$

The latter is used to define the Poisson bracket between two generic phase space functions F and G

$$\{F, G\}_{_{PB}} = \frac{\partial F}{\partial z^a} \Omega^{ab} \frac{\partial G}{\partial z^b} \, . \tag{40}$$

In particular, one finds that the Poisson bracket of the phase space coordinates themselves is given by

$$\{z^a, z^b\}_{_{PB}} = \Omega^{ab} \, . \tag{41}$$

This coincides with standard definitions. The Poisson bracket satisfies the following properties:

$$\begin{aligned}
&\{F, G\}_{_{PB}} = -\{G, F\}_{_{PB}} \qquad \text{(antisymmetry)} \\
&\{F, GH\}_{_{PB}} = \{F, G\}_{_{PB}} H + G\{F, H\}_{_{PB}} \qquad \text{(Leibniz rule)} \\
&\{F, \{G, H\}_{_{PB}}\}_{_{PB}} + \{G, \{H, F\}_{_{PB}}\}_{_{PB}} + \{H, \{F, G\}_{_{PB}}\}_{_{PB}} = 0 \qquad \text{(Jacobi identity)} \, .
\end{aligned} \tag{42}$$

These properties make it consistent to adopt the canonical quantization rules of substituting the fundamental variables z^a by linear operators $\hat{z}^a$ acting on a Hilbert space of physical states, with commutation relations fixed to be $i\hbar$ times the value of the classical Poisson brackets

$$[\hat{z}^a, \hat{z}^b] = i\hbar \Omega^{ab} \, . \tag{43}$$

This prescription is consistent as both sides satisfy the same algebraic properties, listed in (42) for the Poisson brackets.

More generally, phase space functions $F(z)$ are elevated to operators $\hat{F}(\hat{z})$ (after fixing eventual ordering ambiguities) with commutation relations that take the form

$$[\hat{F}(\hat{z}), \hat{G}(\hat{z})] = i\hbar \{F, G\}_{_{PB}} + \text{ higher order terms in } \hbar \, . \tag{44}$$

This set up can be extended to models with Grassmann variables. The basic structure remains unaltered, and one must only take care of signs arising from the anticommuting sector. Let us show how this is done.

We denote collectively the phase space coordinates by $Z^A = (x^i, p_i, \theta^\alpha)$, with (x^i, p_i) the usual Grassmann even phase space variables and θ^α the Grassmann odd variables. We consider a phase space action of the form

$$S[Z^A] = \int dt \left(\frac{1}{2} Z^A (\Omega^{-1})_{AB} \dot{Z}^B - H(Z) \right) \tag{45}$$

where the symplectic term depends on a constant invertible matrix $(\Omega^{-1})_{AB}$ with inverse Ω^{AB}. Again this term must be written splitting the time derivatives democratically between all variables, as in (37). The symplectic term and the hamiltonian are taken to be Grassmann even (i.e. commuting objects). Then, it is seen that Ω^{AB} is antisymmetric in the sector related to the bosonic coordinates, and symmetric in the sector belonging to the Grassmann variables (other off-diagonal entries vanish). That is, denoting the variables by $Z^A = (z^a, \theta^\alpha)$ with z^a bosonic and θ^α fermionic, the matrix Ω^{AB} (as well as its inverse) has a block diagonal form

$$\Omega^{AB} = \begin{pmatrix} \Omega^{ab} & 0 \\ 0 & \Omega^{\alpha\beta} \end{pmatrix} \tag{46}$$

with Ω^{ab} antisymmetric and $\Omega^{\alpha\beta}$ symmetric.

The matrix Ω^{AB} is used to define the Poisson bracket by

$$\{F, G\}_{PB} = \frac{\partial_R F}{\partial Z^A} \Omega^{AB} \frac{\partial_L G}{\partial Z^B} \tag{47}$$

where both right and left derivatives are used. In particular one finds

$$\{Z^A, Z^B\}_{PB} = \Omega^{AB} \; . \tag{48}$$

Phase space functions can be restricted to have a definite Grassmann parity. Given any such function F, we denote its Grassmann parity by $(-1)^{\epsilon_F}$, where $\epsilon_F = 0$ if F is Grassmann even (bosonic function) and $\epsilon_F = 1$ if F is Grassmann odd (fermionic function). Then, one finds that the definition (47) satisfies a graded generalization of the properties in (42), namely

$$\begin{aligned}
&\{F, G\}_{PB} = (-1)^{\epsilon_F \epsilon_G + 1} \{G, F\}_{PB} \\
&\{F, GH\}_{PB} = \{F, G\}_{PB} H + (-1)^{\epsilon_F \epsilon_G} G \{F, H\}_{PB} \\
&\{F, \{G, H\}_{PB}\}_{PB} + (-1)^{\epsilon_F(\epsilon_G + \epsilon_H)} \{G, \{H, F\}_{PB}\}_{PB} + (-1)^{\epsilon_H(\epsilon_F + \epsilon_G)} \{H, \{F, G\}_{PB}\}_{PB} = 0 \; .
\end{aligned} \tag{49}$$

The equations of motion are first order in time. They can be derived minimizing the action and can be expressed in term of the Poisson brackets

$$\dot{Z}^A = \Omega^{AB} \frac{\partial_L H}{\partial Z^B} \qquad \rightarrow \qquad \dot{Z}^A = \{Z^A, H\}_{PB} \; . \tag{50}$$

These are the Hamilton's equations of motion.

The properties of the Poisson brackets make it consistent to adopt the canonical quantization rules, that consist in promoting the phase space coordinates Z^A to operators $\hat{Z}^A$ with commutation/anticommutation relation fixed by their classical Poisson brackets

$$[\hat{Z}^A, \hat{Z}^B\} = i\hbar \{Z^A, Z^B\}_{PB} = i\hbar \Omega^{AB} \tag{51}$$

where we have employed the compact notation

$$[\cdot\,,\cdot\} = \begin{cases} \{\cdot\,,\cdot\} & \text{anticommutator if both variables are fermionic} \\ [\cdot\,,\cdot] & \text{commutator otherwise} \end{cases} \tag{52}$$

which is often called "graded commutator". Indeed, the graded commutator satisfies identities similar to those for the Poisson brackets in (49), and makes it consistent to adopt the given quantization rules.

Our quick exposition becomes clearer by working through some simple examples.

Examples

(i) Single real Grassmann variable ψ ("single Majorana fermion in one dimension"). Taking as phase space lagrangian

$$L = \frac{i}{2}\psi\dot{\psi} - H(\psi) \tag{53}$$

which is formally real and produces equation of motion of the first order in time, one finds $\Omega^{-1} = i$, $\Omega = -i$, and Poisson bracket at equal times $\{\psi, \psi\}_{PB} = -i$. The dynamical variable $\psi(t)$ is often called a Majorana fermion in one dimension, as it satisfies the Dirac equation in one dimension plus a reality condition (akin to the Majorana condition used in four dimensions). One notices that the only possible Grassmann even hamiltonian is a constant, so that the model is rather trivial. One verifies in this example that the phase space can be odd dimensional if Grassmann variables are present. The model is quantized by introducing the hermitian operator $\hat{\psi}$ with anticommutator

$$\{\hat{\psi}, \hat{\psi}\} = \hbar\ . \tag{54}$$

The quantum theory is also trivial, as one represents irreducibly this algebra in a one dimensional Hilbert space, with the operator $\hat{\psi}$ acting as multiplication by the constant $\sqrt{\hbar/2}$. This Hilbert space has no room for any nontrivial dynamics, as there is only the vacuum state.

(ii) Several real Grassmann variables ψ^i ("Majorana fermions in one dimension"). For the case of several real Grassmann variables one may take as phase space lagrangian

$$L = \frac{i}{2}\psi^i\dot{\psi}^i - H(\psi^i) \qquad i = 1, ..., n \tag{55}$$

and one finds $(\Omega^{-1})_{ij} = i\delta_{ij}$, $\Omega^{ij} = -i\delta^{ij}$. The Poisson brackets at equal times read as $\{\psi^i, \psi^j\}_{PB} = -i\delta^{ij}$. Quantization is obtained by considering the anticommutation relations

$$\{\hat{\psi}^i, \hat{\psi}^j\} = \hbar\delta^{ij} \tag{56}$$

which is recognized to be proportional to the Clifford algebra of the gamma matrices, appearing in the Dirac equation in n euclidean dimensions. Indeed setting $\hat{\psi}^i = \sqrt{\hbar/2}\,\gamma^i$ turns the above anticommutation relations into the Clifford algebra

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \tag{57}$$

which is the defining properties of the gamma matrices of the Dirac equation

$$(\gamma^i\partial_i + m)\Psi(x) = 0\ . \tag{58}$$

It is known that the algebra (57) is realized in a complex vector space of dimension $2^{[\frac{n}{2}]}$, where $[\frac{n}{2}]$ indicates the integer part of $\frac{n}{2}$. For example, for $n = 2$ and $n = 3$ the gamma matrices are 2 by 2, for $n = 4$ and $n = 5$ the gamma matrices are 4 by 4, for $n = 6$ and $n = 7$ the gamma matrices are 8 by 8, etc.. One concludes that the operators $\hat{\psi}^i$ are realised as hermitian operators in a Hilbert space of dimensions $2^{[\frac{n}{2}]}$.

(iii) Complex Grassmann variables ψ and $\bar{\psi}$ ("single Dirac fermion in one dimension"). Taking as phase space lagrangian

$$L = i\bar{\psi}\dot{\psi} - H(\psi, \bar{\psi}) \tag{59}$$

one finds $\{\psi, \bar{\psi}\}_{PB} = -i$ as the only nontrivial Poisson bracket between the phase space coordinates $(\psi, \bar{\psi})$. It is quantized by the anticommutator $\{\hat{\psi}, \hat{\psi}^\dagger\} = \hbar$, producing a fermionic annihilation/creation algebra. It is realized in a two dimensional Fock space, as anticipated earlier while discussing the fermionic harmonic oscillator. This model is equivalent to that with two real (Majorana) fermions, seen as the real and the imaginary part of the Dirac fermion. Also, one may straightforwardly consider the theory of a set of several Dirac fermions in one dimension.

These basic examples can be used to construct explicitly the irreducible representations of the gamma matrices in arbitrary dimensions, and check their dimensionality as anticipated above. One proceeds as follows. In even dimensions $n = 2m$ one combines the $2m$ Majorana worldline fermions, corresponding to the gamma matrices, into m pairs of worldline Dirac fermions, that generate a set of m copies of independent, anticommuting creation/annihilation operators. The latter act on the tensor products of m two-dimensional fermionic Fock spaces, each one realizing an independent set of fermionic creation/annihilation operators. This gives a total Hilbert space of 2^m dimensions: indeed for each set of creation/annihilation operators a state can only be empty or filled with the corresponding fermionic excitation. This is in accord with the assertion given above about the dimensionality of the gamma matrices. Adding an extra Majorana fermion corresponds to a Clifford algebra in odd dimensions (i.e. $2m + 1$ dimensions): the related dimension of the Hilbert space does not change as the last Majorana fermion can be realized as proportional to the chirality matrix of the $2m$ dimensional case, which always exists.

2.1 Coherent states

It is useful to introduce coherent states, an overcomplete basis of vectors for the fermionic Fock space described previously, for deriving a path integral for fermionic systems. They provide ket eigenstates of the fermionic operator $\hat{\psi}$ with Grassmann valued eigenvalues. Together with a suitable resolution of the identity, they allow to convert the matrix elements of the quantum evolution operator (transition amplitudes) into a path integral where one sums over Grassmann valued functions. We first review the construction of bosonic coherent states, used in the theory of the harmonic oscillator, to have a guide on the construction for the fermionic case.

In the theory of the harmonic oscillator one introduces coherent states defined as eigenstates of the annihilation operator $\hat{a}$. Let us recall the algebra of the creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$

$$[\hat{a}, \hat{a}^\dagger] = 1 \;, \quad [\hat{a}, \hat{a}] = 0 \;, \quad [\hat{a}^\dagger, \hat{a}^\dagger] = 0 \;. \tag{60}$$

It is realized by operators acting on an infinite dimensional Hilbert space, identified with a Fock space which is constructed as follows. A complete orthonormal basis is obtained by starting from the Fock vacuum $|0\rangle$, defined by the condition $\hat{a}|0\rangle = 0$. The other states of the basis are obtained by acting with the creation operator $\hat{a}^\dagger$ an arbitrary number of times on the Fock vacuum $|0\rangle$

$$
\begin{aligned}
|0\rangle & \quad \text{such that} \quad \hat{a}|0\rangle = 0 \\
|1\rangle &= \hat{a}^\dagger|0\rangle \\
|2\rangle &= \frac{\hat{a}^\dagger}{\sqrt{2}}|1\rangle = \frac{(\hat{a}^\dagger)^2}{\sqrt{2!}}|0\rangle \\
|3\rangle &= \frac{\hat{a}^\dagger}{\sqrt{3}}|2\rangle = \frac{(\hat{a}^\dagger)^3}{\sqrt{3!}}|0\rangle \\
\dots & \\
|n\rangle &= \frac{\hat{a}^\dagger}{\sqrt{n}}|n-1\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle \\
\dots &
\end{aligned} \tag{61}
$$

Normalizing the Fock vacuum to unit norm, $\langle 0|0\rangle = 1$, where $\langle 0| = |0\rangle^\dagger$, one finds that these states are orthonomal

$$\langle m|n\rangle = \delta_{mn} \qquad m, n = 0, 1, 2, \dots \; . \tag{62}$$

Now, choosing a complex number a, one builds the coherent states $|a\rangle$ defined by

$$|a\rangle = e^{a\hat{a}^\dagger}|0\rangle \; . \tag{63}$$

They are eigenstates of the annihilation operator $\hat{a}$

$$\hat{a}|a\rangle = a|a\rangle \; . \tag{64}$$

A way of proving this is by expanding the exponential and viewing $|a\rangle$ as an infinite sum with suitable coefficients of the basis vectors of the Fock space

$$
\begin{aligned}
|a\rangle &= e^{a\hat{a}^\dagger}|0\rangle \\
&= \Big(1 + a\hat{a}^\dagger + \frac{1}{2!}(a\hat{a}^\dagger)^2 + \frac{1}{3!}(a\hat{a}^\dagger)^3 + \dots + \frac{1}{n!}(a\hat{a}^\dagger)^n + \cdots\Big)|0\rangle \\
&= |0\rangle + a|1\rangle + \frac{a^2}{\sqrt{2!}}|2\rangle + \frac{a^3}{\sqrt{3!}}|3\rangle + \dots + \frac{a^n}{\sqrt{n!}}|n\rangle + \cdots \; .
\end{aligned} \tag{65}
$$

In this form it is easy to calculate (using $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$)

$$
\begin{aligned}
\hat{a}|a\rangle &= \hat{a}\Big(|0\rangle + a|1\rangle + \frac{a^2}{\sqrt{2!}}|2\rangle + \frac{a^3}{\sqrt{3!}}|3\rangle + \dots + \frac{a^n}{\sqrt{n!}}|n\rangle + \cdots\Big) \\
&= 0 + a|0\rangle + a^2|1\rangle + \frac{a^3}{\sqrt{2!}}|2\rangle + \dots + \frac{a^n}{\sqrt{(n-1)!}}|n-1\rangle + \cdots \\
&= a\Big(|0\rangle + a|1\rangle + \frac{a^2}{\sqrt{2!}}|2\rangle + \dots + \frac{a^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle + \cdots\Big) \\
&= a|a\rangle \; .
\end{aligned} \tag{66}
$$

A faster way of proving the same result is to recognize that the algebra (60) can be realized by

$$\hat{a}^\dagger \;\to\; \bar{a} \; , \qquad \hat{a} \;\to\; \frac{\partial}{\partial \bar{a}} \tag{67}$$

acting on functions of $\bar a \in \mathbb{C}$, and the result follows straighfowardly[2].

A list of properties that can be proven with similar calculations are

$$
\begin{array}{llll}
(i) & \langle \bar a| = |a\rangle^\dagger = \langle 0| e^{\bar a \hat a} & \Longrightarrow & \langle \bar a|\hat a^\dagger = \langle \bar a|\bar a \\
(ii) & \langle \bar a|a\rangle = e^{\bar a a} & & \text{(scalar product)} \\
(iii) & \mathbb{1} = \displaystyle\int \frac{d\bar a da}{2\pi i}\, e^{-\bar a a}\, |a\rangle\langle \bar a| & & \text{(resolution of the identity)} \\
(iv) & \mathrm{Tr}\, \hat A = \displaystyle\int \frac{d\bar a da}{2\pi i}\, e^{-\bar a a}\, \langle \bar a|\hat A|a\rangle & & \text{(trace of the operator } \hat A) \ .
\end{array}
\qquad (68)
$$

One should note that the set of coherent states form an over-complete basis, in particular they are not orthonormal, in fact $\langle \bar b|a\rangle = e^{\bar b a} \neq 0$. However, it is useful to keep this redundancy. Coherent states may be used to rederive a form of the path integral in phase space in terms of the so-called holomorphic trajectories, corresponding to paths for the a and $\bar a$ complex variables. We will not present it here, but consider only the corresponding fermionic construction which is, mutatis mutandis, analogous.

A similar definition of coherent states can be introduced for fermionic systems. As we have seen the algebra of the anticommutators of the fermionic creation/annihilation operators $\hat\psi^\dagger$ and $\hat\psi$

$$
\{\hat\psi, \hat\psi^\dagger\} = 1\ , \quad \{\hat\psi, \hat\psi\} = 0\ , \quad \{\hat\psi^\dagger, \hat\psi^\dagger\} = 0 \qquad (69)
$$

can be realized by 2×2 matrices acting in the two-dimensional fermionic Fock space generated by the vectors $|0\rangle$ and $|1\rangle$, defined by

$$
\hat\psi|0\rangle = 0\ , \quad |1\rangle = \hat\psi^\dagger|0\rangle\ . \qquad (70)
$$

One defines fermionic coherent states as eigenstates $|\psi\rangle$ of the annihilation operator $\hat\psi$, having the complex Grassmann number ψ as eigenvalue

$$
\hat\psi|\psi\rangle = \psi|\psi\rangle\ . \qquad (71)
$$

The Grassmann numbers, such as ψ and its complex conjugate $\bar\psi$, anticommute between themselves, and we define them to anticommute also with the fermionic operators $\hat\psi^\dagger$ and $\hat\psi$. No confusion should arise between the operators $\hat\psi$ and $\hat\psi^\dagger$ that have a hat, and the complex Grassmann variables ψ and $\bar\psi$, eigenvalues of the eigenstates $|\psi\rangle$ and $\langle\bar\psi|$ respectively, that carry no hat (similarly, in the previous chapter, we indicated position operator, eigenstates and eigenvalues so that $\hat x|x\rangle = x|x\rangle$).

One can prove the following statements

$$
\begin{array}{ll}
(i) & |\psi\rangle = e^{\hat\psi^\dagger \psi}|0\rangle \\
(ii) & \langle\bar\psi| = \langle 0|e^{\bar\psi\hat\psi} \quad \Longrightarrow \quad \langle\bar\psi|\hat\psi^\dagger = \langle\bar\psi|\bar\psi \\
(iii) & \langle\bar\psi|\psi\rangle = e^{\bar\psi\psi} \\
(iv) & \mathbb{1} = \displaystyle\int d\bar\psi d\psi\, e^{-\bar\psi\psi}\, |\psi\rangle\langle\bar\psi| \\
(v) & \mathrm{Tr}\,\hat A = \displaystyle\int d\bar\psi d\psi\, e^{-\bar\psi\psi}\, \langle -\bar\psi|\hat A|\psi\rangle \\
(vi) & \mathrm{Str}\,\hat A \equiv \mathrm{Tr}[(-1)^{\hat F}\hat A] = \displaystyle\int d\bar\psi d\psi\, e^{-\bar\psi\psi}\, \langle\bar\psi|\hat A|\psi\rangle
\end{array}
\qquad (72)
$$

[2] One can represent $|a\rangle$ by $|a\rangle = e^{a\bar a}$ and compute $\hat a|a\rangle = \frac{\partial}{\partial\bar a}e^{a\bar a} = a e^{a\bar a} = a|a\rangle$.

where $\hat{A}$ is an arbitrary bosonic operator.

The proofs can be obtained by explicit calculation. Let us proceed systematically.

(i) One expands the exponential and write the coherent state as

$$\begin{aligned} |\psi\rangle &= e^{\hat{\psi}^\dagger \psi}|0\rangle \\ &= (1+\hat{\psi}^\dagger\psi)|0\rangle = |0\rangle - \psi\hat{\psi}^\dagger|0\rangle \\ &= |0\rangle - \psi|1\rangle \end{aligned} \tag{73}$$

and computes

$$\begin{aligned} \hat{\psi}|\psi\rangle &= \hat{\psi}e^{\hat{\psi}^\dagger\psi}|0\rangle \\ &= \hat{\psi}\Big(|0\rangle - \psi|1\rangle\Big) = -\hat{\psi}\psi|1\rangle = \psi\hat{\psi}|1\rangle = \psi|0\rangle = \psi\Big(|0\rangle - \psi|1\rangle\Big) \\ &= \psi|\psi\rangle \end{aligned} \tag{74}$$

which proves that $|\psi\rangle$ is a coherent state. Note that terms proportional to ψ^2 can be inserted or eliminated at wish, as they vanish due to Grassmann property $\psi^2 = 0$.

(ii) “Bra” coherent state. To prove this relation for a bra coherent state it is sufficient to take the hermitian conjugate of the ket coherent state $|\psi\rangle$. One must remember that the definition of hermitian conjugate reduces to complex conjugation for Grassmann variables and reverses the positions of both variables and operators. For example

$$(\hat{\psi}^\dagger\psi)^\dagger = \bar{\psi}\hat{\psi} \; . \tag{75}$$

(iii) Scalar product. A direct computation (recalling that $\psi^2 = 0$, $\bar{\psi}^2 = 0$ and $\psi\bar{\psi} = -\bar{\psi}\psi$) gives

$$\begin{aligned} \langle\bar{\psi}|\psi\rangle &= \Big(\langle 0| - \langle 1|\bar{\psi}\Big)\Big(|0\rangle - \psi|1\rangle\Big) \\ &= \langle 0|0\rangle + \bar{\psi}\psi\langle 1|1\rangle = 1 + \bar{\psi}\psi \\ &= e^{\bar{\psi}\psi} \; . \end{aligned} \tag{76}$$

We point out that the Grassmann variables are here defined to commute with the Fock vacuum $|0\rangle$, so that they commute with the coherent states, but anticommute with $|1\rangle = \hat{\psi}^\dagger|0\rangle$ (as they anticommute with $\hat{\psi}^\dagger$).

(iv) Resolution of the identity. First of all one must recall that the definition of integration over Grassmann variables makes it identical with differentiation. In particular we use left differentiation, that removes the variable from the left (one must pay attention to signs arising from this operation)

$$\int d\psi \equiv \frac{\partial_{_L}}{\partial\psi} \; , \qquad \int d\bar{\psi} \equiv \frac{\partial_{_L}}{\partial\bar{\psi}} \; . \tag{77}$$

Now a direct calculation shows that

$$\begin{aligned} \int d\bar{\psi}d\psi \, e^{-\bar{\psi}\psi}\,|\psi\rangle\langle\bar{\psi}| &= \int d\bar{\psi}d\psi\,(1-\bar{\psi}\psi)\Big(|0\rangle - \psi|1\rangle\Big)\Big(\langle 0| - \langle 1|\bar{\psi}\Big) \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| \; . \end{aligned} \tag{78}$$

(v) Trace. Given a bosonic operator $\hat{A}$, that commutes with ψ and $\bar{\psi}$, one can verify that

$$
\begin{aligned}
\int d\bar{\psi}d\psi\, e^{-\bar{\psi}\psi}\,\langle -\bar{\psi}|\hat{A}|\psi\rangle &= \int d\bar{\psi}d\psi\,(1-\bar{\psi}\psi)\Big(\langle 0| + \langle 1|\bar{\psi}\Big)\hat{A}\Big(|0\rangle - \psi|1\rangle\Big) \\
&= \int d\bar{\psi}d\psi\,(1-\bar{\psi}\psi)\Big(\langle 0|\hat{A}|0\rangle - \bar{\psi}\psi\langle 1|\hat{A}|1\rangle + \cdots\Big) \\
&= \langle 0|\hat{A}|0\rangle + \langle 1|\hat{A}|1\rangle \\
&= \mathrm{Tr}\,\hat{A}\ . \qquad (79)
\end{aligned}
$$

(vi) Supertrace. An analogous calculation gives

$$
\begin{aligned}
\int d\bar{\psi}d\psi\, e^{-\bar{\psi}\psi}\,\langle \bar{\psi}|\hat{A}|\psi\rangle &= \int d\bar{\psi}d\psi\,(1-\bar{\psi}\psi)\Big(\langle 0| - \langle 1|\bar{\psi}\Big)\hat{A}\Big(|0\rangle - \psi|1\rangle\Big) \\
&= \int d\bar{\psi}d\psi\,(1-\bar{\psi}\psi)\Big(\langle 0|\hat{A}|0\rangle + \bar{\psi}\psi\langle 1|\hat{A}|1\rangle + \cdots\Big) \\
&= \langle 0|\hat{A}|0\rangle - \langle 1|\hat{A}|1\rangle = \mathrm{Tr}[(-1)^{\hat{F}}\hat{A}] \\
&= \mathrm{Str}\,\hat{A}\ . \qquad (80)
\end{aligned}
$$

Here $\hat{F} = \hat{\psi}^\dagger\hat{\psi}$ is the fermion number operator (with eigenvalues $F=0$ for $|0\rangle$ and $F=1$ for $|1\rangle$). The last line gives the definition of the supertrace.

Generalization to more fermionic degrees of freedom is straightofrward.

3 Fermionic path integrals

We now have all the tools to find a path integral representation of the transition amplitude between coherent states $\langle\bar{\psi}_f|e^{-i\hat{H}T}|\psi_i\rangle$, where we have set $\hbar=1$ for notational simplicity. We consider an hamiltonian $\hat{H} = \hat{H}(\hat{\psi}^\dagger, \hat{\psi})$ written in such a way that all creation operators are on the left of the annihilation operators, something that is always possible to achieve using the fundamental anticommutation relations in (69). Note also that for a single pair of fermionic creation/annihilation operators the most general (bosonic) hamiltonian takes the simple form $\hat{H} = \omega\hat{\psi}^\dagger\hat{\psi} + h_0$.

To turn the transition amplitude into a path integral, one divides the total propagation time T into N steps of duration $\epsilon = \frac{T}{N}$, so that $T = N\epsilon$. Using $N-1$ times the decomposition of the identity in terms of coherent states, one gets the following equalites

$$
\begin{aligned}
\langle\bar{\psi}_f|e^{-i\hat{H}T}|\psi_i\rangle &= \langle\bar{\psi}_f|\underbrace{e^{-i\hat{H}\epsilon}e^{-i\hat{H}\epsilon}\cdots e^{-i\hat{H}\epsilon}}_{N\ \text{times}}|\psi_i\rangle \\
&= \langle\bar{\psi}_f|e^{-i\hat{H}\epsilon}\,\mathbb{1}\,e^{-i\hat{H}\epsilon}\,\mathbb{1}\cdots\mathbb{1}\,e^{-i\hat{H}\epsilon}|\psi_i\rangle \\
&= \int\Big(\prod_{k=1}^{N-1} d\bar{\psi}_k d\psi_k\, e^{-\bar{\psi}_k\psi_k}\Big)\prod_{k=1}^{N}\langle\bar{\psi}_k|e^{-i\hat{H}\epsilon}|\psi_{k-1}\rangle \qquad (81)
\end{aligned}
$$

where we have defined $\psi_0 \equiv \psi_i$ and $\bar{\psi}_N \equiv \bar{\psi}_f$. For $\epsilon\to 0$ one can approximate the elementary transition amplitudes as

$$
\begin{aligned}
\langle\bar{\psi}_k|e^{-i\hat{H}(\hat{\psi}^\dagger,\hat{\psi})\epsilon}|\psi_{k-1}\rangle &= \langle\bar{\psi}_k|\Big(1-i\hat{H}(\hat{\psi}^\dagger,\hat{\psi})\epsilon+\cdots\Big)|\psi_{k-1}\rangle \\
&= \langle\bar{\psi}_k|\psi_{k-1}\rangle - i\epsilon\langle\bar{\psi}_k|\hat{H}(\hat{\psi}^\dagger,\hat{\psi})|\psi_{k-1}\rangle+\cdots \\
&= \Big(1-i\epsilon H(\bar{\psi}_k,\psi_{k-1})+\cdots\Big)\langle\bar{\psi}_k|\psi_{k-1}\rangle \\
&= e^{-i\epsilon H(\bar{\psi}_k,\psi_{k-1})}e^{\bar{\psi}_k\psi_{k-1}}\ .
\end{aligned}
\tag{82}
$$

The substitution $\hat{H}(\hat{\psi}^\dagger,\hat{\psi})\to H(\bar{\psi}_k,\psi_{k-1})$ follows from the ordering of the hamiltonian specified previously ($\hat{\psi}^\dagger$ on the left and $\hat{\psi}$ on the right). This allows one to act with the creation operator on a bra eigenstate, and with the annihilation operator on a ket eigenstate, so that all operators in the hamitonian gets substituted by the respective eigenvalues, producing a function of these Grassmann numbers. This way the hamiltonian operator $\hat{H}(\hat{\psi}^\dagger,\hat{\psi})$ gets substituted by the hamiltonian function $H(\bar{\psi}_k,\psi_{k-1})$. These approximations are valid for $N\to\infty$, i.e. $\epsilon\to 0$. Substituting (82) in (81) one finds

$$
\begin{aligned}
\langle\bar{\psi}_f|e^{-i\hat{H}T}|\psi_i\rangle &= \lim_{N\to\infty}\int\Big(\prod_{k=1}^{N-1}d\bar{\psi}_k d\psi_k\, e^{-\bar{\psi}_k\psi_k}\Big)e^{i\sum_{k=1}^{N}[-i\bar{\psi}_k\psi_{k-1}-H(\bar{\psi}_k,\psi_{k-1})\epsilon]} \\
&= \lim_{N\to\infty}\int\Big(\prod_{k=1}^{N-1}d\bar{\psi}_k d\psi_k\Big)e^{i\sum_{k=1}^{N}[i\bar{\psi}_k\frac{(\psi_k-\psi_{k-1})}{\epsilon}-H(\bar{\psi}_k,\psi_{k-1})]\epsilon+\bar{\psi}_N\psi_N} \\
&= \int D\bar{\psi}D\psi\, e^{i\int_0^T dt[i\bar{\psi}\dot{\psi}-H(\bar{\psi},\psi)]+\bar{\psi}(T)\psi(T)} = \int D\bar{\psi}D\psi\, e^{iS[\bar{\psi},\psi]}\ .
\end{aligned}
\tag{83}
$$

This is the path integral for one complex fermionic degree of freedom. We recognize in the exponent a discretization of the classical action

$$
\begin{aligned}
S[\bar{\psi},\psi] &= \int_0^T dt\,[i\bar{\psi}\dot{\psi}-H(\bar{\psi},\psi)]-i\bar{\psi}(T)\psi(T) \\
&\to \sum_{k=1}^{N}\epsilon\,[i\bar{\psi}_k\frac{(\psi_k-\psi_{k-1})}{\epsilon}-H(\bar{\psi}_k,\psi_{k-1})]-i\bar{\psi}_N\psi_N
\end{aligned}
\tag{84}
$$

where $T=N\epsilon$ is the total propagation time. The discrete values ψ_k and $\bar{\psi}_k$ are those corresponding to the values of the continuous functions evaluated at times $t=k\epsilon$, i.e. $\psi_k=\psi(k\epsilon)$ and $\bar{\psi}_k=\bar{\psi}(k\epsilon)$. The last way of writing the amplitude in (83) is symbolic and indicates the formal sum over all paths $\bar{\psi}(t),\psi(t)$ such that $\psi(0)=\psi_0\equiv\psi_i$ and $\bar{\psi}(T)=\bar{\psi}_N\equiv\bar{\psi}_f$, weighed by the exponential of i times the classical action $S[\bar{\psi},\psi]$. Note that the action contains the boundary term $-i\bar{\psi}(T)\psi(T)$, which is essential for formulating a variational principle where one fixes the initial value of the function $\psi(t)$ and the final value of the function $\bar{\psi}(t)$ as boundary data (i.e. $\psi(0)=\psi_i$ and $\bar{\psi}(T)=\bar{\psi}_f$).

Trace

One can now produce a path integral expression for the trace of the transition amplitude $e^{-i\hat{H}T}$. Using the expression of the trace in the coherent state basis, and the path integral representation of the transition amplitude found previously, one finds

$$
\begin{aligned}
\mathrm{Tr}[e^{-i\hat{H}T}] &= \int d\bar{\psi}_0 d\psi_0\, e^{-\bar{\psi}_0\psi_0}\, \langle -\bar{\psi}_0| e^{-i\hat{H}T}|\psi_0\rangle \\
&= \lim_{N\to\infty}\int \Big(\prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k\Big) e^{i\sum_{k=1}^{N}[i\bar{\psi}_k \frac{(\psi_k-\psi_{k-1})}{\epsilon} - H(\bar{\psi}_k,\psi_{k-1})]\epsilon} \\
&= \int_A D\bar{\psi}D\psi\, e^{iS[\bar{\psi},\psi]} \qquad (85)
\end{aligned}
$$

where we have identified $\bar{\psi}_N = -\bar{\psi}_0$ and $\psi_N = -\psi_0$, and used that the exponential $e^{-\bar{\psi}_0\psi_0}$ due to taking the trace cancels the boundary term $e^{\bar{\psi}_N\psi_N}$. Note that with this identification the path integral measure can also be written as $\prod_{k=1}^{N} d\bar{\psi}_k d\psi_k$. In the continuum limit one finds a sum on all antiperiodic paths i.e. such that $\psi(T) = -\psi(0)$ and $\bar{\psi}(T) = -\bar{\psi}(0)$ (A stands for antiperiodic boundary conditions). This representation finds obvious applications in statistical mechanical problems involving fermions.

Supertrace

Similarly, the supertrace is calculated by

$$
\begin{aligned}
\mathrm{Str}[e^{-i\hat{H}T}] &= \int d\bar{\psi}_0 d\psi_0\, e^{-\bar{\psi}_0\psi_0}\, \langle \bar{\psi}_0| e^{-i\hat{H}T}|\psi_0\rangle \\
&= \lim_{N\to\infty}\int \Big(\prod_{k=0}^{N-1} d\bar{\psi}_k d\psi_k\Big) e^{i\sum_{k=1}^{N}[i\bar{\psi}_k \frac{(\psi_k-\psi_{k-1})}{\epsilon} - H(\bar{\psi}_k,\psi_{k-1})]\epsilon} \\
&= \int_P D\bar{\psi}D\psi\, e^{iS[\bar{\psi},\psi]} \qquad (86)
\end{aligned}
$$

where we have now identified $\bar{\psi}_N = \bar{\psi}_0$ and $\psi_N = \psi_0$. Again the term $e^{-\bar{\psi}_0\psi_0}$ due to taking the supertrace cancels the boundary term $e^{\bar{\psi}_N\psi_N}$. In the continuum limit the sum is over all periodic trajectories defined by the boundary conditions $\psi(T) = \psi(0)$ and $\bar{\psi}(T) = \bar{\psi}(0)$ (P stands for periodic boundary conditions).

We have derived the path integral for fermionic systems from the operatorial formulation using a time slicing of the total propagation time. This produces a discretization of the classical action and defines concretely the meaning of the path integral once written in the continuum notation (i.e. it provides a regularization). We have discussed just a simple model with one complex degree of freedom, $\psi(t)$ and its complex conjugate $\bar{\psi}(t)$, that may be called a Dirac fermion in one dimension. The extension to several complex degrees of freedom is immediate.

The case of Majorana fermions

A subtle situation arises if one imposes a reality condition on the fermionic variable $\psi(t)$. In a sense this is the minimal possible model for fermions. Since $\bar{\psi}(t) = \psi(t)$, one may call it a Majorana fermion in one dimension, because of similar definitions for fermionic fields in four dimensions. Its action takes the simple form

$$
S[\psi] = \int dt\, \frac{i}{2}\psi\dot{\psi}\ . \qquad (87)
$$

It is formally real and non vanishing because of the Grassmann character of the variable, but there exists no nontrivial, local and even term that can be written down as hamiltonian. Also, local boundary terms cannot be introduced, as the obvious one would vanish for a single anticommuting variable. Non local boundary terms, consistent with the necessity of imposing boundary conditions equal in number to the number of degrees of freedom (one in our case), are however possible. As already mentioned, canonical quantization gives rise to the anticommutation relation

$$\{\hat{\psi}, \hat{\psi}\} = 1 \tag{88}$$

which may be realized as multiplication by a number, $\hat{\psi} = \frac{1}{\sqrt{2}}$, acting on a one dimensional Hlibert space. There is no room for any dynamics and the model is empty.

For a number $n > 1$ of Majorana fermions ψ^i, $i = 1, ..., n$, one can write down an action containing a nontrivial hamiltonian

$$S[\psi^i] = \int dt \left(\frac{i}{2} \psi^i \dot{\psi}^i - H(\psi^i) \right) . \tag{89}$$

Its canonical quantization gives rise to the algebra

$$\{\hat{\psi}^i, \hat{\psi}^j\} = \delta^{ij} . \tag{90}$$

For an even number of Majorana fermions, say $n = 2m$, one can realize it by pairing together the Majorana fermions to obtain m complex Dirac fermions, rewrite the anticommutator algebra in the new basis, and obtain a set of independent fermionic creation/annihilation operators acting on a fermionic Fock space. Then one may proceed in constructing the path integral as described previously. This procedure is sometimes called "fermion halving", as one halves the number of fermions at the price of making them complex. Local boundary terms in the action can be introduced in this basis and are related to the external coherent states used in the derivation of the path integral expression for the transition amplitude. However, this procedure may hide symmetries manifest in the Majorana basis.

To avoid this last problem, one may instead add a second set of (free) Majorana fermions χ^i to be able to make n complex combinations $\Psi^i = \frac{1}{\sqrt{2}}(\psi^i + i\chi^i)$, and with these Dirac fermions proceed again as before in the construction of the path integral. The fermions χ^i are just free spectators as the hamiltonian does not contain them. However, one must discuss carefully how to fix the external boundary condition to describe the original physical degrees of freedom only. On the circle (trace or supertrace) it is easy to fix this problem: one fixes appropriately the overall normalization of the path integral to correspond to the correct dimensions of the physical Hilbert space. Indeed the ψ^i have their Hilbert space of dimensions $2^{\frac{n}{2}}$, which differs from the dimensions of the full unphysical Hilbert space of the Ψ^i that is 2^n. In this sense this procedure is sometimes called "fermion doubling", as the number of Majorana fermions is doubled.

Finally, if the number of Majorana fermions is odd, say $n = 2m + 1$, one can treat the first $2m$ fermions in either one of the two ways, but for the last one one must necessarily use the fermion doubling procedure. An important application of Majorana fermions is in worldline descriptions of spin 1/2 particles in spacetimes of dimensions $D > 1$, and in particular in $D = 4$. Another method of treating this case make use of more sophisticated procedures, and use the symbol map qunatization, akin to the Weyl quantization scheme. It will not be described at this point, but we use it later on in the treatement of the fermion propagator in $D = 4$ dimensions.

To summarize, we have discussed the path integral quantization of fermionic theories with one-dimensional Majorana fermions ψ^i

$$S[\psi] = \int dt \left(\frac{i}{2} \psi^i \dot{\psi}^i - H(\psi^i) \right) \quad \rightarrow \quad \int D\psi \; e^{iS[\psi]} \tag{91}$$

and one-dimensional Dirac fermions $\psi^i, \bar\psi_i$

$$S[\psi,\bar\psi] = \int dt\, \Big(i\bar\psi_i\dot\psi^i - H(\psi^i,\bar\psi_i)\Big) \quad \to \quad \int D\bar\psi D\psi\; e^{iS[\psi,\bar\psi]} \tag{92}$$

whose path integrals are concretely made sense of by using the time slicing discretization. Other regularizations may be developed as well. They are useful for working directly in the continuum limit and will be discussed in later chapters.

Finally, one can perform a Wick rotation to an euclidean time τ by $t \to -i\tau$, and derive euclidean path integrals of the forms

$$\int D\psi\; e^{-S_E[\psi]} \qquad \text{and} \qquad \int D\bar\psi D\psi\; e^{-S_E[\psi,\bar\psi]} \tag{93}$$

where now

$$S_E[\psi] = \int dt\, \Big(\frac{1}{2}\psi^i\dot\psi^i + H(\psi^i)\Big) \qquad \text{and} \qquad S_E[\psi,\bar\psi] = \int dt\, \Big(\bar\psi_i\dot\psi^i + H(\psi^i,\bar\psi_i)\Big)\,. \tag{94}$$

In this form they will be used in subsequent parts of the book.

3.1 Determinants

Let us now compute some simple functional determinants for fermionic and bosonic path integrals. They reproduce the infinite product representation of sine and cosine, as found by Euler

$$\cos x = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\pi^2\left(n-\frac{1}{2}\right)^2}\right)\,, \qquad \sin x = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right)\,. \tag{95}$$

Let us start with the fermionic harmonic oscillator with action (26) and quantum hamiltonian

$$\hat H = \omega\Big(\hat\psi^\dagger\hat\psi - \frac{1}{2}\Big)\,. \tag{96}$$

We start by computing the trace of $e^{-\beta\hat H}$, the partition function. The path integral computing the trace has antiperiodic boundary conditions on the Grassmann variables ψ and $\bar\psi$ (with period β)

$$\mathrm{Tr}\, e^{-\beta\hat H} = \int_A D\bar\psi D\psi\, e^{-S[\psi,\bar\psi]} \tag{97}$$

and euclidean action

$$S[\psi,\bar\psi] = \int_0^\beta d\tau\, (\bar\psi\dot\psi + \omega\bar\psi\psi)\,. \tag{98}$$

The trace is easily computed in the two-dimensional Fock space

$$\mathrm{Tr}\, e^{-\beta\hat H} = \sum_{n=0}^{1}\langle n|e^{-\beta\hat H}|n\rangle = e^{\frac{\beta\omega}{2}} + e^{-\frac{\beta\omega}{2}} = 2\cosh\frac{\beta\omega}{2} \tag{99}$$

while the fermionic path integral produces a functional determinant over antiperiodic functions

$$\int_A D\bar\psi D\psi\, e^{-S[\psi,\bar\psi]} = \mathrm{Det}_A\,(\partial_\tau+\omega) = \mathrm{Det}_A\,(\partial_\tau)\frac{\mathrm{Det}_A\,(\partial_\tau+\omega)}{\mathrm{Det}_A\,(\partial_\tau)}\,. \tag{100}$$

The absolute normalization is usually difficult to obtain with a path integrals, while a ratio of determinants is more easily computed. The normalization is nevertheless fixed to be $\mathrm{Det}_A\left(\partial_\tau\right) = 2$ (the dimension of the fermionic Hilbert space, $\mathrm{Tr}\,\mathbb{1}$), while the ratio of determinants is computed by expressing them as an infinite products of eigenvalues. A basis of antiperiodic functions with period β is given by

$$f_r(\tau) = e^{2\pi i r \frac{\tau}{\beta}} \qquad r \in \mathbb{Z} + \frac{1}{2} \,. \tag{101}$$

which are also eigenfunctions of $\partial_\tau + \omega$

$$(\partial_\tau + \omega) f_r(\tau) = \left(\frac{2\pi i r}{\beta} + \omega\right) f_r(\tau) \,. \tag{102}$$

Thus we compute

$$\begin{aligned}
\frac{\mathrm{Det}_A\left(\partial_\tau + \omega\right)}{\mathrm{Det}_A\left(\partial_\tau\right)} &= \frac{\prod_{r\in\mathbb{Z}+\frac{1}{2}}\left(\frac{2\pi i r}{\beta} + \omega\right)}{\prod_{r\in\mathbb{Z}+\frac{1}{2}}\left(\frac{2\pi i r}{\beta}\right)} = \prod_{r\in\mathbb{Z}+\frac{1}{2}}\left(1 + \frac{\beta\omega}{2\pi i r}\right) \\
&= \prod_{n=0}^{\infty}\left(1 + \frac{\beta\omega}{2\pi i\left(n+\frac{1}{2}\right)}\right)\left(1 - \frac{\beta\omega}{2\pi i\left(n+\frac{1}{2}\right)}\right) \\
&= \prod_{n=0}^{\infty}\left(1 + \frac{(\beta\omega)^2}{4\pi^2\left(n+\frac{1}{2}\right)^2}\right) = \cosh\frac{\beta\omega}{2}
\end{aligned} \tag{103}$$

which indeed produces

$$\int_A D\bar{\psi} D\psi \, e^{-S[\psi,\bar{\psi}]} = \mathrm{Det}_A\left(\partial_\tau + \omega\right) = 2\cosh\frac{\beta\omega}{2} \tag{104}$$

as expected from the operatorial trace calculation.

We now turn to the more sublte case of the bosonic harmonic oscillator with action in eq. (25) and quantum hamiltonian

$$\hat{H} = \omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right) \,. \tag{105}$$

The path integral computing the partition function has periodic boundary conditions

$$\mathrm{Tr}\, e^{-\beta \hat{H}} = \int_P D\bar{a} Da \, e^{-S[a,\bar{a}]} \tag{106}$$

and euclidean action

$$S[a,\bar{a}] = \int_0^\beta d\tau \left(\bar{a}\dot{a} + \omega \bar{a} a\right) \,. \tag{107}$$

The left hand side of (106) is directly computed in the bosonic Fock space

$$\begin{aligned}
\mathrm{Tr}\, e^{-\beta\hat{H}} &= \sum_{n=0}^{\infty} \langle n | e^{-\beta\hat{H}} | n\rangle = e^{-\frac{\beta\omega}{2}} \sum_{n=0}^{\infty} e^{-\beta\omega n} = \frac{e^{-\frac{\beta\omega}{2}}}{1 - e^{-\beta\omega}} \\
&= \frac{1}{e^{\frac{\beta\omega}{2}} - e^{-\frac{\beta\omega}{2}}} = \frac{1}{2\sinh\frac{\beta\omega}{2}}
\end{aligned} \tag{108}$$

while the right hand side of (106) produces (the inverse of) a functional determinant over periodic functions

$$\int_P D\bar{a}Da\, e^{-S[a,\bar{a}]} = \mathrm{Det}_P^{-1}\left(\partial_\tau + \omega\right) \tag{109}$$

Now fixing the normalization is a bit more subtle, as there are zero modes. Comparison with the calculation on the Hilbert space is the safest way to fix it. The determinant needed is on periodic functions of period β

$$f_n(\tau) = e^{2\pi i n \frac{\tau}{\beta}} \qquad n \in \mathbb{Z} \tag{110}$$

which are also eigenfunctions of $\partial_\tau + \omega$

$$(\partial_\tau + \omega) f_n(\tau) = \left(\frac{2\pi i n}{\beta} + \omega\right) f_r(\tau) \tag{111}$$

and we regulate the determinant as

$$\mathrm{Det}_P\left(\partial_\tau + \omega\right) = \frac{\mathrm{Det}_P\left(\partial_\tau + \omega\right)}{\mathrm{Det}'_P\left(\partial_\tau\right)} \mathrm{Det}'_P\left(\partial_\tau\right) N \tag{112}$$

where the prime means exclusion of the zero mode (the eigenfunction with zero eigenvalue), and N a normalization for the zero mode that should be fixed appropriately. We thus compute

$$\begin{aligned}
\frac{\mathrm{Det}_P\left(\partial_\tau + \omega\right)}{\mathrm{Det}'_P\left(\partial_\tau\right)} &= \frac{\prod_{n\in\mathbb{Z}}\left(\frac{2\pi i n}{\beta} + \omega\right)}{\prod_{n\in\mathbb{Z};n\neq 0}\left(\frac{2\pi i n}{\beta}\right)} = \omega \prod_{n\in\mathbb{Z};n\neq 0}\left(1 + \frac{\beta\omega}{2\pi i n}\right) \\
&= \omega \prod_{n=1}^{\infty}\left(1 + \frac{\beta^2\omega^2}{4\pi^2 n^2}\right) = \omega \frac{\sinh\frac{\beta\omega}{2}}{\frac{\beta\omega}{2}}
\end{aligned} \tag{113}$$

and fixing the nomalization such that $N\,\mathrm{Det}'_P\left(\partial_\tau\right) = \beta$ produces the expected partition function

$$\int_P D\bar{a}Da\, e^{-S[a,\bar{a}]} = \mathrm{Det}_P^{-1}\left(\partial_\tau + \omega\right) = \frac{1}{2\sinh\frac{\beta\omega}{2}}\ . \tag{114}$$

For the fermionic oscillator, the calculation of the supertrace is also of interest

$$\mathrm{Tr}\,[(-1)^{\hat{F}} e^{-\beta\hat{H}}] = \int_P D\bar{\psi}D\psi\, e^{-S[\psi,\bar{\psi}]} \tag{115}$$

where $\hat{F} = \hat{\psi}^\dagger\hat{\psi}$, and the boundary conditions are now periodic. On the Hilbert spaces we directly compute

$$\mathrm{Tr}\,[(-1)^{\hat{F}} e^{-\beta\hat{H}}] = e^{\frac{\beta\omega}{2}} - e^{-\frac{\beta\omega}{2}} = 2\sinh\frac{\beta\omega}{2} \tag{116}$$

while the fermionic path integral gives just the inverse of the bosonic one in (114), that is a determinant rather then an inverse determinant,

$$\int_P D\bar{\psi}D\psi\, e^{-S[\psi,\bar{\psi}]} = \mathrm{Det}_P\left(\partial_\tau + \omega\right) = 2\sinh\frac{\beta\omega}{2}\ . \tag{117}$$

Note that in normalizing the fermionic determinant with the corresponding one with $\omega = 0$ one finds again zero modes but of fermionic nature that could be treated as before. The final result shows these would be zero modes: it vanishes for $\omega \to 0$ as the integration over the Grassmann coordinates of the zero modes are not saturated by any Grassmann number (similarly, its bosonic cousin (114) would diverge for $\omega \to 0$, this is interpreted as an integration over the zero modes that do not carry any action, there is no gaussian damping and the divergence is interpreted as the volume of $\mathbb{R}$).

3.2 Correlation functions

Correlation functions are defined as normalized averages of the dynamical variables. Again, one may introduce a generating functional by adding sources to the path integral. Using an hypercondensed notation, and denoting all fermionic functions by ψ^i and corresponding sources by η_i (taking values in a Grassmann algebra), one may write down the generating functional of correlation functions

$$Z[\eta] = \int D\psi \; e^{iS[\psi]+i\eta_i\psi^i} \; . \tag{118}$$

As an example, the two point function is given by

$$\langle \psi^i \psi^j \rangle = \frac{\int D\psi \; \psi^i \psi^j \; e^{iS[\psi]}}{\int D\psi \; e^{iS[\psi]}} = \frac{1}{Z[0]} \Big(\frac{1}{i}\Big)^2 \frac{\delta^2 Z[\eta]}{\delta\eta_i \delta\eta_j}\bigg|_{\eta=0} \; . \tag{119}$$

In a free theory, identified by a quadratic action of the form $S[\psi] = -\frac{1}{2}\psi^i K_{ij} \psi^j$ with K_{ij} an antisymmetric matrix, one may formally compute the path integral with sources by gaussian integration (after completing squares in terms of $\psi^i + G^{ij}\eta_j$, and making use of the transitional invariance of the measure), thus obtaining an answer of the form

$$Z[\eta] = \det{}^{\frac{1}{2}}(K_{ij}) \; e^{-\frac{i}{2}\eta_i G^{ij} \eta_j} \tag{120}$$

where G^{ij} is the inverse of K_{ij}, also an antisymmetric matrix. One finds

$$\langle \psi^i \psi^j \rangle = -iG^{ij} \tag{121}$$

where G^{ij} is interpreted as a Green function in quantum mechanical applications. To check the overall normalization one must be careful with signs arising from the anticommuting character of the Grassmann variables and from the antisymmetric properties of K_{ij} and G^{ij}.

Similar formulae may be written down for complex fermions (though they are contained in the above formula as well) and in the euclidean time obtained after a Wick rotation. Let us just write down as an example the essential formulae for last case

$$\begin{aligned} Z_E[\eta,\bar\eta] &= \int D\bar\psi D\psi \; e^{-S_E[\psi,\bar\psi]+\bar\eta_i\psi^i+\bar\psi_i\eta^i} \\ \langle \psi^i \bar\psi_j \rangle &= -\frac{1}{Z_E}\frac{\delta}{\delta\bar\eta_i}\frac{\delta}{\delta\eta^j} Z_E[\eta,\bar\eta]\bigg|_{\eta=\bar\eta=0} \end{aligned} \tag{122}$$

and for a free action

$$\begin{aligned} S_E[\psi,\bar\psi] &= \bar\psi_i K^i{}_j \psi^j \\ Z_E[\eta,\bar\eta] &= \int D\bar\psi D\psi \; e^{-S_E[\psi]+\bar\eta_i\psi^i+\bar\psi_i\eta^i} = \det(K^i{}_j)\, e^{\bar\eta_i G^i{}_j \eta^j} \\ \langle \psi^i \bar\psi_j \rangle &= G^i{}_j \end{aligned} \tag{123}$$

where the second one has been computed by completing squares (using the inverse $G^i{}_j$ of $K^i{}_j$). Eventually one must take into account the chosen boundary conditions on the path integral, and use the corresponding Green functions $G^i{}_j$. The whole set of generating functions described already for the bosonic case may be introduced here as well. We leave their derivation as an exercize for the reader.

4 Supersymmetric quantum mechanics

Very special quantum mechanical models containing bosons and fermions may exhibit the property of supersymmetry, a particular symmetry that finds various applications in theoretical physics. In quantum field theories supersymmetry relates particles with integer spins (bosons) to particles with half-integer spins (fermions). It is used to find extensions of the standard model of elementary particles that softens hierarchical problems related to the mass of the Higgs boson. More generally, it helps improving the ultraviolet behavior of quantum field theories and gravitational theories (supergravities). It is also a fundamental ingredient of string theory.

At the quantum mechanical level, supersymmetry appears in worldline models for particles with spin. It finds beautiful applications in mathematical physics, as for example in producing a simple and elegant proof of index theorems, that relate topological properties of differential manifolds to local properties.

In the following section we present a class of quantum mechanical models, including the supersymmetric harmonic oscillator, that exemplifies various aspects of supersymmetry in one of its simplest contexts, that of quantum mechanics. After the introduction of the model, we analyze and verify some important consequences of supersymmetry. Then, we identify its classical action and use it in the path integral quantization to calculate a topological invariant associated to families of supersymmetric theories, the so-called Witten index.

A cautionary note: once should observe that in the first quantized description of particles two different spaces are present: (i) the worldline, that for pedagogical reasons may be considered as a 0+1 dimensional space-time, and (ii) the target 3+1 dimensional space-time, on which the particle propagates and which is the embedding space of the worldline. One may have supersymmetries on both spaces: (i) supersymmetry on the worldline, which is the case we are going to analyze in detail, useful for describing spinning particles. The states called bosonic and fermionic are not related to the spin of the particle in target space (which is fixed), but find a precise definition in the full Hilbert space. (ii) Supersymmetry in target space. Quantum mechanical models carrying this symmetry are often called superparticles, and describe a multiplet of particles containing bosons and fermions in equal numbers, where the bosons are particle with integer spin and fermions particles with half-integer spin. Superparticles are not discussed in this book.

N=2 supersymmetric model

We introduce here a class of supersymmetric models that describe the motion of a non-relativistic particle of unit mass on the line $\mathbb{R}$. The particle carries an intrinsic spin that gives it two possible polarizations, and thus one may think of a non-relativistic electron moving on a line. It is described by hermitian bosonic operators, the position $\hat{x}$ and the momentum $\hat{p}$, satisfying the usual commutation relation

$$[\hat{x}, \hat{p}] = i \tag{124}$$

augmented by fermionic annhilation/creation operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ that satisfy anticommutation relations

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = 1 \, . \tag{125}$$

Other independent graded commutators vanish.

The dynamics is specified by an hamiltonian that depends on a function $W(x)$, called the prepotential, through its first and second derivatives $W'(x)$ and $W''(x)$

$$\hat{H} \;=\; \frac{1}{2}\hat{p}^2 + \frac{1}{2}\Big(W'(\hat{x})\Big)^2 + \frac{1}{2}W''(\hat{x})(\hat{\psi}^\dagger\hat{\psi} - \hat{\psi}\hat{\psi}^\dagger)\;. \tag{126}$$

In addition the model has conserved charges

$$\begin{aligned} \hat{Q} &= (\hat{p} + iW'(\hat{x}))\hat{\psi} \\ \hat{Q}^\dagger &= (\hat{p} - iW'(\hat{x}))\hat{\psi}^\dagger \\ \hat{F} &= \hat{\psi}^\dagger\hat{\psi} \end{aligned} \tag{127}$$

that indeed commute with the hamiltonian, as may be verified by an explicit calculation

$$[\hat{Q},\hat{H}] = [\hat{Q}^\dagger,\hat{H}] = [\hat{F},\hat{H}] = 0\;. \tag{128}$$

The charges $\hat{Q}$ and $\hat{Q}^\dagger$ are the supercharges that generate supersymmetry transformations. They are fermionic operators. There are two of them, $\hat{Q}$ and $\hat{Q}^\dagger$, or equivalently the real and the imaginary part of $\hat{Q}$. This means that there are two supersymmetries, thus the name $N = 2$ supersymmetry. The hamiltonian may look complicated, but in supersymmetric models it is related to the anticommutator of the supercharges, which are more basic objects. In the present case one has

$$\{\hat{Q},\hat{Q}^\dagger\} = 2\hat{H}\;, \qquad \{\hat{Q},\hat{Q}\} = \{\hat{Q}^\dagger,\hat{Q}^\dagger\} = 0\;, \qquad [\hat{H},\hat{Q}] = 0\;. \tag{129}$$

These relations contain the hallmark of supersymmetry: the hamiltonian, which is the generator of time translations, is related to the anticommutator of the supersymmetry charges, i.e. a time translation is obtained by the composition of two supersymmetry transformations.

In the present case there is also a conserved fermion number operator $\hat{F}$, that is seen to satisfy

$$[\hat{Q},\hat{F}] = \hat{Q}\;, \quad [\hat{Q}^\dagger,\hat{F}] = -\hat{Q}^\dagger\;. \tag{130}$$

To summarize, the $N = 2$ supersymmetric model is characterized by a $N = 2$ supersymmetry algebra, a superalgebra with bosonic ($\hat{H}$, $\hat{F}$) and fermionic ($\hat{Q}$, $\hat{Q}^\dagger$) charges with the following non vanishing independent graded commutation relations

$$\{\hat{Q},\hat{Q}^\dagger\} = 2\hat{H}\;, \quad [\hat{Q},\hat{F}] = \hat{Q}\;, \quad [\hat{Q}^\dagger,\hat{F}] = -\hat{Q}^\dagger\;. \tag{131}$$

The operators of the model are realized explicitly in a Hilbert space constructed as follows. The operators $\hat{x}$ and $\hat{p}$ are realized in the usual way on the Hilbert space of square integrable functions $L_2(\mathbb{R})$ by multiplication $\hat{x} \to x$ and differentiation $\hat{p} \to -i\frac{\partial}{\partial x}$. As for the fermionic operators $\hat{\psi}$ e $\hat{\psi}^\dagger$, their algebra identifies them as fermionic annihilation/creation operators realized on a two dimensional Fock space $\mathcal{F}_2$, with basis $|0\rangle$ and $|1\rangle$, defined by

$$\begin{aligned} &\hat{\psi}|0\rangle = 0, \qquad \langle 0|0\rangle = 1 \qquad \text{(Fock vacuum)} \\ &|1\rangle = \hat{\psi}^\dagger|0\rangle \qquad \text{(state with one fermionic exitation)} \end{aligned} \tag{132}$$

as described earlier. We recall that one cannot add additional excitations, as the creation operator satisfies $\{\psi^\dagger,\psi^\dagger\} = 0$, i.e. $(\psi^\dagger)^2 = 0$, so that $|2\rangle = \hat{\psi}^\dagger|1\rangle = (\hat{\psi}^\dagger)^2|0\rangle = 0$. Thus, one has a Pauli exclusion principle. The two states obtained, with $|0\rangle$ considered bosonic and $|1\rangle$

considered fermionic, may be interpreted as describing the two possibile polarizations of the particle mentioned above.

The full Hilbert space is the direct product of the two Hilbert spaces described above, that is $\mathcal{H} = L_2(\mathbb{R}) \otimes \mathcal{F}_2 = L_2(\mathbb{R}) \oplus L_2(\mathbb{R})$, and one may realize the states by wave functions with two components

$$\Psi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} \tag{133}$$

on which the basic operators act as follows

$$\begin{aligned} \hat{x} &\longrightarrow x \\ \hat{p} &\longrightarrow -i\frac{\partial}{\partial x} \\ \hat{\psi} &\longrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \hat{\psi}^\dagger &\longrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \end{aligned} \tag{134}$$

Then, one finds the following explicit realization of the conserved charges

$$\begin{aligned} \hat{Q} &= \begin{pmatrix} 0 & -i\partial_x + iW'(x) \\ 0 & 0 \end{pmatrix} \\ \hat{Q}^\dagger &= \begin{pmatrix} 0 & 0 \\ -i\partial_x - iW'(x) & 0 \end{pmatrix} \\ \hat{H} &= \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\partial_x^2 + \frac{1}{2}(W'(x))^2 - \frac{1}{2}W''(x) & 0 \\ 0 & -\frac{1}{2}\partial_x^2 + \frac{1}{2}(W'(x))^2 + \frac{1}{2}W''(x) \end{pmatrix} \\ \hat{F} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \end{aligned} \tag{135}$$

The hamiltonian is diagonal in the Fock space basis, and contains the two hamiltonians $H_\mp$ with potentials $V_\mp = \frac{1}{2}(W'(x))^2 \mp \frac{1}{2}W''(x)$. We assume them to have a suitable asymptotic behavior that confines the particle, so that the energy eigenstates are bound and have normalized wave functions.

Having defined $\hat{F}$ as the fermion number, we see that the states of the form

$$\Psi_b(x) = \begin{pmatrix} \varphi_1(x) \\ 0 \end{pmatrix}$$

are bosonic as they have fermion number $F = 0$, while states of the form

$$\Psi_f(x) = \begin{pmatrix} 0 \\ \varphi_2(x) \end{pmatrix}$$

are fermionic as they have fermion number $F = 1$. These are consistent worldline assignments of the adjectives bosonic and fermionic. Note that there is no concept of rotations, and thus of spin, on the 0+1 dimensional spacetime given by the worldline: the 0-dimensional space is just a point. From the target space perspective there is no supersymmetry: the model is describing a non relativistic particle of unit mass that moves on a line and has two possible polarizations of its intrinsic spin. The explicit representation of the various operators can be used to verify again the supersymmetric algebra.

The Schrödinger equation takes the standard form

$$i\frac{\partial}{\partial t}\Psi = \hat{H}\Psi \tag{136}$$

and one may solve it to study the model. However many properties can be deduced without finding the explicit solution by using the symmetries .

In the following we analyze some general properties of the energy eigenstates that follows from suspersymmetry, and then verify those properties directly in our model.

General properties of supersymmetry

Let us discuss general properties of supersymmetry that follows form its algebraic structure. We assume that the supersymmetric model has an hamiltonian $\hat{H}$, one hermitian supersymmetry charge $\hat{Q}$, and the parity operator $(-1)^{\hat{F}}$ satisfying the following algebra

$$\{\hat{Q},\hat{Q}\} = 2\hat{H}, \qquad \{(-1)^{\hat{F}},\hat{Q}\} = 0, \qquad [(-1)^{\hat{F}},\hat{H}] = 0\ . \tag{137}$$

This algebra is general enough to derive properties of supersymmetric systems in any dimension.

Our explicit quantum mechanical model has a $N = 2$ susy algebra with hamiltonian $\hat{H}$, two supersymmetry charges $\hat{Q}$ and $\hat{Q}^\dagger$, and fermion number operator $\hat{F}$. It certainly satisfies the requirements stated above. One may equivalently use any one of its two hermitian supercharges $\hat{Q}_1$ and $\hat{Q}_2$, defined by setting $Q = \frac{1}{\sqrt{2}}(\hat{Q}_1+i\hat{Q}_2)$. We satisy the above requirements by choosing for example Q_1. Also, one may exponentiate the fermion number operator to obtain the unitary operator $e^{i\alpha\hat{F}}$ that performs phase rotations belonging to the group $U(1)$. For $\alpha = \pi$ one has the parity operator $(-1)^{\hat{F}}$ needed to meet the above requirements. In the representation of eq. (33) one finds

$$e^{i\alpha\hat{F}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}, \qquad (-1)^{\hat{F}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{138}$$

We also assume a discrete energy spectrum to guarantee that the energy eigenstes are normalizable, and of course the existence of an Hilbert space $\mathcal{H}$ with a positive definite inner product. One can prove the following properties:

Property 1: *The hamiltonian $\hat{H}$ is positive definite.*

It means that for any vector $|\Psi\rangle$ of the Hilbert space $\mathcal{H}$

$$\langle\Psi|\hat{H}|\Psi\rangle \geq 0\ . \tag{139}$$

In fact, using $\hat{H} = \frac{1}{2}\{\hat{Q},\hat{Q}\} = \hat{Q}^2$ with a hermitian $\hat{Q}$ one calculates

$$\langle\Psi|\hat{H}|\Psi\rangle = \langle\Psi|\hat{Q}\hat{Q}|\Psi\rangle = |\hat{Q}|\Psi\rangle|^2 \geq 0 \tag{140}$$

as the norm is positive definite on the Hilbert space. In particular, one deduces that any energy eigenstate with energy E must have $E \geq 0$.

Proprerty 2: *Any state $|\Psi_0\rangle$ with $E = 0$ is necessarily a ground state and supersymmetric.*

Indeed, from the previous calculation one finds that the value $E = 0$ is the lowest energy admissible, so that if a state with such an energy exists, it must be a ground state. In addition, from $\langle\Psi_0|\hat{H}|\Psi_0\rangle = 0$, and using eq. (140), one finds that

$$\hat{Q}|\Psi_0\rangle = 0 \tag{141}$$

as the Hilbert space has a positive definite norm, and the only zero norm state is the null vector. Thus $|\Psi_0\rangle$ is a supersymmetric state, a state invariant under supersymmetry transformations generated by $\hat{Q}$.

Property 3: *Energy levels with $E \neq 0$ are degenerate.*

This is proved by showing that for any "bosonic" state ($(-1)^F = 1$) with $E \neq 0$ there must exists a "fermionic" state ($(-1)^F = -1$) with the same energy, and viceversa. Let us consider a bosonic state $|b\rangle$ (assumed to be normalizable) with energy $E \neq 0$

$$\begin{aligned} \hat{H}|b\rangle &= E|b\rangle\ , \qquad E \neq 0\ , \qquad \langle b|b\rangle = ||b\rangle|^2 = 1 \\ (-1)^{\hat{F}}|b\rangle &= |b\rangle\ . \end{aligned} \tag{142}$$

Then one can construct

$$|f'\rangle = \hat{Q}|b\rangle \tag{143}$$

and show that this is an energy eigenstate of opposite fermionic number

$$\hat{H}|f'\rangle = E|f'\rangle\ , \qquad (-1)^{\hat{F}}|f'\rangle = -|f'\rangle\ . \tag{144}$$

Indeed, using $[\hat{H}, \hat{Q}] = 0$ one finds

$$\hat{H}|f'\rangle = \hat{H}\hat{Q}|b\rangle = \hat{Q}\hat{H}|b\rangle = \hat{Q}E|b\rangle = E|f'\rangle \tag{145}$$

and similarly, using $\{(-1)^{\hat{F}}, \hat{Q}\} = 0$, one finds

$$(-1)^{\hat{F}}|f'\rangle = (-1)^{\hat{F}}\hat{Q}|b\rangle = -\hat{Q}(-1)^{\hat{F}}|b\rangle = -\hat{Q}|b\rangle = -|f'\rangle\ . \tag{146}$$

The state is properly normalized by $|f\rangle = \frac{1}{\sqrt{E}}|f'\rangle$, which is well-defined since $E \neq 0$, so that $\langle f|f\rangle = 1$. Thus, the states $|b\rangle$ and $|f\rangle$ are degenerate in energy and have opposite fermionic parity: they form a two-dimensional representation of the supersymmetry algebra.

The same procedure can be repeated starting from a fermionic eigenstes with $E \neq 0$ to construct the bosonic energy eigenstate partner. One deduces that all energy levels with $E \neq 0$ are degenerate and with an equal number of bosonic and fermionic states, while states with $E = 0$ are supersymmetric (singlets under supersymmetry) and, if they exists, are possible ground states of the model.

The Witten index $\mathrm{Tr}(-1)^{\hat{F}}$

It is useful to define the Witten index as the number of bosonic minus the number of fermionic ground states of vanishing energy

$$n_b^{(E=0)} - n_f^{(E=0)} = \mathrm{Tr}_{(E=0)}(-1)^{\hat{F}} \tag{147}$$

here written also as the trace of $(-1)^{\hat{F}}$ on the subspace of the Hilbert space formed by the states with $E = 0$.

Typically, one rewrite this as $\mathrm{Tr}(-1)^{\hat{F}}$, with the trace extended over the full Hilbert space, as positive energy states cancels pairwise in the sum and do not contribute

$$\mathrm{Tr}(-1)^{\hat{F}} = n_b^{(E=0)} - n_f^{(E=0)} \ . \tag{148}$$

However, this rewriting is formal as the infinite sum does not converge absolutely. To make sure that the cancellation of positive energy states is achieved orderly, one should regulate the Witten index, as for example by defining it as

$$\mathrm{Tr}[(-1)^F e^{-\beta\hat{H}}] \tag{149}$$

that is found to be independent of the regulating parameter β: indeed, using the complete basis of energy eigenstates one calculates

$$\begin{aligned} \mathrm{Tr}[(-1)^{\hat{F}} e^{-\beta\hat{H}}] &= \sum_k \langle k|(-1)^{\hat{F}} e^{-\beta E_k}|k\rangle \\ &= n_b^{(E=0)} - n_f^{(E=0)} + e^{-\beta E_1} - e^{-\beta E_1} + e^{-\beta E_2} - e^{-\beta E_2} + \dots \\ &= n_b^{(E=0)} - n_f^{(E=0)} \ . \end{aligned} \tag{150}$$

The limit $\beta \to 0$ defines $\mathrm{Tr}(-1)^{\hat{F}}$ properly, though the limit is not really necessary, as the sum is independent of β.

The Witten index has topological properties in the sense that it is invariant under reasonable deformations of the parameters of the theory (e.g. one may vary the coupling constants without modifying the asymptotic behavior of the potential): in fact a single state cannot leave the sector $E = 0$ as it must have a partner to form a doublet degenerate in energy. Only pairs of states can leave the zero energy level by acquiring a small value of the energy, so that they can form a supersymmetry doublet containing one bosonic and one fermionic state. Viceversa, states with $E \neq 0$ are paired and the pair contains one boson and one fermion: if by varying the parameters of the theory they acquire the exact value $E = 0$ they do not modify the value of the Witten index.

Calculation of the Witten index

Let us calculate the Witten index in the class of models with $N = 2$ supersymmetries presented above. Vacuum states $|\Psi_0\rangle$ with $E = 0$ satisfy $\hat{H}|\Psi_0\rangle = 0$ so that

$$\langle\Psi_0|\hat{H}|\Psi_0\rangle = \frac{1}{2}\langle\Psi_0|(\hat{Q}\hat{Q}^\dagger + \hat{Q}^\dagger\hat{Q})|\Psi_0\rangle = \frac{1}{2}|\hat{Q}^\dagger|\Psi_0\rangle|^2 + \frac{1}{2}|\hat{Q}|\Psi_0\rangle|^2 = 0 \tag{151}$$

and they must be supersymmetric

$$\hat{Q}|\Psi_0\rangle = 0 \ , \qquad \hat{Q}^\dagger|\Psi_0\rangle = 0 \ . \tag{152}$$

Let us solve these equations. Setting

$$\Psi_0(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

we translate the equation $\hat{Q}|\Psi_0\rangle = 0$ as

$$\begin{aligned} \hat{Q}\Psi_0(x) &= \Big((\hat{p} + iW'(\hat{x}))\hat{\psi}\Big)\Psi_0(x) = \begin{pmatrix} 0 & -i\partial_x + iW'(x) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \\ &= \begin{pmatrix} -i\partial_x f_2(x) + iW'(x)f_2(x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \tag{153}$$

which is solved by the function

$$f_2(x) = c_2 e^{W(x)} \tag{154}$$

with c_2 a constant, while $f_1(x)$ remains arbitrary. Similarly the equation $\hat{Q}^\dagger|\Psi_0\rangle = 0$ takes the form

$$\begin{aligned}
\hat{Q}^\dagger \Psi_0(x) &= \left((\hat{p} - iW'(\hat{x}))\hat{\psi}^\dagger\right)\Psi_0(x) = \begin{pmatrix} 0 & 0 \\ -i\partial_x - iW'(x) & 0 \end{pmatrix}\begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -i\partial_x f_1(x) - iW'(x) f_1(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned} \tag{155}$$

which is solved by

$$f_1(x) = c_1 e^{-W(x)} \tag{156}$$

with c_1 a constant, while $f_2(x)$ remains arbitrary.

The combined equations are then solved by

$$\Psi_0(x) = \begin{pmatrix} c_1 e^{-W(x)} \\ c_2 e^{+W(x)} \end{pmatrix} . \tag{157}$$

Requiring the wave functions to be normalizable fixes the constants c_1 and c_2. There are essentially three different cases to consider:

1. <u>Case $W(x) \stackrel{x\to\pm\infty}{\longrightarrow} \infty$</u>

In such a situation the wave function (157) must have $c_2 = 0$ to be normalizable. Thus there is only one ground state of vanishing energy. It is a boson, as its fermionic number vanishes

$$\hat{F}\Psi_0(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 e^{-W(x)} \\ 0 \end{pmatrix} = 0 \tag{158}$$

and thus $(-1)^F = 1$. The Witten index is calculated to be $\mathrm{Tr}(-1)^{\hat{F}} = 1$.

2. <u>Case $W(x) \stackrel{x\to\pm\infty}{\longrightarrow} -\infty$</u>

Now eq. (157) must have $c_1 = 0$ to be normalizable. There is again only one ground state of vanishing energy, and is fermionic. Indeed it has fermion number $F = 1$ and parity $(-1)^F = -1$, so that the Witten index is $\mathrm{Tr}(-1)^{\hat{F}} = -1$.

3. <u>Case $W(x) \stackrel{x\to\pm\infty}{\longrightarrow} \pm\infty$ or $W(x) \stackrel{x\to\pm\infty}{\longrightarrow} \mp\infty$</u>

In this case $c_1 = c_2 = 0$, so that there are no normalizables solutions with $E = 0$. The Witten index vanishes, $\mathrm{Tr}(-1)^{\hat{F}} = 0$.

Let us also make a brief comment. If there are no states with $E = 0$ one says that supersymmetry is spontaneously broken: the ground state is not invariant under supersymmetry transformations. This happens in the third case above. The consequences of spontaneous breaking of supersymmetry are particularly dramatic in quantum field theories, where a fermionic massless excitation called goldstino appears. In general, a vanishing value of the Witten index does not allow to reach any conclusion, as there might exist several ground states with $E = 0$, half of them bosonic and half of them fermionic, that balance each other to produce a vanishing Witten index. On the other hand, if the Witten index is different form zero supersymmetry certainly cannot be spontaneously broken: there always exists at least one ground state with $E = 0$, which is necessarily invariant under supersymmetry. The concept of spontaneous breaking of symmetries is of fundamental importance in QFT applications.

We may verify the topological properties of the Witten index in the previous model: one may modify at will the prepotential $W(x)$ without altering its asymptotic behavior, but these deformations do not modify the value of $\mathrm{Tr}(-1)^{\hat{F}}$ as the ground states are only fixed by the asymptotics of $W(x)$.

Classical action

One needs the classical action to be able to study the quantum model through path integrals. The action makes use of Grassmann variables for treating the worldline fermions at the classical level. Classical models are useful to identify and construct quantum models. In our case we have started directly with the quantum theory. Now we present its classical action, show that it is supersymmetric, and that its canonical quantization gives rise precisely to the previous quantum theory.

The action is given by

$$S[x,\psi,\bar{\psi}] = \int dt \left[\frac{1}{2}\dot{x}^2 - \frac{1}{2}\Big(W'(x)\Big)^2 + i\bar{\psi}\dot{\psi} - W''(x)\bar{\psi}\psi\right] \tag{159}$$

where $x(t)$ is the coordinate of the particles in one dimension, while $\psi(t)$ and its complex conjugate $\bar{\psi}(t)$ are Grassmann variables that describe additional degrees of freedom[3]. When the prepotential $W(x)$ satisfies suitable asymptotic conditions it guarantees that the energy eigenstates are bound states at the quantum level, and thus normalizable. In particular, the choice $W(x) = \frac{1}{2}\omega x^2$ identifies the so-called supersymmetric harmonic oscillator.

The equations of motion are easily obtained by extremizing the action

$$\begin{aligned}
\frac{\delta S}{\delta x(t)} = 0 \qquad &\Rightarrow \qquad \ddot{x} + W'(x)W''(x) + W'''(x)\bar{\psi}\psi = 0 \\
\frac{\delta S}{\delta \bar{\psi}(t)} = 0 \qquad &\Rightarrow \qquad i\dot{\psi} - W''(x)\psi = 0 \\
\frac{\delta S}{\delta \psi(t)} = 0 \qquad &\Rightarrow \qquad i\dot{\bar{\psi}} + W''(x)\bar{\psi} = 0\ .
\end{aligned} \tag{160}$$

Let us now analyse the symmetries. A first obvious symmetry is the one related to the time translational invariance, induced by $t \to t' = t - a$. This arises as the lagrangian depends on time only implicitly through the dynamical variables. Infinitesimally, this symmetry acts on the dynamical variables as

$$\begin{aligned}
\delta_{_T} x &= a\dot{x} \\
\delta_{_T} \psi &= a\dot{\psi} \\
\delta_{_T} \bar{\psi} &= a\dot{\bar{\psi}}
\end{aligned} \tag{161}$$

and implies conservation of the energy H.

[3]Alternatively, one may use the real Grassmann variables ψ_1 and ψ_2, proportional to the real and imaginary part of ψ as defined by setting $\psi = (\psi_1 + i\psi_2)/\sqrt{2}$.

There are also $U(1)$ phase transformations of the complex Grassman variables that leave the action invariant: using an infinitesimal parameter α they read

$$\begin{aligned}
\delta_{_F} x &= 0 \\
\delta_{_F} \psi &= i\alpha\psi \\
\delta_{_F} \bar{\psi} &= -i\alpha\bar{\psi}
\end{aligned} \tag{162}$$

and imply conservation of the fermion number F.

Finally, the action is invariant under infinitesimal supersymmetry transformations (with ϵ and $\bar{\epsilon}$ Grassmann constants) given by

$$\begin{aligned}
\delta_{_S} x &= i\epsilon\bar{\psi} + i\bar{\epsilon}\psi \\
\delta_{_S} \psi &= -\epsilon(\dot{x} - iW'(x)) \\
\delta_{_S} \bar{\psi} &= -\bar{\epsilon}(\dot{x} + iW'(x))
\end{aligned} \tag{163}$$

which gives rise to conserved charges Q and $\bar{Q}$ called supercharges. This is the most laborious symmetry to verify.

An application of the Noether's theorem gives the following explicit expressions for the conserved charges

$$\begin{aligned}
H &= \frac{1}{2}\dot{x}^2 + \frac{1}{2}\Big(W'(x)\Big)^2 + W''(x)\bar{\psi}\psi \\
F &= \bar{\psi}\psi \\
Q &= (\dot{x} + iW'(x))\psi \\
\bar{Q} &= (\dot{x} - iW'(x))\bar{\psi}\ .
\end{aligned} \tag{164}$$

One may verify that the commutator of two supersymmetry transformations generates a time translation. Calculating it onto the dynamical variable x one obtains

$$[\delta_{_S}(\epsilon_1), \delta_S(\epsilon_2)]x = -a\dot{x} = \delta_{_T}(a)x \tag{165}$$

with $a = 2i(\bar{\epsilon}_1\epsilon_2 - \bar{\epsilon}_2\epsilon_1)$. This is the characterizing property of supersymmetry: the composition of two supersymmetries generate a time translation. The same property can be verified on the variables ψ and $\bar{\psi}$ (in this case on the right hand side there appears also terms proportional to the equations of motion, which however vanish on the physical paths).

Hamiltonian formalism

To perform canonical quantization we have to reformulate the model in phase space. The conjugate momentum to the variable x is obtained as usual by

$$p = \frac{\partial L}{\partial \dot{x}} = \dot{x} \tag{166}$$

As for the Grassmann variables, they have equations of motions that are first order in time, so that they are already in a hamiltonian form. The momentum conjugate to ψ is proportional to $\bar{\psi}$ as

$$\pi = \frac{\partial_{_L} L}{\partial\dot{\psi}} = -i\bar{\psi} \tag{167}$$

where ∂_L denotes left differentiation (one commutes the variable to the left and then removes it). The corresponding hamiltonian is given by the Legendre transform of the lagrangian

$$
\begin{aligned}
H &= \dot{x}p + \dot{\psi}\pi - L \\
&= \frac{1}{2}p^2 + \frac{1}{2}\Big(W'(x)\Big)^2 + W''(x)\bar{\psi}\psi
\end{aligned}
\tag{168}
$$

and coincides with the energy calculated in (164) once written in terms of the phase space variables. The phase space action is given by

$$
S[x,p,\psi,\bar{\psi}] = \int dt\, L = \int dt\, (p\dot{x} + i\bar{\psi}\dot{\psi} - H)\ . \tag{169}
$$

From the previous discussion on the hamiltonian formalism, one knows how to construct the Poisson brackets. Up to boundary terms, the action can be rewritten so to expose the matrices of the symplectic term

$$
S[x,p,\psi,\bar{\psi}] = \int dt\, \left[\frac{1}{2}(x,p)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} + \frac{1}{2}(\psi,\bar{\psi})\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} \dot{\psi} \\ \dot{\bar{\psi}} \end{pmatrix} - H\right] \tag{170}
$$

and one finds that the Poisson bracket for arbitrary phase space functions A and B is given by

$$
\{A,B\}_{PB} = \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - i\frac{\partial_R A}{\partial \psi}\frac{\partial_L B}{\partial \bar{\psi}} - i\frac{\partial_R A}{\partial \bar{\psi}}\frac{\partial_L B}{\partial \psi}\ . \tag{171}
$$

It satisfies the properties listed in eq. (49) when the functions A and B have definite Grassmann parity. In particular, the independent and non vanishing Poisson brackets between the fundamental variables are given by

$$
\{x,p\}_{PB} = 1\ , \quad \{\psi,\bar{\psi}\}_{PB} = -i \tag{172}
$$

which are antisymmetric for commuting variables and symmetric for anticommuting ones.

The Noether charges calculated previously can be written as functions on phase space as

$$
\begin{aligned}
H &= \frac{1}{2}p^2 + \frac{1}{2}\Big(W'(x)\Big)^2 + W''(x)\bar{\psi}\psi \\
F &= \bar{\psi}\psi \\
Q &= (p + iW'(x))\psi \\
\bar{Q} &= (p - iW'(x))\bar{\psi}
\end{aligned}
\tag{173}
$$

and satisfy a Poisson bracket algebra whose non vanishing terms are given by

$$
\{Q,\bar{Q}\}_{PB} = -2iH\ , \quad \{F,Q\}_{PB} = iQ\ , \quad \{F,\bar{Q}\}_{PB} = -i\bar{Q}\ . \tag{174}
$$

The first relation shows that the composition of two supersymmetries generates a translation in time. This is the classical $N=2$ susy algebra.

Canonical quantization is now straightforward, and one recognizes the same quantum theory discussed earlier. In particular, the quantum $N=2$ susy algebra (129) is reproduced. One may notice that an ordering ambiguity emerging in defining the quantum Hamiltonian (the last term proportional to $\bar{\psi}\psi$) is fixed precisely in order to mantain the susy algebra at the quantum level.

Path integrals and Witten index

One can now give a path integral representation of the Witten index. The regulated form of the index $\mathrm{Tr}[(-1)^{\hat F} e^{-\beta \hat H}]$ can be obtained by a Wick rotation $T \to -i\beta$ of the transition amplitude $e^{-i\hat H T}$. The corresponding path integral has an euclidean action S_E, obtained from (159) by Wick rotation ($iS[x,\psi,\bar\psi] \to -S_E[x,\psi,\bar\psi]$)

$$S_E[x,\psi,\bar\psi] = \int_0^\beta d\tau \left[\frac{1}{2}\dot x^2 + \frac{1}{2}\Big(W'(x)\Big)^2 + \bar\psi\dot\psi + W''(x)\bar\psi\psi\right] . \tag{175}$$

Also, one must recall that a trace in the Hilbert space is calculated by a path integral with periodic boundary conditions (P) for the bosonic variables, and antiperiodic boundary conditions (A) for the fermionic ones

$$\mathrm{Tr}\, e^{-\beta \hat H} = \int_P Dx \int_A D\psi \mathcal{D}\bar\psi \; e^{-S_E[x,\psi,\bar\psi]} . \tag{176}$$

The insertion of the operator $(-1)^{\hat F}$ creates instead a supertrace, and has the effect of changing the boundary conditions on the fermions from antiperiodic to periodic

$$\mathrm{Tr}\,[(-1)^{\hat F} e^{-\beta \hat H}] = \int_P Dx D\psi D\bar\psi \; e^{-S_E[x,\psi,\bar\psi]} . \tag{177}$$

Later on we sketch the calculation of the Witten index using this path integral representation. The calculation is simplified by using the fact that the Witten index is invariant under continuous deformations of the parameters of the theory, that can be used to our advantage. One can deform the prepotential as

$$W(x) \to \lambda W(x) \tag{178}$$

with positive λ, and then take the limit $\lambda \to \infty$. The final result must be independent of λ, and indeed one finds that the index is computed by

$$\mathrm{Tr}\,[(-1)^{\hat F} e^{-\beta \hat H}] = \sum_{\{x_0\}} \frac{W''(x_0)}{|W''(x_0)|} \tag{179}$$

where the sum is over all critical points $\{x_0\}$, defined as the points of local maxima and minima that satisfy $W'(x_0) = 0$. This result exemplify how the index connects topological properties to local properties (the critical points).

It is easy to see that this result coincides with the one obtained earlier from the explicit analysis of the ground states with $E = 0$: for case 1 there is always one more minimum than maxima, so that $\mathrm{Tr}\,(-1)^{\hat F} = 1$, for case 2 there is always one more maximum than minima, so that $\mathrm{Tr}\,(-1)^{\hat F} = -1$, for case 3 the number of maxima and minima concide and $\mathrm{Tr}\,(-1)^{\hat F} = 0$.

A direct identification of the ground states with $E = 0$ may be difficult to carry out in more complicated models, and the path integral representation gives an easier tool to compute the Witten index.

Sketch of the calculation

One must sum over all periodic trajectories, i.e. trajectories that close on themselves after an euclidean time β. We indicate such trajectories by $[x(\tau), \psi(\tau), \bar\psi(\tau)]$. The leading contribution to the path integral for $\lambda \to \infty$ is associated to the constant trajectories $[x_0, 0, 0]$, where the constants x_0 are the critical points of the prepotential $W(x)$, defined by the equation $W'(x_0) = 0$. They solve the equations of motion, are obviously periodic, and make the potential terms vanish also in the limit $\lambda \to 0$. Thus, the leading classical approximation to the path integral is given by

$$\mathrm{Tr}\,[(-1)^{\hat{F}} e^{-\beta \hat{H}}] \sim \sum_{\{x_0\}} e^{-S_E[x_0,0,0]} = \sum_{\{x_0\}} e^{-\beta \frac{\lambda^2}{2}(W'(x_0))^2} = \sum_{\{x_0\}} 1 \; .$$

This is not yet correct. One must add the semiclassical corrections due to the quantum fluctuations around the "classical vacua" $[x_0 + \delta x(\tau), \psi(\tau), \bar{\psi}(\tau)]$ that are identified by considering the quadratic part in $[\delta x(\tau), \psi(\tau), \bar{\psi}(\tau)]$ from the expansion of the action around $[x_0, 0, 0]$. These corrections correspond to calculating gaussian path integrals and give rise to functional determinants. Let us try to calculate them. Expanding the action in a Taylor series around the trajectory $[x_0, 0, 0]$ and keeping only the quadratic part in $[\delta x(\tau), \psi(\tau), \bar{\psi}(\tau)]$, one finds

$$\begin{aligned} S_E[x,\psi,\bar{\psi}] &= \int_0^\beta d\tau \left[\frac{1}{2}(\partial_\tau \delta x)^2 + \frac{\lambda^2}{2}\Big(W'(x_0) + W''(x_0)\delta x + \cdots\Big)^2 + \bar{\psi}\dot{\psi} + \lambda(W''(x_0) + \cdots)\bar{\psi}\psi\right] \\ &= \int_0^\beta d\tau \left[\frac{1}{2}\delta x[-\partial_\tau^2 + \lambda^2 (W''(x_0))^2]\delta x + \bar{\psi}[\partial_\tau + \lambda W''(x_0)]\psi\right] + \cdots \; . \end{aligned} \tag{180}$$

The gaussian path integral over $[\delta x(\tau), \psi(\tau), \bar{\psi}(\tau)]$ produces

$$\begin{aligned} \mathrm{Tr}\,[(-1)^{\hat{F}} e^{-\beta \hat{H}}] &= \sum_{\{x_0\}} \int_P D\delta x D\psi D\bar{\psi}\; e^{-S_E[x_0+\delta x,\psi,\bar{\psi}]} \\ &= \sum_{\{x_0\}} \frac{\mathrm{Det}_P\,[\partial_\tau + \lambda W''(x_0)]}{\mathrm{Det}_P^{1/2}\,[-\partial_\tau^2 + \lambda^2 (W''(x_0))^2]} \; . \end{aligned} \tag{181}$$

These determinants are defined by the product of the eigenvalues. A basis of periodic functions with period β is given by

$$f_n(\tau) = e^{\frac{2\pi i n \tau}{\beta}} \qquad n \in Z \; . \tag{182}$$

They are also the eigenfunctions of the differential operators that appear in (181)

$$\begin{aligned} &[\partial_\tau + \lambda W''(x_0)] f_n(\tau) = \lambda_n f_n(\tau) \quad \Rightarrow \quad \lambda_n = \frac{2\pi i n}{\beta} + \lambda W''(x_0) \\ &[-\partial_\tau^2 + (\lambda W''(x_0))^2] f_n(\tau) = \Lambda_n f_n(\tau) \quad \Rightarrow \quad \Lambda_n = \Big(\frac{2\pi n}{\beta}\Big)^2 + (\lambda W''(x_0))^2 \end{aligned}$$

so that

$$\begin{aligned} \frac{\mathrm{Det}_P\,[\partial_\tau + \lambda W''(x_0)]}{\mathrm{Det}_P^{1/2}\,[-\partial_\tau^2 + \lambda^2 (W''(x_0))^2]} &= \prod_{n\in Z} \frac{\lambda_n}{\Lambda_n^{1/2}} \\ &= \prod_{n\in Z} \frac{\frac{2\pi i n}{\beta} + \lambda W''(x_0)}{\left[\Big(\frac{2\pi n}{\beta}\Big)^2 + (\lambda W''(x_0))^2\right]^{1/2}} \\ &= \frac{\lambda W''(x_0)}{|\lambda W''(x_0)|} \prod_{n>0} \frac{[\frac{2\pi i n}{\beta} + \lambda W''(x_0)][-\frac{2\pi i n}{\beta} + \lambda W''(x_0)]}{\Big(\frac{2\pi n}{\beta}\Big)^2 + (\lambda W''(x_0))^2} \\ &= \frac{W''(x_0)}{|W''(x_0)|} \; . \end{aligned} \tag{183}$$

The result is indeed independent of λ, and we have obtained the following value of the Witten index

$$\mathrm{Tr}\,[(-1)^{\hat{F}} e^{-\beta \hat{H}}] = \sum_{\{x_0\}} \frac{W''(x_0)}{|W''(x_0)|} \ . \tag{184}$$

<u>$N = 2$, $D = 1$ superspace</u>

Superspace is a useful construction that gives a geometrical interpretation of supersymmetry. It allows to formulate theories that are manifestly supersymmetric. It is constructed by adding anticommuting coordinates to the usual space-time coordinates, and supersymmetry transformations are intepreted as arising from translations in the anticommuting directions. We exemplify this construction for the previous mechanical model with $N = 2$ supersymmetry, considered as a theory in $(0 + 1)$ space-time dimensions. The extension to $(3 + 1)$ space-time dimensions is conceptually similar, but algebraically more demanding. The $N = 1$ superspace is also presented, as it is sometimes used in worldline description of spin 1/2 fields.

Time translational invariance

To appreciate the ideas underlying the construction of superspace, it is useful to review in a critical way the main points that guarantee the construction of actions invariant under translations in time (which we interpret as a space with one dimension). They will guide us in the construction of superspace, where supersymmetry is interpreted geometrically as a suitable translation along Grassmann directions.

In the case of a single degree of freedom carried by the variable $x(t)$, an infinitesimal time translation ($t \to t - a$) is given by a transport term

$$\delta_{\scriptscriptstyle T} x(t) \equiv x'(t) - x(t) = a\dot{x}(t) = (-iaH)x(t) \tag{185}$$

where in the last form we have used the differential operator $H \equiv i\frac{\partial}{\partial t}$. This operator is seen as the generator of a one parameter group of abelian transformations, the group of time translations that is isomorphic to $\mathbb{R}$ (the real numbers with the addition as group product), and indeed it satisfies the corresponding Lie algebra

$$[H, H] = 0 \ . \tag{186}$$

A finite transformation can be obtained by exponentiation $x'(t) = e^{-iaH}x(t) = x(t + a)$. Time derivatives of $x(t)$ transform similarly, with a rule having the same general structure

$$\delta_{\scriptscriptstyle T} \dot{x}(t) \equiv \dot{x}'(t) - \dot{x}(t) = \frac{\partial}{\partial t}(\delta_{\scriptscriptstyle T} x(t)) = a\ddot{x}(t) = (-iaH)\dot{x}(t) \tag{187}$$

i.e. they are generated by the same differential operator.

Invariant actions can be obtained as integrals in time of a lagrangian that depends on time only implicitly, through the dynamical variables and their derivatives only,

$$S[x] = \int dt\, L(x, \dot{x}) \ . \tag{188}$$

Indeed, as consequence of (185) and (187) it follows that

$$\begin{aligned} \delta_{\scriptscriptstyle T} L(x, \dot{x}) &= \delta_{\scriptscriptstyle T} x \frac{\partial L}{\partial x} + \delta_{\scriptscriptstyle T} \dot{x} \frac{\partial L}{\partial \dot{x}} = a\frac{dx}{dt}\frac{\partial L}{\partial x} + a\frac{d\dot{x}}{dt}\frac{\partial L}{\partial \dot{x}} \\ &= a\frac{d}{dt}L(x, \dot{x}) = \frac{d}{dt}\left(aL(x, \dot{x})\right) \end{aligned} \tag{189}$$

so that

$$\delta_T S[x] = \int dt\, \delta_T L(x,\dot{x}) = \int dt\, \frac{d}{dt}\left(aL(x,\dot{x})\right) = 0 \tag{190}$$

which vanishes up to boundary terms. This is enough to prove invariance.

$N = 2$ superspace

The superspace $\mathbb{R}^{1|2}$ ($N = 2$ superspace in $d = 1$) is defined by its coordinates

$$(t, \theta, \bar{\theta}) \in \mathbb{R}^{1|2} \tag{191}$$

where $(\theta, \bar{\theta})$ are complex Grassmann variables. The generator of time translations is again the differential operator

$$H = i\frac{\partial}{\partial t}\ . \tag{192}$$

In addition, one introduces the differential operators

$$Q = \frac{\partial}{\partial\theta} + i\bar{\theta}\frac{\partial}{\partial t}\ , \qquad \bar{Q} = \frac{\partial}{\partial\bar{\theta}} + i\theta\frac{\partial}{\partial t} \tag{193}$$

that realize the algebra of $N = 2$ supersymmetry

$$\{Q, \bar{Q}\} = 2H\ , \qquad \{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0\ , \quad [Q, H] = [\bar{Q}, H] = 0\ . \tag{194}$$

Translations generated by these differential operators on functions of superspace produce supersymmetry transformations.

Superfields

Superfields are defined to begin with as functions of superspace, and are used as dynamical variables in the construction of supersymmetric models. In addition they are required to have specific transformation rules under supersymmetry, as we shall discuss. Let us consider the example of a scalar superfield $X(t, \theta, \bar{\theta})$, taken to be Grassmann even. Expanded in components (that is performing a Taylor expansion in the anticommuting variables) it reads

$$X(t, \theta, \bar{\theta}) = x(t) + i\theta\psi(t) + i\bar{\theta}\bar{\psi}(t) + \theta\bar{\theta}F(t)\ . \tag{195}$$

The last component is traditionally called F (and should not be confused with the fermion number). Supersymmetry transformations on superfields are generated by the differential operators Q and $\bar{Q}$

$$\delta_S X(t, \theta, \bar{\theta}) = (\bar{\epsilon}Q + \epsilon\bar{Q})X(t, \theta, \bar{\theta}) \tag{196}$$

where ϵ and $\bar{\epsilon}$ are Grassmann parameters. This definition generalizes the similar one for infinitesimal time translation given in eq. (185). Expanding both sides in components one obtains

$$\begin{aligned} \delta_S x &= i\epsilon\bar{\psi} + i\bar{\epsilon}\psi \\ \delta_S \psi &= -\epsilon(\dot{x} + iF) \\ \delta_S \bar{\psi} &= -\bar{\epsilon}(\dot{x} - iF) \\ \delta_S F &= \bar{\epsilon}\dot{\psi} - \epsilon\dot{\bar{\psi}} \end{aligned} \tag{197}$$

to be compared with (163). In general, one call superfields only those functions of superspace that transform as (196) under supersymmetry transformation.

Covariant derivatives and supersymmetric actions

To identify invariant actions it is useful to introduce the covariant derivatives (covariant under supersymmetry transformations), defined by

$$D = \frac{\partial}{\partial\theta} - i\bar{\theta}\frac{\partial}{\partial t} \,, \qquad \bar{D} = \frac{\partial}{\partial\bar{\theta}} - i\theta\frac{\partial}{\partial t} \tag{198}$$

and characterized by the fundamental property of anticommuting with the generators of supersymmetry in (193)

$$\{D, Q\} = \{D, \bar{Q}\} = \{\bar{D}, Q\} = \{\bar{D}, \bar{Q}\} = 0 \,. \tag{199}$$

One may note that D and $\bar{D}$ differ from the operators Q and $\bar{Q}$ only by the sign of the second term, and satisfy the algebra

$$\{D, \bar{D}\} = -2i\partial_t \,, \qquad \{D, D\} = \{\bar{D}, \bar{D}\} = 0 \,. \tag{200}$$

Thanks to these properties, covariant derivatives of superfields are again superfields, meaning that they transform under supersymmerty as the original superfield in (196)

$$\delta_s(DX) = D(\delta_s X) = D\left((\bar{\epsilon}Q + \epsilon\bar{Q})X\right) = (\bar{\epsilon}Q + \epsilon\bar{Q})DX \,. \tag{201}$$

Similarly for $\bar{D}X$. As a consequence a lagrangian $L(X, DX, \bar{D}X)$, function of the superspace point only implicitly through dynamical superfields and their covariant derivatives, (that is without any explicit dependence on the point of superspace) transforms as

$$\delta_s L(X, DX, \bar{D}X) = (\bar{\epsilon}Q + \epsilon\bar{Q})L(X, DX, \bar{D}X) \,. \tag{202}$$

Actions defined by integrating over the whole superspace

$$S[X] = \int dt d\bar{\theta} d\theta \, L(X, DX, \bar{D}X) \tag{203}$$

are manifestly invariant (up to boundary terms), as they transform as integrals of total derivatives.

In particular, the model described by

$$S[X] = \int dt d\bar{\theta} d\theta \left(\frac{1}{2}DX\bar{D}X + W(X)\right) \tag{204}$$

that depends on an arbitrary function $W(x)$ is manifestly supersymmetric. A direct integration over the anticommuting coordinates of superspace shows that it reproduces the action in eq. (159). In a more elegant fashion, one uses the algebra of covariant derivatives to obtain the same final result. Let us show this in a telegraphic way. Up to total derivatives one may write

$$\begin{aligned}
S[X] &= \int dt d\bar{\theta} d\theta \left(\frac{1}{2}DX\bar{D}X + W(X)\right) \\
&= \int dt \bar{D} D \left(\frac{1}{2}DX\bar{D}X + W(X)\right)\Big|_{\theta,\bar{\theta}=0} \\
&= \int dt \left(-\frac{1}{2}\bar{D}DXD\bar{D}X - iDX\bar{D}\dot{X} + W''(X)\bar{D}XDX + W'(X)\bar{D}DX\right)\Big|_{\theta,\bar{\theta}=0}
\end{aligned}$$

and using the projections on the first component of the various superfields

$$\begin{aligned} X\Big|_{\theta,\bar\theta=0} &= x(t) \\ DX\Big|_{\theta,\bar\theta=0} &= i\psi(t) \\ \bar{D}X\Big|_{\theta,\bar\theta=0} &= i\bar\psi(t) \\ D\bar{D}X\Big|_{\theta,\bar\theta=0} &= -(F+i\dot{x}) \\ \bar{D}DX\Big|_{\theta,\bar\theta=0} &= F-i\dot{x} \end{aligned}$$

one finds (up to total derivatives)

$$S[X] = \int dt \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}F^2 + i\bar\psi\dot\psi + W'(x)F - W''(x)\bar\psi\psi \right) . \tag{205}$$

Eliminating the auxiliary field F through its algebraic equation of motion ($F = -W'(x)$) one recovers the action in (159), which is thus guaranteed to be supersymmetric by the superspace construction.

$N = 1$, $D = 1$ superspace

A superspace can also be constructed for $N = 1$ supersymmetry, with a real (i.e. hermitian) supercharge. It is used in worldline descriptions of spin 1/2 fields, where the Γ^i matrices appearing in the Dirac equations are realized on the worldline by real Grassmann variables ψ^i (worldline Majorana fermions). Indeed, the simple model

$$S[x,\psi] = \int dt \left(\frac{1}{2}\dot{x}^i\dot{x}^i + \frac{i}{2}\psi^i\dot\psi^i \right) \tag{206}$$

has $N = 1$ supersymmetry and appears in first quantized descriptions of the Dirac field. One can give a superspace construction to prove its supersymmetry. The $N = 1$ supersymmetry algebra $\{Q, Q\} = 2H$ is realized by the differential operators

$$Q = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial t} , \qquad H = i\frac{\partial}{\partial t} \tag{207}$$

that act on functions of a superspace with coordinates (t, θ). The susy covariant derivative is given by

$$D = \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t} \tag{208}$$

and anticommutes with Q. Using the superfields

$$X^i(t,\theta) = x^i(t) + i\theta\psi^i(t) \tag{209}$$

one can construct in superspace the manifestly supersymmetric action

$$S[X] = \frac{i}{2}\int dt d\theta D X^i \dot{X}^i \tag{210}$$

which reduces to (206) when passing to the superfield components. The susy transformations are realized geometrically by

$$\delta_{_S} X^i = \epsilon Q X^i \tag{211}$$

where ϵ is an infinitesimal real Grassmann parameter. In components they reduce to

$$\delta_{_S} x^i = i\epsilon\psi^i \ , \qquad \delta_{_S}\psi^i = -\epsilon\dot{x}^i \ . \tag{212}$$

They are easily seen to leave the action (206) invariant. An application of the Noether method shows that the associated conserved charge is $Q = \psi^i \dot{x}^i$.